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ON THE ANALOG OF SHEPHARD PROBLEM FOR THE L_p–PROJECTION BODY

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Abstract. For $p \geqslant 1$, Lutwak, Yang and Zhang introduced the concept of L_p-projection body. In this paper, we develop a Fourier analytic approach in the L_p-Brunn-Minkowski theory. We consider the question of whether $\Pi_p K \subseteq \Pi_p L$ implies $\Omega_p(K) \leqslant \Omega_p(L)$. We also formulate and solve an analog of the Shephard problem for the L_p-projection body.

1. Introduction

Let $\mathscr{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors and the set of origin-symmetric convex bodies in $\mathbb{R}^n$, we write $\mathscr{K}_o^n$ and $\mathscr{K}_s^n$, respectively. Let S^{n-1}, B denote the unit sphere and the standard unit ball in Euclidean space $\mathbb{R}^n$, respectively. Denote by $\mathtt{Vol}_n(K)$ the n-dimensional volume of body K, for the standard unit ball B in $\mathbb{R}^n$, denote $\mathtt{Vol}_n(B) = \omega_n$.

If $K \in \mathscr{K}^n$, its support function, $h_K = h(K,\cdot) : \mathbb{R}^n \longrightarrow (0,\infty)$, is defined by $h(K,u) = \max\{u \cdot x : x \in K\}$, $u \in S^{n-1}$. Where $x \cdot y$ denotes the standard inner product of x and y.

If K is a compact star-shaped (about the origin) in $\mathbb{R}^n$, its radial function $\rho(K,\cdot)$ is defined by $\rho(K,u) = \max\{\lambda \geqslant 0 : \lambda u \in K\}, u \in S^{n-1}$. If $\rho(K,u)$ is positive and continuous, K will be called a star body (about the origin). Let S_o^n denote the set of star bodies (about the origin) in $\mathbb{R}^n$. Two star bodied K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

For $K \in \mathscr{K}_o^n$, the polar body of K, K^*, is defined by $K^* = \{x \in \mathbb{R}^n : x \cdot y \leqslant 1, y \in K\}$. Obviously, we have $\rho(K^*,\cdot) = 1/h(K,\cdot)$ and $(K^*)^* = K$.

The projection body was introduced at the turn of the previous century by Minkowski. For $K \in \mathscr{K}^n$, the projection body of K, ΠK, is a centrally symmetric convex body whose support function is given by (see [3])

$$h(\Pi K,\theta) = \mathtt{Vol}_{n-1}(K|\theta^{\perp}) = \frac{1}{2}\int_{S^{n-1}} |\theta \cdot u| dS(K,u), \quad \text{for all } \theta \in S^{n-1},$$

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where $\mathtt{vol}_{n-1}$ denotes $(n-1)$-dimensional volume, $K|\theta^{\perp}$ denotes the image of the orthogonal projection of K onto the codimensional 1 subspace orthogonal to θ, and $S(K,\cdot)$ is the surface area measure of K.

For the projection body, Koldobsky, Ryabogin and Zvavitch (see [4]) proved that if the surface area measure of a convex body K is absolutely continuous, then

$$h(\Pi K,\theta) = -\frac{1}{\pi}\widehat{f(K,\cdot)}(\theta), \quad \forall\theta \in S^{n-1}, \tag{1.1}$$

where $f(K,\cdot)$ is the curvature function of the body K, we may extended $f(K,\cdot)$ to a homogeneous of degree $-n-1$ function on $\mathbb{R}^n$, and $\widehat{f(K,\cdot)} := f(K,\cdot)^{\wedge}$ is Fourier transform of function $f(K,\cdot)$, which is in the sense of distributions. It turns out that this formula can serve as the main tool in the study of different problems concerning the volumes of projections. In particular, it can be applied to the following Shephard problem of the projection body.

Let K,L be origin-symmetric convex bodies in $\mathbb{R}^n$ and suppose that, for every $\theta \in S^{n-1}$,

$$\Pi K \subseteq \Pi L. \tag{1.2}$$

Does it follow that

$$\mathtt{Vol}_n(K) \leqslant \mathtt{Vol}_n(L)? \tag{1.3}$$

This problem was solved independently by Petty [14] and Schneider [16], who showed that the answer is affirmative if $n \leqslant 2$ and negative if $n \geqslant 3$. It is also well known [16] that the Shephard problem has an affirmative answer if L is a projection body.

