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VOLTERRA COMPOSITION OPERATORS FROM WEIGHTED–TYPE SPACES TO BLOCH–TYPE SPACES AND MIXED NORM SPACES

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Abstract. The boundedness and compactness of the Volterra composition operator from a class of weighted-type spaces to Bloch-type spaces and mixed norm spaces on the unit ball are completely characterized in this paper.

1. Introduction

Let $\mathbb{B}_n$ be the unit ball of $\mathbb{C}^n$ and let S be its boundary of $\mathbb{B}_n$. Let $H(\mathbb{B}_n)$ denote the space of all holomorphic functions in $\mathbb{B}_n$. Let $H^\infty = H^\infty(\mathbb{B}_n)$ denote the space of all bounded holomorphic functions on $\mathbb{B}_n$. For $f \in H(\mathbb{B}_n)$, let (see [34])

$$\Re f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z)$$

stand for the radial derivative of $f \in H(\mathbb{B}_n)$.

For a fixed $k \in \mathbb{N}$, an $f \in H(\mathbb{B}_n)$ is said to belong to the weighted-type space, denoted by $H_{\log_k} = H_{\log_k}(\mathbb{B}_n)$, if (see [21])

$$\|f\|_{H_{\log_k}} = \sup_{z\in\mathbb{B}_n} (1-|z|^2)\Big(\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|z|^2}\Big)|f(z)| < \infty,$$

where $e^{[k]}$ is defined inductively by $e^{[1]} = e$, $e^{[k]} = e^{e^{[k-1]}}$ and

$$\ln^{[j]} z = \underbrace{\ln\cdots\ln}_{j \text{ times}} z.$$

$H_{\log_k}$ is a Banach space with the norm $\|\cdot\|_{H_{\log_k}}$. Here we naturally introduce the little weighted-type space, denoted by $H_{\log_k,0}$, which is the subspace of $H_{\log_k}$ consisting of those $f \in H_{\log_k}$ such that

$$\lim_{|z|\to 1} (1-|z|^2)\Big(\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|z|^2}\Big)|f(z)| = 0.$$

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Let ϕ be a positive continuous function on $[0,1)$. ϕ is called normal, if there exist positive numbers s and t, $0<s<t$, and $\delta\in[0,1)$ such that (see, for example, [23, 27])

$$\frac{\phi(r)}{(1-r)^s} \text{ is decreasing on } [\delta,1) \text{ and } \lim_{r\to 1}\frac{\phi(r)}{(1-r)^s}=0;$$

$$\frac{\phi(r)}{(1-r)^t} \text{ is increasing on } [\delta,1) \text{ and } \lim_{r\to 1}\frac{\phi(r)}{(1-r)^t}=\infty.$$

Let μ be a normal function on $[0,1)$. An $f\in H(\mathbb{B}_n)$ is said to belong to the Bloch-type space, denoted by $\mathscr{B}_\mu=\mathscr{B}_\mu(\mathbb{B}_n)$, if

$$\|f\|_{\mathscr{B}_\mu}=|f(0)|+\sup_{z\in\mathbb{B}_n}\mu(|z|)|\Re f(z)|<\infty.$$

Under the above norm, $\mathscr{B}_\mu$ becomes a Banach space. The little Bloch-type space, denoted by $\mathscr{B}_{\mu,0}$, is the space of all $f\in H(\mathbb{B}_n)$ such that

$$\lim_{|z|\to 1}\mu(|z|)|\Re f(z)|=0.$$

Let ϕ be a normal function on $[0,1)$. For $0<p,q<\infty$, the mixed norm space $H(p,q,\phi)=H(p,q,\phi)(\mathbb{B}_n)$ consists of all $f\in H(\mathbb{B}_n)$ such that

$$\|f\|_{H(p,q,\phi)}=\left(\int_0^1 M_q^p(f,r)\frac{\phi^p(r)}{1-r}dr\right)^{1/p}<\infty, \tag{1}$$

where

$$M_q(f,r)=\left(\int_S |f(r\zeta)|^q d\sigma(\zeta)\right)^{1/q}.$$

Let φ be a holomorphic self-map of $\mathbb{B}_n$. The composition operator C_φ is defined by

$$(C_\varphi f)(z)=(f\circ\varphi)(z), \quad f\in H(\mathbb{B}_n).$$

The study of composition operators become fairly active since it provides connections between operator theory and complex analysis and help us to gain deeper understanding of both areas. A linear operator is said to be bounded if the image of a bounded set is a bounded set, while a linear operator is compact if it takes bounded sets to sets with compact closure. The book [3] contains plenty of information on this topic.

