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PHILOS TYPE CRITERIA FOR SECOND–ORDER HALF–LINEAR DYNAMIC EQUATIONS

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Abstract. Some new criteria for the oscillation of solutions of the second-order half-linear dynamic equation

$$\left(a\left(x^{\Delta}\right)^{\alpha}\right)^{\Delta}(t)+q(t)x^{\alpha}(t)=0$$

are established when $\int^{\infty} a^{-1/\alpha}(s)\Delta s=\infty$.

1. Introduction

This paper is concerned with the oscillation of solutions of the second-order half-linear dynamic equation

$$\left(a(x^{\Delta})^{\alpha}\right)^{\Delta}(t)+q(t)x^{\alpha}(t)=0, \tag{1.1}$$

where a and q are real-valued positive rd-continuous functions on a time scale $\mathbb{T}\subset\mathbb{R}$ with $\sup\mathbb{T}=\infty$, and α is the ratio of two positive odd integers.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$, and as oscillation of solutions is our main concern, we make the assumption that $\sup\mathbb{T}=\infty$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on real numbers $\mathbb{R}$. The forward and backward jump operators $\sigma,\rho:\mathbb{T}\to\mathbb{T}$ are defined by

$$\sigma(t)=\inf\{s\in\mathbb{T}:\ s>t\}\quad\text{and}\quad\rho(t)=\sup\{s\in\mathbb{T}:\ s<t\},$$

where the supremum of the empty set is defined to be the infimum of $\mathbb{T}$. A point $t\in\mathbb{T}$ is said to be right-scattered if $\sigma(t)>t$ and right-dense if $\sigma(t)=t$, and $t\in\mathbb{T}$ with $t>\inf\mathbb{T}$ is said to be left-scattered if $\rho(t)<t$ and left-dense if $\rho(t)=t$. A function $g:\mathbb{T}\to\mathbb{R}$ is said to be rd-continuous provided g is continuous at right-dense points and has finite left-hand limits at left-dense points in $\mathbb{T}$. The graininess function μ for

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a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$, and for every function $f : \mathbb{T} \to \mathbb{R}$, the notation f^σ means the composition $f \circ \sigma$.

We recall that a solution of equation (1.1) is said to be oscillatory on $[t_0, \infty)_\mathbb{T}$ if it is neither eventually positive nor eventually negative. Otherwise, the solution is said to be nonoscillatory. Equation (1.1) is said to be oscillatory if all of its solutions are oscillatory.

Recently much attention has focused on dynamic equations on time scales, and we refer the reader to the landmark paper of Hilger [21] for a comprehensive treatment of the subject. Since then, several authors have expounded on various aspects of this new theory, see [5], the references cited therein, and the monographs [12, 13]. In recent years there has been much research activity concerning the oscillation and nonoscillation of solutions of dynamic equations on time scales. We refer the reader to [2, 3, 5, 6, 11–19]. If $\mathbb{T} = \mathbb{R}$ and $\alpha = 1$, then equation (1.1) reduces to

$$(ax')'(t) + q(t)x(t) = 0. \tag{1.2}$$

For (1.2), numerous oscillation and nonoscillation criteria have been established, see [1, 8–10, 22, 23]. It is known [22], when $a(t) \equiv 1$, the condition

$$\limsup_{t \to \infty} \frac{1}{t^m} \int_{t_0}^{t} (t-s)^m q(s) ds = \infty, \tag{1.3}$$

where $m > 1$ is an integer, plays an important rôle in the oscillation of all solutions of (1.2). In recent years, improvements of condition (1.3) for the continuous case $\mathbb{T} = \mathbb{R}$ and the discrete case $\mathbb{T} = \mathbb{Z}$ were obtained in [4, 7–10, 23]. The purpose of this paper is to proceed further in this direction and establish some time scale analogues of the results presented in [4, 7–10]. This paper supplements [3], where related oscillation criteria for (1.1) recently have been established.