Ryabogin and Zvavitch (see [15]) extended the above facts to the Lp-projection body which was introduced by Lutwak, Yang and Zhang (see [9]). For each $K \in \mathscr{K}_o^n$ and real $p \geqslant 1$, the L_p-projection body of K, $\Pi_p K$, be an origin-symmetric convex body whose support function is given by

$$h(\Pi_p K,x)^p = \frac{1}{2n}\int_{S^{n-1}} |x\cdot u|^p dS_p(K,u), \quad x \in \mathbb{R}^n. \tag{1.4}$$

Here $S_p(K,\cdot)$ is L_p-surface area measure of K.

In [15], authors extended the analog of (1.1) as follows:

$$h(\Pi_p K,\xi)^p = \frac{2\pi C_p}{n}\widehat{f_p(K,\cdot)}(\xi). \tag{1.5}$$

Here p is not an even integer, $f_p(K,\cdot)$ is L_p-curvature function of the body K, and C_p is a constant depending only on p , i.e.

$$C_p = \frac{2^{p+1}\sqrt{\pi}\Gamma((p+1)/2)}{\Gamma(-p/2)} = -2\Gamma(p+1)\sin\frac{\pi p}{2}$$

is positive for each $p \in (4k-2,4k)$ and negative for $p \in (4k,4k+2)$.

From formula (1.5) and together with Lutwak's generalized Minkowski theorem, Ryabogin and Zvavitch (see [15]) obtained a generalization of the following Shephard problem for the Lp-projection body:

Shephard problem for L_p-projection body. Consider two origin-symmetric convex bodies K and L in $\mathbb{R}^n$. Fix $p \geqslant 1$ and suppose that

$$\Pi_p K \subseteq \Pi_p L. \tag{1.6}$$

Does it follow that

$$\mathtt{Vol}_n(K) \leqslant \mathtt{Vol}_n(L), \qquad \text{for } 1 \leqslant p < n,$$

and

$$\mathtt{Vol}_n(K) \geqslant \mathtt{Vol}_n(L), \qquad \text{for } n < p\ ?$$

In the case $p = 1$, condition (1.6) is equivalent to (1.2), and the answer is affirmative if $n \leqslant 2$ and negative if $n \geqslant 3$. It was proved in [15] that the answer is negative for any $n \geqslant 2$ and $p > 1$. Actually, it was proved that the Shephard problem for the L_p-projection body has an affirmative answer if L is a L_p-projection body, and that the existence of the body which is not a L_p-projection body leads to a counterexample.

In this paper, we prove that the analog of Shephard's problem for the L_p-projection body. For $p \geqslant 1$, let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$, and both of them have a positive continuous curvature function, and $\Lambda_p L$ be L_p-curvature image of body L (see Section 2 for definition), for which $(\mathbb{R}^n, \|\cdot\|_{\Lambda_p L})$ embeds in L_p, and

$$\Pi_p K \subseteq \Pi_p L.$$

Then

$$\Omega_p(K) \leqslant \Omega_p(L).$$

Where $\Omega_p(K)$ be L_p-affine surface area of K (see Section 2 for definition). On the other hand, let K be an origin-symmetric convex body with a positive continuous curvature function in $\mathbb{R}^n$, and $\Lambda_p K$ be an infinitely smooth origin-symmetric strictly convex body in $\mathbb{R}^n$, for which $(\mathbb{R}^n, \|\cdot\|_{\Lambda_p K})$ does not embed in L_p. Then there exists an origin-symmetric convex body L in $\mathbb{R}^n$ leads to a counterexample.

Using conclusion relating to isometric embedding, the conclusion of above can be reformulated as follows theorem.

THEOREM 1.1. *Let $p \geqslant 1, K, L$ be infinitely smooth origin-symmetric convex bodies in $\mathbb{R}^n$ with positive continuous curvature function such that $\Pi_p K \subseteq \Pi_p L$. Then if $p = 1$, $\Omega_p(K) \leqslant \Omega_p(L)$ holds surely in $\mathbb{R}^2$, while if $p > 1$, this is no longer true in dimensions $n \geqslant 2$.*

2. Preliminaries

2.1. L_p-Mixed Volume

Firey [2] extended the concept of Minkowski linear combination. For $p \geqslant 1$, $K, L \in \mathscr{K}_o^n$ and $\alpha, \beta > 0$, the Firey L_p-combination $\alpha K +_p \beta L \in \mathscr{K}_o^n$ is defined by

$$h(\alpha K +_p \beta L, \cdot)^p = \alpha h(K, \cdot)^p + \beta h(L, \cdot)^p.$$

where "$\cdot$" in $\varepsilon \cdot L$ denotes the Firey scalar multiplication. For $p=1$, $K+_p \varepsilon \cdot L$ is just the Minkowski linear combination of K and L.

Lutwak (see [8,10]) showed that the Firey L_p-combination lead to a Brunn-Minkowski theory for $p \geqslant 1$. He introduced the notion of L_p-mixed volume as follows: For $K,L \in \mathscr{K}_o^n$ and $p \geqslant 1$, L_p-mixed volume of K and L, $V_p(K,L)$, is defined by[10]

$$\frac{n}{p}V_p(K,L) = \lim_{\varepsilon \to 0} \frac{V(K+_p \varepsilon L) - V(K)}{\varepsilon}.$$

Lutwak (see [10]) further proved that for each $K \in \mathscr{K}_o^n$, there exists a positive Borel measure $S_p(K,\cdot)$ on S^{n-1} so that

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} h(L,u)^p dS_p(K,u), \tag{2.1}$$

for all $L \in \mathscr{K}_o^n$. It turns out that the measure $S_p(K,\cdot)$ is absolutely continuous with respect to $S(K,\cdot)$, and has the Radon-Nikodym derivative

$$\frac{dS_p(K,\cdot)}{dS(K,\cdot)} = h^{1-p}(K,\cdot).$$

If $S_p(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S, we have

$$\frac{dS_p(K,\cdot)}{dS} = f_p(K,\cdot). \tag{2.2}$$

Together with (2.1) and (2.2), we have

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} h(L,u)^p f_p(K,u) dS(u), \tag{2.3}$$

for all $L \in \mathscr{K}_o^n$. In particular

$$\mathtt{Vol}_n(K) = \frac{1}{n}\int_{S^{n-1}} h(K,u)^p f_p(K,u) dS(u). \tag{2.4}$$

If a body K has both L_p-curvature and the curvature functions, then (see [8])

$$f_p(K,\cdot) = h(K,\cdot)^{1-p} f(K,\cdot).$$

Lutwak (see [10]) generalized the Brunn-Minkowski inequality to the case of L_p-mixed volumes: For $K,L \in \mathscr{K}_o^n$, $p \geqslant 1$, then

$$V_p(K,L)^n \geqslant \mathtt{Vol}_n(K)^{n-p} \mathtt{Vol}_n(L)^p.$$

He also proved a generalization of the classical Minkowski theorem, which states that given $p>0, p \neq n$, and a continuous even function $g: S^{n-1} \to \mathbb{R}^+$, there exists a unique convex body K such that $f_p(K,\cdot) = g$.

2.2. L_p-Affine Surface Area and ith L_p-Mixed Affine Surface Area

Let $\mathscr{F}_o^n, \mathscr{F}_s^n$ denote the set of all bodies in $\mathscr{K}_o^n, \mathscr{K}_s^n$, respectively, and both of them have a positive continuous curvature function.

Lutwak (see [8]) introduced the concept of L_p-affine surface area and can be described as follows: For $K \in \mathscr{F}_o^n$, L_p-affine surface area of K, $\Omega_p(K)$, is defined by

$$\Omega_p(K) = \int_{S^{n-1}} f_p(K,u)^{\frac{n}{n+p}} dS(u). \tag{2.5}$$

Further, Lutwak (see [8]) introduced the notion of L_p-mixed affine surface area. For $p \geqslant 1$, L_p-mixed affine surface area of $K_1, \cdots, K_n \in \mathscr{F}_o^n$, $\Omega_p(K_1, \cdots, K_n)$, is defined by

$$\Omega_p(K_1, \cdots, K_n) = \int_{S^{n-1}} [f_p(K_1,u) \cdots f_p(K_n,u)]^{\frac{1}{n+p}} dS(u). \tag{2.6}$$

In (2.6), let $K_1 = \cdots = K_{n-i} = K$ and $K_{n-i+1} = \cdots = K_n = L (i = 0, \cdots, n)$, then $\Omega_{p,i}(K,L) = \Omega_p(K, \cdots, K, L, \cdots, L)$, with $n-i$ copies of K, and i copies of L. If i is any real, Wang and Leng [17] define that: For $K, L \in \mathscr{F}_o^n, p \geqslant 1, i \in \mathbb{R}$, the ith L_p-mixed affine surface area of K and L, $\Omega_{p,i}(K,L)$, is defined by

$$\Omega_{p,i}(K,L) = \int_{S^{n-1}} f_p(K,u)^{\frac{n-i}{n+p}} f_p(L,u)^{\frac{i}{n+p}} dS(u). \tag{2.7}$$

Specially, for the case $i = -p$, we have that

$$\Omega_{p,-p}(K,L) = \int_{S^{n-1}} f_p(K,u) f_p(L,u)^{\frac{-p}{n+p}} dS(u). \tag{2.8}$$

If $p = 1$, then $\Omega_{1,-1}(K,L)$ is just $\Omega_{-1}(K,L)$ (see[11]).