2. Preliminary Results

For a function $f : \mathbb{T} \to \mathbb{R}$, the (delta) derivative $f^\Delta(t)$ at $t \in \mathbb{T}$ is defined to be the number (if it exists) such that for all $\varepsilon > 0$, there is a neighborhood $\mathscr{U}$ of t with

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| < \varepsilon|\sigma(t) - s| \quad \text{for all} \quad s \in \mathscr{U}.$$

If the (delta) derivative $f^\Delta(t)$ exists for all $t \in \mathbb{T}$, then we say that f is (delta) differentiable on $\mathbb{T}$.

We will make use of the following product and quotient rules [12, Theorem 1.20] for the derivatives of the product fg and the quotient f/g (where $gg^\sigma \neq 0$) of two (delta) differentiable functions f and g,

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma, \quad \left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma} \tag{2.1}$$

as well as of the chain rule [12, Theorem 1.90] for the derivative of the composite function $f \circ g$ for a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ and a (delta) differentiable function $g : \mathbb{T} \to \mathbb{R}$,

$$(f \circ g)^{\Delta} = \left\{ \int_0^1 f'(g + h\mu g^{\Delta}) dh \right\} g^{\Delta}. \tag{2.2}$$

For $b, c \in \mathbb{T}$ and a differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_b^c f^{\Delta}(t)\Delta t = f(c) - f(b),$$

and infinite integrals are defined as

$$\int_b^{\infty} f(t)\Delta t = \lim_{c \to \infty} \int_b^c f(t)\Delta t.$$

An integration by parts formula reads

$$\int_b^c f(t)g^{\Delta}(t)\Delta t = f(t)g(t)\Big|_b^c - \int_b^c f^{\Delta}(t)g(\sigma(t))\Delta t. \tag{2.3}$$

Note that in the case $\mathbb{T} = \mathbb{R}$, we have

$$\sigma(t) = \rho(t) = t, \quad \mu(t) \equiv 0, \quad f^{\Delta}(t) = f'(t), \quad \int_b^c f(t)\Delta t = \int_b^c f(t)dt,$$

and in the case $\mathbb{T} = \mathbb{Z}$, we have

$$\sigma(t) = t + 1, \quad \rho(t) = t - 1, \quad \mu(t) \equiv 1, \quad f^{\Delta}(t) = \Delta f(t) := f(t+1) - f(t),$$

and (if $b < c$)

$$\int_b^c f(t)\Delta t = \sum_{t=b}^{c-1} f(t).$$

For more discussion on time scales, we refer the reader to [12, 13].

Finally, we recall the following lemma from [20] which will be needed for the proof of Theorem 3.2 in the next section.

LEMMA 2.1. *If X and Y are nonnegative and $\gamma > 1$, then*

$$X^{\gamma} - \gamma X Y^{\gamma - 1} + (\gamma - 1)Y^{\gamma} \geqslant 0,$$

where equality holds if and only if $X = Y$.

3. Main Results

In this section, we shall give some new Philos and Kamenev type oscillation criteria for the solutions of equation (1.1). To do so, we define $\mathscr{D}$ as follows: $H \in \mathscr{D}$ provided $H : [t_0,\infty)_{\mathbb{T}} \times [t_0,\infty)_{\mathbb{T}} \to \mathbb{R}$ satisfies

$$H(t,t) \geqslant 0 \quad \text{for} \quad t \geqslant t_0, \quad H(t,s) > 0 \quad \text{for} \quad t > s \geqslant t_0,$$

and $H(t,s)$ is (delta) differentiable with respect to s.

THEOREM 3.1. *Assume that*

$$\int_{t_0}^{\infty} a^{-1/\alpha}(s)\Delta s = \infty, \tag{3.1}$$

$H \in \mathscr{D}$, and suppose that there exists a positive (delta) differentiable function ξ such that, for $t \geqslant s \geqslant t_1 \in [t_0,\infty)_{\mathbb{T}}$, with

$$h(t,s) := H(t,s)\frac{\xi^{\Delta}(s)}{\xi^{\sigma}(s)} + H^{\Delta_s}(t,s),$$

we have

$$h(t,s) \geqslant 0 \quad \textit{for} \quad t \geqslant s \geqslant t_1 \in [t_0,\infty)_{\mathbb{T}} \tag{3.2}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} [H(t,s)\xi(s)q(s) - h(t,s)(\xi(s)\eta^{\alpha}(s))^{\sigma}]\Delta s = \infty, \tag{3.3}$$