In (2.7), let $L = B$ and we write

$$\Omega_{p,i}(K,B) = \Omega_{p,i}(K), \tag{2.9}$$

From (2.2), we get $f_p(B,\cdot) = 1$, which together with (2.7) and (2.9) lead to

$$\Omega_{p,i}(K) = \int_{S^{n-1}} f_p(K,u)^{\frac{n-i}{n+p}} dS(u), \tag{2.10}$$

where $\Omega_{p,i}(K)$ is called the ith L_p-mixed affine surface area of $K \in \mathscr{F}_o^n$.

The Minkowski inequality for the ith L_p-mixed affine surface area is shown as follows (see[17]):

If $K, L \in \mathscr{F}_o^n, p \geqslant 1, i \in \mathbb{R}$, then for $i < 0$ or $i > n$,

$$\Omega_{p,i}(K,L)^n \geqslant \Omega_p(K)^{n-i} \Omega_p(L)^i; \tag{2.11}$$

for $0 < i < n$,

$$\Omega_{p,i}(K,L)^n \leqslant \Omega_p(K)^{n-i} \Omega_p(L)^i, \tag{2.12}$$

with equality in every inequality for $p = 1$ if and only if K and L are homothetic, for $n \neq p > 1$ if and only if K and L are dilater. For $i = 0$ or $i = n$, (2.11) (or (2.12)) is identical.

2.3. L_p-Curvature Image

Lutwak (see [8]) introduced the notion of L_p-curvature image and can be described as follows: For each $K \in \mathscr{F}_o^n$ and real $p \geqslant 1$, define $\Lambda_p K \in \mathscr{S}_o^n$ be a star body (about the origin) in $\mathbb{R}^n$, the L_p-curvature image of K, by

$$\rho(\Lambda_p K, \cdot)^{n+p} = \frac{\mathtt{Vol}_n(\Lambda_p K)}{\omega_n} f_p(K, \cdot). \tag{2.13}$$

Note that for $p = 1$, this definition differs from the definition of classical curvature image (see [8]).

For L_p-curvature image and L_p-affine surface area, we have the following result: If $K \in \mathscr{F}_o^n, p \geqslant 1$, then

$$\mathtt{Vol}_n(\Lambda_p K)^{\frac{p}{n+p}} = \frac{1}{n} \omega_n^{-\frac{n}{n+p}} \Omega_p(K). \tag{2.14}$$

3. Main Results and its Proofs

The Minkowski functional of a star-shaped origin-symmetric body $K \subset \mathbb{R}^n$ is defined as

$$\|x\|_K = \min\{a \geqslant 0, x \in aK\}.$$

The function $\rho_K(x) = \|x\|_K^{-1}$ is called the radial function of K. If $x \in S^{n-1}, \rho_K(x)$ is the distance from the origin to the boundary of K in the direction of x. We denote by $(\mathbb{R}^n, \|\cdot\|_K)$ the Euclidean space equipped with the Minkowski functional of the body K. Clearly, $(\mathbb{R}^n, \|\cdot\|_K)$ is a normed space if and only if the body K is convex.

A well-known result (see [1,p.189] or [5, Section 6.1]) is that a space $(\mathbb{R}^n, \|\cdot\|)$ embeds into $L_p, p > 0$ if and only if there exists a finite Borel measure μ on the unit space, such that

$$\|x\|^p = \int_{S^{n-1}} |x \cdot \xi|^p d\mu(\xi), \tag{3.1}$$

for every $x \in \mathbb{R}^n$. On the other hand, this can be considered as the definition of embedding in $L_p, -1 < p < 0$ (see [6]).

It was proved in [7] that a space $(\mathbb{R}^n, \|\cdot\|)$ embeds isometrically in $L_p, p > 0, p \neq 2, 4, \cdots$, if and only if the Fourier transform $\Gamma(-p/2)(\|\cdot\|_K^p)^\wedge$ is a positive distribution on $\mathbb{R}^n \backslash \{0\}$. If $-n < p < 0$, a similar fact was proved in [6]: a space $(\mathbb{R}^n, \|\cdot\|)$ embeds in L_p if and only if the Fourier transform $(\|.\|^p)^\wedge$ of $\|\cdot\|^p$ is a positive distribution in the whole $\mathbb{R}^n$.

Now, we first prove a result as follows:

THEOREM 3.1. *Consider $p \geqslant 1$. Let $K, L \in \mathscr{F}_s^n$ and $\Lambda_p L \in \mathscr{S}_o^n$, so that $(\mathbb{R}^n, \|\cdot\|_{\Lambda_p L})$ embeds in L_p and*

$$\Pi_p K \subseteq \Pi_p L.$$

Then $\Omega_p(K) \leqslant \Omega_p(L)$.

Proof. Since $(\mathbb{R}^n, \|\cdot\|_{\Lambda_p L})$ embeds in L_p, there exists a finite Borel measure $\mu_{\Lambda_p L}$ on the unit sphere S^{n-1} such that

$$\|x\|^p_{\Lambda_p L} = \int_{S^{n-1}} |x\cdot\xi|^p d\mu_{\Lambda_p L}(\xi), \quad x\in\mathbb{R}^n.$$

Note that $\Pi_p K \subseteq \Pi_p L$ can be written as

$$\frac{1}{2n}\int_{S^{n-1}} |x\cdot\xi|^p f_p(K,x)dx \leqslant \frac{1}{2n}\int_{S^{n-1}} |x\cdot\xi|^p f_p(L,x)dx, \quad \xi\in S^{n-1}.$$

Integrating both sides of the last inequality over S^{n-1} with measure $\mu_{\Lambda_p L}$, we get

$$\int_{S^{n-1}}\int_{S^{n-1}} |x\cdot\xi|^p f_p(K,x)dxd\mu_{\Lambda_p L}(\xi) \leqslant \int_{S^{n-1}}\int_{S^{n-1}} |x\cdot\xi|^p f_p(L,x)dxd\mu_{\Lambda_p L}(\xi).$$

Applying Fubini's Theorem, we have

$$\int_{S^{n-1}} \|x\|^p_{\Lambda_p L} f_p(K,x)dx \leqslant \int_{S^{n-1}} \|x\|^p_{\Lambda_p L} f_p(L,x)dx, \tag{3.2}$$

note that $\|x\|_{\Lambda_p L} = h(\Lambda_p^* L, x)$, therefore, (3.2) can be rewritten as

$$\int_{S^{n-1}} h(\Lambda_p^* L,x)^p f_p(K,x)dx \leqslant \int_{S^{n-1}} h(\Lambda_p^* L,x)^p f_p(L,x)dx. \tag{3.3}$$

On the other hand, from (2.5), (2.8) and (2.13), we have

$$\begin{aligned}
\Omega_p(L) &= \int_{S^{n-1}} f_p(L,x)^{\frac{n}{n+p}} dx \\
&= \int_{S^{n-1}} f_p(L,x) f_p(L,x)^{-\frac{p}{n+p}} dx \\
&= \Big(\frac{\mathtt{Vol}_n(\Lambda_p L)}{\omega_n}\Big)^{\frac{p}{n+p}} \int_{S^{n-1}} f_p(L,x)\rho(\Lambda_p L,x)^{-p} dx \\
&= \Big(\frac{\mathtt{Vol}_n(\Lambda_p L)}{\omega_n}\Big)^{\frac{p}{n+p}} \int_{S^{n-1}} h(\Lambda_p^* L,x)^p f_p(L,x)dx,
\end{aligned}$$

and

$$\begin{aligned}
\Omega_{p,-p}(K,L) &= \int_{S^{n-1}} f_p(K,x) f_p(L,x)^{-\frac{p}{n+p}} dx \\
&= \Big(\frac{\mathtt{Vol}_n(\Lambda_p L)}{\omega_n}\Big)^{\frac{p}{n+p}} \int_{S^{n-1}} h(\Lambda_p^* L,x)^p f_p(K,x)dx.
\end{aligned}$$