where $\eta(t) = \left(\int_{t_1}^{t} a^{-1/\alpha}(s)\Delta s\right)^{-1}$. *Then equation* (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of equation (1.1) on $[t_0,\infty)_{\mathbb{T}}$, say (without loss of generality), $x(t) > 0$ for $t \geqslant t_1 \geqslant t_0$. We shall first show that

$$x^{\Delta}(t) > 0 \quad \text{for} \quad t \geqslant t_1. \tag{3.4}$$

From equation (1.1), we have

$$\left(a(x^{\Delta})^{\alpha}\right)^{\Delta}(t) = -q(t)x^{\alpha}(t) \leqslant 0 \quad \text{for} \quad t \geqslant t_1.$$

Thus, if (3.4) does not hold, then it follows that

$$a(x^{\Delta})^{\alpha}(t) \leqslant a(x^{\Delta})^{\alpha}(t_1) =: c < 0,$$

which implies

$$x^{\Delta}(t) \leqslant \left(\frac{c}{a(t)}\right)^{1/\alpha} \quad \text{for} \quad t \geqslant t_1.$$

Now an integration from t_1 to t yields

$$x(t) \leqslant x(t_1) + c^{1/\alpha} \int_{t_1}^{t} a^{-1/\alpha}(s)\Delta s \to -\infty \quad \text{as} \quad t \to \infty,$$

which contradicts the fact that $x(t) > 0$ for $t \geqslant t_1$. Thus, (3.4) is true. Define now

$$w := \xi \frac{a(x^\Delta)^\alpha}{x^\alpha} \quad \text{on} \quad [t_1, \infty)_{\mathbb{T}}.$$

Then, it follows that on $[t_1, \infty)_{\mathbb{T}}$ we have

$$\begin{aligned} w^\Delta &= \left(\frac{\xi}{x^\alpha}\right)^\Delta \left(a(x^\Delta)^\alpha\right)^\sigma + \frac{\xi}{x^\alpha}\left(a(x^\Delta)^\alpha\right)^\Delta \\ &= -\xi q + \left(a(x^\Delta)^\alpha\right)^\sigma \left[\frac{\xi^\Delta x^\alpha - \xi (x^\alpha)^\Delta}{x^\alpha (x^\sigma)^\alpha}\right] \\ &= -\xi q + \frac{\xi^\Delta}{\xi^\sigma} w^\sigma - \xi \frac{(x^\alpha)^\Delta}{x^\alpha} \frac{\left(a(x^\Delta)^\alpha\right)^\sigma}{(x^\sigma)^\alpha} \qquad (3.5) \\ &\leqslant -\xi q + \frac{\xi^\Delta}{\xi^\sigma} w^\sigma, \qquad (3.6) \end{aligned}$$

where we have used that $(x^\alpha)^\Delta \geqslant 0$, which follows by an application of the chain rule (2.2):

$$(x^\alpha)^\Delta = \alpha x^\Delta \int_0^1 (x + \mu h x^\Delta)^{\alpha-1} dh \geqslant \alpha x^\Delta \int_0^1 x^{\alpha-1} dh = \alpha x^{\alpha-1} x^\Delta. \qquad (3.7)$$

Next, we have for $t \geqslant t_1$

$$\begin{aligned} x(t) &= x(t_1) + \int_{t_1}^t x^\Delta(s)\Delta s \\ &= x(t_1) + \int_{t_1}^t a^{-1/\alpha}(s) \left(a(s)(x^\Delta(s))^\alpha\right)^{1/\alpha} \Delta s \\ &\geqslant \left(\int_{t_1}^t a^{-1/\alpha}(s)\Delta s\right) \left(a(t)(x^\Delta(t))^\alpha\right)^{1/\alpha} \end{aligned}$$

so that

$$a(t)\left(\frac{x^\Delta(t)}{x(t)}\right)^\alpha \leqslant \left(\int_{t_1}^t a^{-1/\alpha}(s)\Delta s\right)^{-\alpha} = \eta^\alpha(t) \quad \text{for} \quad t \geqslant t_1. \qquad (3.8)$$