This gives the formulas:

$$\Omega_p(L) = \Big(\frac{\mathtt{Vol}_n(\Lambda_p L)}{\omega_n}\Big)^{\frac{p}{n+p}} \int_{S^{n-1}} h(\Lambda_p^* L,x)^p f_p(L,x)dx, \tag{3.4}$$

and

$$\Omega_{p,-p}(K,L) = \left(\frac{\mathtt{Vol}_n(\Lambda_p L)}{\omega_n}\right)^{\frac{p}{n+p}} \int_{S^{n-1}} h(\Lambda_p^* L, x)^p f_p(K,x)dx. \tag{3.5}$$

Therefore, (3.3) can be rewritten as

$$\Omega_{p,-p}(K,L) \leqslant \Omega_p(L).$$

Using (2.11), we have

$$\Omega_p(K) \leqslant \Omega_p(L).$$

The proof of Theorem 3.1 is completed. □

In order to show a negative counterpart of Theorem 3.1, we need a lemma. Now recall a version of Parseval's formula on the sphere, it is proved first by Koldobsky, Ryabogin, Zvavitch in the case $p = 1$)(see [4]) and Ryabogin, Zvavitch in the case $p > 1$ (see [15]).

LEMMA 3.2. *Let $p \geqslant 1$. If $K, L \subset \mathbb{R}^n$ be two infinitely smooth origin-symmetric convex bodies, and both of them have a positive continuous curvature function. Then $\widehat{h_K^p}(\theta)$ and $f_p(L,\cdot)^\wedge(\theta)$ are continuous function on S^{n-1} and*

$$\int_{S^{n-1}} \widehat{h_K^p}(\theta)\widehat{f_p(L,\cdot)}(\theta)d\theta = (2\pi)^n \int_{S^{n-1}} h_K^p(\xi) f_p(L,\xi)d\xi.$$

Now we prove the negative counterpart of Theorem 3.1.

THEOREM 3.3. *Let $K \in \mathscr{F}_s^n$, and $\Lambda_p K$ be an infinitely smooth origin-symmetric strictly convex body in $\mathbb{R}^n$, for which $(\mathbb{R}^n, \|\cdot\|_{\Lambda_p K})$ does not embed in $L_p, p \geqslant 1$. Then there exists an origin-symmetric convex body L in $\mathbb{R}^n$ such that*

$$\Pi_p K \subseteq \Pi_p L,$$

but

$$\Omega_p(K) > \Omega_p(L).$$

Proof. First we consider that the case $p \geqslant 1$ is not even integer. Since $(\mathbb{R}^n, \|\cdot\|_{\Lambda_p K})$ does not embed in L_p, there exists a $\xi \in S^{n-1}$ such that $\Gamma(-p/2)(\|x\|_{\Lambda_p K}^p)^\wedge(\xi)$ is negative, for more detail see [7]. Because $\Gamma(-p/2)(\|x\|_{\Lambda_p K}^p)^\wedge(\theta)$ is a continuous function on S^{n-1}, there exists a neighborhood of ξ where it is negative. Define

$$\Omega = \{\theta \in S^{n-1} : \Gamma(-p/2)(\|x\|_{\Lambda_p K}^p)^\wedge(\theta) < 0\}.$$

Consider a function $\upsilon \in C^\infty(S^{n-1})$ such that $C_p\upsilon$ is non-positive even function supported on Ω and υ is not identically zero. We can extend υ to a homogeneous function $|x|_2^p \upsilon(x/|x|_2)$ of degree p on $\mathbb{R}^n$. From this we know that the Fourier transform of $|x|_2^p \upsilon(x/|x|_2)$ is a homogeneous function of degree $-n-p$: $(|x_2|^p \upsilon(x/|x|_2))^\wedge = |x|_2^{-n-p} g(x/|x|_2)$, where g is an infinitely smooth function on S^{n-1} (see the proof of Lemma 3.16 from [5]).

Since g is bounded on S^{n-1} and $f_p(K,\theta)=h_K^{1-p}(\theta)f(K,\theta)>0$, then an $\varepsilon>0$ small enough can be choosed, so that for every $\theta\in S^{n-1}$ and $|x|_2>0$,

$$f_p(L,|x|_2\theta)=f_p(K,|x|_2\theta)-\varepsilon|x|_2^{-n-p}g(x/|x|_2)>0. \tag{3.6}$$

Form this and Lutwak's (see [12]) extension of the Minkowski's existence theorem, we know that $f_p(L,\cdot)$ defines an unique origin-symmetric convex body $L\subset\mathbb{R}^n$.

Now multiply both sides by C_p and apply the Fourier transform, we get

$$C_p\widehat{f_p(K,\cdot)}(y)=C_p\widehat{f_p(L,\cdot)}(y)+\varepsilon(2\pi)^2|y|_2^pC_p\upsilon\Big(\frac{y}{|y|_2}\Big)\leqslant C_p\widehat{f_p(L,\cdot)}(y),$$

from (1.5), this is equivalent to $\Pi_pK\subseteq\Pi_pL$.

On the other hand, denote

$$\lambda_p=\Big(\frac{\mathtt{Vol}_n(\Lambda_pK)}{\omega_n}\Big)^{\frac{p}{n+p}}.$$

Since $C_p\upsilon$ is supported and non-positive in the set S^{n-1} where $\Gamma(-p/2)(\|x\|^p_{\Lambda_pK})^\wedge(\theta)<0$, apply Lemma 3.2 and (3.4), we have that

$$\begin{aligned}
&\lambda_pC_p\Gamma(-p/2)\int_{S^{n-1}}(\|\cdot\|^p_{\Lambda_pK})^\wedge(\theta)\widehat{f_p(L,\cdot)}(\theta)d\theta\\
=&\ \lambda_pC_p\Gamma(-p/2)\int_{S^{n-1}}(\|\cdot\|^p_{\Lambda_pK})^\wedge(\theta)\widehat{f_p(K,\cdot)}(\theta)d\theta\\
&+\lambda_pC_p\Gamma(-p/2)\int_{S^{n-1}}(\|\cdot\|^p_{\Lambda_pK})^\wedge(\theta)\varepsilon(2\pi)^2\upsilon(\theta)d\theta\\
>&\ \lambda_pC_p\Gamma(-p/2)\int_{S^{n-1}}(\|\cdot\|^p_{\Lambda_pK})^\wedge(\theta)\widehat{f_p(K,\cdot)}(\theta)d\theta\\
=&\ \lambda_pC_p\Gamma(-p/2)\int_{S^{n-1}}\widehat{h^p_{\Lambda_p^*K}}(\theta)\widehat{f_p(K,\cdot)}(\theta)d\theta\\
=&\ (2\pi)^nC_p\Gamma(-p/2)\lambda_p\int_{S^{n-1}}h^p_{\Lambda_p^*K}(\xi)f_p(K,\xi)d\xi\\
=&\ (2\pi)^nC_p\Gamma(-p/2)\Omega_p(K).
\end{aligned}$$

Again using Parseval's formula in Lemma 3.2 and combining (3.5), we obtain that

$$\begin{aligned}
&\lambda_pC_p\Gamma(-p/2)\int_{S^{n-1}}(\|\cdot\|^p_{\Lambda_pK})^\wedge(\theta)\widehat{f_p(L,\cdot)}(\theta)d\theta\\
=&\ \lambda_pC_p\Gamma(-p/2)\int_{S^{n-1}}\widehat{h^p_{\Lambda_p^*K}}(\theta)\widehat{f_p(L,\cdot)}(\theta)d\theta\\
=&\ (2\pi)^nC_p\Gamma(-p/2)\lambda_p\int_{S^{n-1}}h^p_{\Lambda_p^*K}(\xi)f_p(L,\xi)d\xi\\
=&\ (2\pi)^nC_p\Gamma(-p/2)\Omega_{p,-p}(L,K).
\end{aligned}$$

Combination of the above two inequalities, we can get

$$C_p\Gamma(-p/2)\Omega_{p,-p}(L,K)>C_p\Gamma(-p/2)\Omega_p(K).$$

Note that $p \geqslant 1, C_p\Gamma(-p/2)$ is negative all along, thus

$$\Omega_{p,-p}(L,K) < \Omega_p(K).$$

This together with (2.11), hence

$$\Omega_p(K) > \Omega_p(L).$$

Next, we show that the answer is always negative if $p \geqslant 1$ is an even integer. It turns out that for any body $K \subset \mathbb{R}^n$ there exists a body L in $\mathbb{R}^n$ such that the L_p-projections of bodies K and L are equal but their L_p-affine surface area are different.

Let p be an even integer. Then $|x\cdot\xi|^p = (x\cdot\xi)^p$, and there exists a nonzero continuous even function g on S^{n-1} such that (see [15])

$$\int_{S^{n-1}} |x\cdot\xi|^p g(x)dx = 0, \quad \forall\xi \in S^{n-1}. \tag{3.7}$$

Indeed, if $p = 2k$, then $(x\cdot\xi)^{2k}$ is a polynomial of degree $2k$ with coefficients depending on ξ. So, it is enough to construct a nontrivial even function g, satisfying

$$\int_{S^{n-1}} x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n} g(x)dx = 0,$$

for all integer powers $0 \leqslant i_j \leqslant 2k$ such that $i_1 + i_2 + \cdots + i_n = 2k$. Taking $g(x) = \sum_{l=1}^{m} c_l x_1^{2l}$ and solving the system of linear equations, one can find a nontrivial solution $c_1, c_2, \cdots, c_m$ provided m is big enough.