From (3.6), it follows that

$$\int_{t_1}^t H(t,s) w^\Delta(s)\Delta s \leqslant -\int_{t_1}^t H(t,s)\xi(s) q(s)\Delta s + \int_{t_1}^t H(t,s) \frac{\xi^\Delta(s)}{\xi^\sigma(s)} w^\sigma(s)\Delta s.$$

Using the integration by parts formula (2.3), we find

$$H(t,s)w(s)\Big|_{s=t_1}^{s=t} \leqslant -\int_{t_1}^t H(t,s)\xi(s)q(s)\Delta s + \int_{t_1}^t \left[H(t,s)\frac{\xi^\Delta(s)}{\xi^\sigma(s)} + H^{\Delta_s}(t,s)\right] w^\sigma(s)\Delta s. \qquad (3.9)$$

Using (3.8) in (3.9), we have

$$-H(t,t_1)w(t_1) \leqslant -\int_{t_1}^{t} H(t,s)\xi(s)q(s)\Delta s + \int_{t_1}^{t} \left[H(t,s)\frac{\xi^{\Delta}(s)}{\xi^{\sigma}(s)} + H^{\Delta_s}(t,s)\right](\xi(s)\eta^{\alpha}(s))^{\sigma}\Delta s,$$

and therefore

$$\infty > w(t_1) \geqslant \frac{1}{H(t,t_1)}\int_{t_1}^{t}[H(t,s)\xi(s)q(s) - h(t,s)(\xi(s)\eta^{\alpha}(s))^{\sigma}]\Delta s.$$

Taking the lim sup of both sides of the above inequality as $t \to \infty$, we obtain a contradiction to condition (3.3). This completes the proof.

Next, we establish the following result.

THEOREM 3.2. *Let the hypotheses of Theorem* 3.1 *hold and condition* (3.3) *be replaced by*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[H(t,s)\xi(s)q(s) - \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{(h(t,s))^{\alpha+1}}{(h^*(t,s))^{\alpha}}\right]\Delta s = \infty, \qquad (3.10)$$

where h is as in Theorem 3.1*, and for $t > s \geqslant t_1 \in [t_0,\infty)_{\mathbb{T}}$,*

$$h^*(t,s) := \alpha H(t,s)a^{-1/\alpha}(s)\frac{\xi(s)}{(\xi^{\sigma}(s))^{1+1/\alpha}}.$$

Then equation (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geqslant t_1 \geqslant t_0$. As in the proof of Theorem 3.1, we obtain (3.5). Now, note that we have on $[t_1,\infty)_{\mathbb{T}}$

$$\frac{x^{\Delta}}{x} = \xi^{-1/\alpha}a^{-1/\alpha}w^{1/\alpha} \quad \text{and} \quad w^{\sigma} \leqslant \frac{\xi^{\sigma}}{\xi}w. \qquad (3.11)$$

Using (3.7) and (3.11) in inequality (3.5), we get

$$w^{\Delta} \leqslant -\xi q + \frac{\xi^{\Delta}}{\xi^{\sigma}}w^{\sigma} - \alpha a^{-1/\alpha}\frac{\xi}{(\xi^{\sigma})^{1+1/\alpha}}(w^{\sigma})^{1+1/\alpha} \quad \text{on} \quad [t_1,\infty)_{\mathbb{T}}. \qquad (3.12)$$

From (3.12), it follows that

$$\int_{t_1}^{t} H(t,s)w^{\Delta}(s)\Delta s \leqslant -\int_{t_1}^{t} H(t,s)\xi(s)q(s)\Delta s + \int_{t_1}^{t} H(t,s)\frac{\xi^{\Delta}(s)}{\xi^{\sigma}(s)}w^{\sigma}(s)\Delta s - \int_{t_1}^{t}\alpha H(t,s)a^{-1/\alpha}(s)\frac{\xi(s)}{(\xi^{\sigma}(s))^{1+1/\alpha}}(w^{\sigma}(s))^{1+1/\alpha}\Delta s,$$

and thus, as in the proof of Theorem 3.1, we see that

$$-H(t,t_1)w(t_1) \leqslant -\int_{t_1}^{t} H(t,s)\xi(s)q(s)\Delta s + \int_{t_1}^{t} h(t,s)w^{\sigma}(s)\Delta s - \int_{t_1}^{t} h^*(t,s)(w^{\sigma}(s))^{1+1/\alpha}\Delta s. \tag{3.13}$$