Consider a convex body K containing the origin in their interiors in $\mathbb{R}^n$ with a positive continuous curvature function, such that $\Lambda_p^* K$ is an origin-symmetric convex body with a strictly positive L_p-curvature function (i.e. $f_p(\Lambda_p^*K,\xi) > 0, \forall\xi \in S^{n-1}$). We may assume that

$$\int_{S^{n-1}} h(\Lambda_p^*K,\xi)^p g(\xi)d\xi \geqslant 0,$$

(otherwise consider $-g(\xi)$ instead of $g(\xi)$). Choose $\varepsilon > 0$ such that

$$f_p(K,\xi) - \varepsilon g(\xi) > 0, \quad \forall\xi \in S^{n-1}.$$

Using the existence theorem for L_p-curvature functions (see [9]), we conclude that there exists an origin-symmetric convex body L in $\mathbb{R}^n$ such that

$$f_p(L,\xi) = f_p(K,\xi) - \varepsilon g(\xi). \tag{3.8}$$

Now multiply both sides by $|x\cdot\xi|^p$ and integrating, then

$$\int_{S^{n-1}} |x\cdot\xi|^p f_p(L,\xi)d\xi = \int_{S^{n-1}} |x\cdot\xi|^p f_p(K,\xi)d\xi - \varepsilon\int_{S^{n-1}} |x\cdot\xi|^p g(\xi)d\xi.$$

Applying (3.7) and (1.4), we get that $h(\Pi_p L,x) = h(\Pi_p K,x)$, i.e. $\Pi_p L = \Pi_p K$. Using (3.4) and (3.5) in (3.8), we have

$$\begin{aligned}\Omega_p(K) &= \lambda_p \int_{S^{n-1}} h(\Lambda_p^* K,x)^p f_p(K,x)dx \\ &= \lambda_p \int_{S^{n-1}} h(\Lambda_p^* K,x)^p f_p(L,x)dx + \varepsilon\lambda_p \int_{S^{n-1}} h(\Lambda_p^* K,x)^p g(x)dx \\ &\geqslant \lambda_p \int_{S^{n-1}} h(\Lambda_p^* K,x)^p f_p(L,x)dx \\ &= \Omega_{p,-p}(L,K).\end{aligned}$$

From the argument from Theorem 3.1 we get that

$$\Omega_p(K) \geqslant \Omega_p(L).$$

If $\Omega_p(K) = \Omega_p(L)$, then the equality of (2.11) is hold, and L and K are dilates (see [17]), i.e., $K = L$, but from (3.10), this is a contradiction with the uniqueness of L_p-curvature function. The proof of Theorem 3.3 is completed. □

Together with Theorem 3.1 and Theorem 3.3, an immediate result can be obtained:

COROLLARY 3.4. *If $p \geqslant 1$, then $\Pi_p K \subseteq \Pi_p L$ implies that $\Omega_p(K) \leqslant \Omega_p(L)$ if and only if for every $Q \in \mathscr{F}_s^n$, $(\mathbb{R}^n, \|\cdot\|_{\Lambda_p Q})$ is isometric embedding to a subspace of L_p.*

Because of all the 2-dimensional space $(\mathbb{R}^2, \|\cdot\|)$ must be isometric embedding to a subspace of L_1 (see [7]), while all n-dimensional space $(\mathbb{R}^n, \|\cdot\|)$ be not surely isometric embedding to a subspace of $L_p (p > 1, n \geqslant 2)$. Therefore, from Corollary 3.4, we get immediately Theorem 1.1.

REMARK. From (2.14), we saw that $\Omega_p(K) \leqslant \Omega_p(L)$ is equivalent to $\mathtt{Vol}_n(\Lambda_p K) \leqslant \mathtt{Vol}_n(\Lambda_p L)$ in Theorem 3.1, while $\Omega_p(K) > \Omega_p(L)$ is equivalent to $\mathtt{Vol}_n(\Lambda_p K) > \mathtt{Vol}_n(\Lambda_p L)$ in Theorem 3.3.

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