Set

$$X = (h^*(t,s))^{\alpha/(\alpha+1)}w^{\sigma}(s) \quad \text{and} \quad Y = \left(\frac{\alpha}{\alpha+1}\frac{h(t,s)}{(h^*(t,s))^{\alpha/(\alpha+1)}}\right)^{\alpha}$$

in Lemma 2.1 (with $\gamma = (\alpha+1)/\alpha > 1$) to conclude that

$$\int_{t_1}^{t}\left[h(t,s)w^{\sigma}(s) - h^*(t,s)(w^{\sigma}(s))^{1+1/\alpha}\right]\Delta s \leqslant \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\int_{t_1}^{t}\frac{(h(t,s))^{\alpha+1}}{(h^*(t,s))^{\alpha}}\Delta s. \tag{3.14}$$

Using (3.14) in (3.13), we obtain

$$\infty > w(t_1) \geqslant \frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[H(t,s)\xi(s)q(s) - \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{(h(t,s))^{\alpha+1}}{(h^*(t,s))^{\alpha}}\right]\Delta s.$$

Taking the $\limsup$ of both sides of the above inequality as $t \to \infty$, we obtain a contradiction to condition (3.10). This completes the proof.

Finally, we shall establish the following result.

THEOREM 3.3. *Let the hypotheses of Theorem* 3.1 *hold and conditions* (3.2) *and* (3.3) *be replaced by*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[H(t,s)\xi(s)q(s) - \frac{h^2(t,s)}{4\overline{h}(t,s)}\right]\Delta s = \infty, \tag{3.15}$$

where

$$\overline{h}(t,s) := \alpha H(t,s)a^{-1/\alpha}(s)\frac{\xi(s)}{(\xi^{\sigma}(s))^2}(\eta^{\sigma}(s))^{1-\alpha}, \quad t > s \geqslant t_1$$

and h *and* η *are as in Theorem* 3.1. *Then equation* (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geqslant t_1 \geqslant t_0$. As in the proof of Theorem 3.2, we obtain (3.12), which can be rewritten as

$$w^{\Delta} \leqslant -\xi q + \frac{\xi^{\Delta}}{\xi^{\sigma}}w^{\sigma} - \alpha a^{-1/\alpha}\frac{\xi}{(\xi^{\sigma})^{1+1/\alpha}}(w^{\sigma})^{1/\alpha-1}(w^{\sigma})^2 \quad \text{on} \quad [t_1,\infty)_{\mathbb{T}}. \tag{3.16}$$

Now, by (3.11),

$$w^{1/\alpha-1} = a^{1/\alpha-1}\xi^{1/\alpha-1}\left(\frac{x}{x^{\Delta}}\right)^{\alpha-1} \quad \text{on} \quad [t_1,\infty)_{\mathbb{T}}. \tag{3.17}$$

Using (3.8) in (3.17), we get

$$w^{1/\alpha-1} \geqslant \xi^{1/\alpha-1} a^{1/\alpha-1} a^{(\alpha-1)/\alpha} \eta^{1-\alpha} = \xi^{1/\alpha-1} \eta^{1-\alpha} \quad \text{on} \quad [t_1,\infty)_{\mathbb{T}}. \tag{3.18}$$

Employing (3.18) in (3.16), we find

$$w^{\Delta} \leqslant -\xi q + \frac{\xi^{\Delta}}{\xi^{\sigma}} w^{\sigma} - \alpha a^{-1/\alpha} \frac{\xi}{(\xi^{\sigma})^2} (\eta^{\sigma})^{1-\alpha} (w^{\sigma})^2 \quad \text{on} \quad [t_1,\infty)_{\mathbb{T}}. \tag{3.19}$$

From (3.19), it follows that

$$\begin{aligned}\int_{t_1}^{t} H(t,s) w^{\Delta}(s)\Delta s \leqslant & -\int_{t_1}^{t} H(t,s)\xi(s)q(s)\Delta s + \int_{t_1}^{t} H(t,s)\frac{\xi^{\Delta}(s)}{\xi^{\sigma}(s)} w^{\sigma}(s)\Delta s \\ & - \int_{t_1}^{t} \alpha H(t,s) a^{-1/\alpha}(s) \frac{\xi(s)}{(\xi^{\sigma}(s))^2} (\eta^{\sigma}(s))^{1-\alpha} (w^{\sigma}(s))^2 \Delta s.\end{aligned}$$

Now as in the proof of Theorem 3.1, we find that

$$\begin{aligned}-H(t,t_1)w(t_1) &\leqslant -\int_{t_1}^{t} H(t,s)\xi(s)q(s)\Delta s + \int_{t_1}^{t} h(t,s) w^{\sigma}(s)\Delta s - \int_{t_1}^{t} \overline{h}(t,s)(w^{\sigma}(s))^2 \Delta s \\ &= -\int_{t_1}^{t} H(t,s)\xi(s)q(s)\Delta s + \int_{t_1}^{t} \frac{h^2(t,s)}{4\overline{h}(t,s)}\Delta s \\ &\qquad - \int_{t_1}^{t} \left(\sqrt{\overline{h}(t,s)} w^{\sigma}(s) - \frac{h(t,s)}{2\sqrt{\overline{h}(t,s)}} \right)^2 \Delta s\end{aligned}$$

and thus

$$\infty > w(t_1) \geqslant \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[H(t,s)\xi(s)q(s) - \frac{h^2(t,s)}{\overline{h}(t,s)} \right] \Delta s.$$

Taking the $\limsup$ of both sides of the above inequality as $t \to \infty$, we obtain a contradiction to (3.15). This completes the proof.

4. Remarks and Examples

REMARK 4.1. Oscillation criteria similar to those established in Section 3 can also be obtained by replacing $H(t,s)$ with $(t-s)^m$, $m \in \mathbb{N}$. Then we have

$$H^{\Delta_s}(t,s) = \sum_{\nu=0}^{m-1} (t-\sigma(s))^{\nu} (t-s)^{m-\nu-1}.$$

Now, Theorem 3.1 can be reformulated as in the following corollary. Other criteria can be obtained similarly. We omit the details here.

COROLLARY 4.2. *Let condition* (3.1) *hold and assume that there exists a positive (delta) differentiable function* ξ *such that, for* $t \geqslant s \geqslant t_1 \in [t_0,\infty)_{\mathbb{T}}$, *with*

$$h(t,s) := (t-s)^m \frac{\xi^{\Delta}(s)}{\xi^{\sigma}(s)} + \sum_{\nu=0}^{m-1} (t-\sigma(s))^{\nu}(t-s)^{m-\nu-1},$$

we have

$$h(t,s) \geqslant 0 \quad \text{for} \quad t \geqslant s \geqslant t_1 \in [t_0,\infty)_{\mathbb{T}}$$

and

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_1}^{t} [(t-s)^m \xi(s) q(s) - h(t,s)(\xi(s)\eta^{\alpha}(s))^{\sigma}]\Delta s = \infty,$$

where $m \in \mathbb{N}$ *and* η *is as in Theorem* 3.1. *Then equation* (1.1) *is oscillatory.*

The following result is an immediate consequence of Theorem 3.1.

COROLLARY 4.3. *Let condition* (3.1) *hold,* $H \in \mathscr{D}$, *and suppose that there exists a positive (delta) differentiable function* ξ *such that for* $t \geqslant s \geqslant t_1 \in [t_0,\infty)_{\mathbb{T}}$, *condition* (3.2) *holds. If*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} H(t,s)\xi(s)q(s)\Delta s = \infty$$

and

$$\lim_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} h(t,s)(\xi(s)\eta^{\alpha}(s))^{\sigma}\Delta s < \infty,$$

where η *is as in Theorem* 3.1, *then equation* (1.1) *is oscillatory.*

Similar corollaries can be drawn from Theorems 3.2 and 3.3. The details are left to the reader.

EXAMPLE 4.4. From the above results, we can derive some new oscillation criteria for equation (1.1) on different types of time scales.

1. If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$, $\mu(t) = 0$, $\xi^{\Delta} = \xi'$, and $H^{\Delta_s}(t,s) = \partial H(t,s)/\partial s$. As an example, condition (3.10) becomes

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[H(t,s)\xi(s)q(s) - \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{(h(t,s))^{\alpha+1}}{(h^*(t,s))^{\alpha}} \right] ds = \infty, \tag{4.1}$$

where

$$h(t,s) = \frac{\partial}{\partial s} H(t,s) + \frac{\xi'(s)}{\xi(s)} H(t,s)$$

and

$$h^*(t,s) = \alpha H(t,s) a^{-1/\alpha}(s) \frac{\xi(s)}{(\xi^{\sigma}(s))^{1+1/\alpha}}$$

for $t \geqslant s \geqslant t_1 \geqslant t_0$.

Note that when $\xi(t) \equiv 1$, $a(t) \equiv 1$, and $\alpha = 1$, then condition (4.1) reduces to the results of Philos [23]. If in addition, $H(t,s) = (t-s)^m$, $m \in \mathbb{N}$ and $m > 1$, then condition (4.1) reduces to the results of Kamenev [22].

2. If $\mathbb{T} = \mathbb{Z}$, then $\xi^\Delta(n) = \Delta\xi(n) = \xi(n+1) - \xi(n)$ and $H^{\Delta_s}(m,n) = \Delta_2 H(m,n) := H(m,n+1) - H(m,n)$. In this discrete case, as an example, condition (3.15) takes the form

$$\limsup_{m\to\infty} \frac{1}{H(m,n_0)} \sum_{n=n_0}^{m-1} \left[H(m,n)\xi(n)q(n) - \frac{h^2(m,n)}{4\overline{h}(m,n)} \right] = \infty, \tag{4.2}$$

where

$$h(m,n) = \Delta_2 H(m,n) + \frac{\Delta\xi(n)}{\xi(n+1)} H(m,n)$$

and

$$\overline{h}(m,n) = \alpha H(m,n) a^{-1/\alpha}(n) \frac{\xi(n)}{(\xi(n+1))^2} (\eta(n+1))^{1-\alpha}$$

for $m \geqslant n \geqslant n_0$.

Note that condition (4.2) is new for the oscillation of equation (1.1) with $\mathbb{T} = \mathbb{Z}$ and is also new for the special cases when $a(t) \equiv 1$, or $a(t) \equiv 1$ and $\xi(t) \equiv 1$.

3. If $\mathbb{T} = \theta\mathbb{Z}$, with $\theta > 0$, then $\sigma(t) = t + \theta$, $\mu(t) = \theta$, $\xi^\Delta(t) = \Delta_\theta \xi(t) := [\xi(t+\theta) - \xi(t)]/\theta$, and $H^{\Delta_s}(t,s) = \Delta_{\theta 2} H(t,s) := [H(t,s+\theta) - H(t,s)]/\theta$. In this case, condition (3.3) becomes for $t,s \in \theta\mathbb{Z}$,

$$\limsup_{n\to\infty} \frac{1}{H(\theta n, \theta n_1)} \sum_{k=n_1 \geqslant n_0}^{n-1} \big[H(\theta n, \theta k)\xi(\theta k) q(\theta k) \\ - h(\theta n, \theta k)(\xi((k+1)\theta)\eta^\alpha((k+1)\theta)) \big] = \infty,$$

where

$$h(t,s) = \Delta_{\theta 2} H(t,s) + \frac{\Delta_\theta \xi(s)}{\xi(s+\theta)} H(t,s).$$

4. We can employ other types of time scales, e.g., $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, $\mathbb{T} = \mathbb{N}_0^2$, etc., see [12, 13]. The details are left to the reader.

REMARK 4.5. We note that the results of this paper are presented in a form which is essentially new and of high degree of generality. Some of the obtained results improve, unify and contain well-known results which appeared in the literature particularly for the cases $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, see [4, 7–10].

REMARK 4.6. The results of this paper can be extended to more general dynamic equations with deviating arguments of the form

$$\left(a(x^\Delta)^\alpha \right)^\Delta (t) + q(t) x^\beta[\tau(t)] = 0,$$

where β is the ratio of positive odd integers and $\tau : \mathbb{T} \to \mathbb{T}$ satisfies $\lim_{t\to\infty} \tau(t) = \infty$. The details are left to the reader.

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