Suppose that φ is a holomorphic self-map of $\mathbb{B}_n$ and $g\in H(\mathbb{B}_n)$. In this paper we consider the operator $T_{g,\varphi}$, which is defined as follows

$$T_{g,\varphi}f(z)=\int_0^1 f(\varphi(tz))\Re g(tz)\frac{dt}{t}, \qquad f\in H(\mathbb{B}_n),\ z\in\mathbb{B}_n. \tag{2}$$

The operator $T_{g,\varphi}$ is called the Volterra composition operator, which is first defined in [39] and studied in [35, 36]. See [25, 26, 27, 28, 29, 30] for the boundedness and compactness of a related operator on some holomorphic function spaces in the unit ball. In the setting of the unit disk D, the Volterra composition operator $T_{g,\varphi}$ was introduced by S. Li and studied in [9].

In the setting of the unit ball, let $\varphi(z) = z$. We get $T_{g,z} = T_g$, i.e.

$$T_g f(z) = \int_0^1 f(tz)\Re g(tz)\frac{dt}{t}, \qquad f \in H(\mathbb{B}_n), \ z \in \mathbb{B}_n.$$

The operator T_g is called the Riemann-Stieltjes operator (or the extended Cesàro operator), which was introduced in [4] and studied in [1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 24, 31, 32, 33, 36, 37, 38].

In this paper, we give some sufficient and necessary conditions for the boundedness and compactness of Volterra composition operators from $H_{\log_k}$ to Bloch-type spaces and mixed norm spaces.

Throughout the paper, constants are denoted by C, they are positive and may not be the same in every occurrence. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \leqslant A \leqslant CB$.

2. Main results and proofs

In this section, we give our main results and their proofs. Before stating these results, we need some auxiliary results, which are incorporated in the lemmas which follow.

LEMMA 1. *Assume that $0 < p,q < \infty$, μ and ϕ are normal on $[0,1)$, $g \in H(\mathbb{B}_n)$ and φ is a holomorphic self-map of $\mathbb{B}_n$. Let $Y = \mathscr{B}_\mu$ or $H(p,q,\phi)$. Then $T_{g,\varphi} : H_{\log_k} \to Y$ is compact if and only if $T_{g,\varphi} : H_{\log_k} \to Y$ is bounded and for any bounded sequence $(f_i)_{i\in\mathbb{N}}$ in $H_{\log_k}$ which converges to zero uniformly on compact subsets of $\mathbb{B}_n$ as $i \to \infty$, we have $\|T_{g,\varphi} f_i\|_Y \to 0$ as $i \to \infty$.*

Here we omit the proof of Lemma 1, since it follows by standard arguments of the proof of Proposition 3.11 of [3], as well as the proof of the corresponding result in [10].

The following lemma can be found in [21].

LEMMA 2. *There exists $N = N(n)$ and functions $f_1, \cdots, f_N \in H_{\log_k}(\mathbb{B}_n)$ such that*

$$\sum_{m=1}^{N} |f_m(z)| \geqslant \frac{C}{(1-|z|^2)\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|z|^2}}, \ z \in \mathbb{B}_n, \tag{3}$$

for some positive constant $C > 0$.

LEMMA 3. *A closed set K in $\mathscr{B}_{\mu,0}$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} \mu(|z|)|f(z)| = 0.$$

The proof of Lemma 3 is similar to the proof of Lemma 1 in [22]. We omit the details.

Now we are in a position to state and prove our main results.

THEOREM 1. *Assume that* μ *is normal on* $[0,1)$, $g \in H(\mathbb{B}_n)$ *and* φ *is a holomorphic self-map of* $\mathbb{B}_n$. *Then the following statements are equivalent.*

(i) $T_{g,\varphi} : H_{\log_k} \to \mathscr{B}_\mu$ *is bounded;*

(ii) $T_{g,\varphi} : H_{\log_k,0} \to \mathscr{B}_\mu$ *is bounded;*

(iii)

$$M := \sup_{z\in\mathbb{B}_n} \frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|\varphi(z)|^2}} < \infty. \tag{4}$$

Moreover, when $T_{g,\varphi} : H_{\log_k} \to \mathscr{B}_\mu$ *is bounded, it holds*

$$\|T_{g,\varphi}\| \asymp \sup_{z\in\mathbb{B}_n} \frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|\varphi(z)|^2}}. \tag{5}$$

Proof. $(iii) \Rightarrow (i)$. Assume that (4) holds. Then, for any $f \in H_{\log_k}$, we have

$$\begin{aligned}\|T_{g,\varphi}f\|_{\mathscr{B}_\mu} &= |(T_{g,\varphi}f)(0)| + \sup_{z\in\mathbb{B}_n} \mu(|z|)|\Re(T_{g,\varphi}f)(z)| \\ &= \sup_{z\in\mathbb{B}_n} \mu(|z|)|\Re g(z)||f(\varphi(z))| \\ &\leqslant \|f\|_{H_{\log_k}} \sup_{z\in\mathbb{B}_n} \frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|\varphi(z)|^2}} < \infty. \end{aligned} \tag{6}$$

Here we used the facts that $(T_{g,\varphi}f)(0) = 0$ and

$$\Re(T_{g,\varphi}f)(z) = \Re g(z) f(\varphi(z)).$$

Therefore the operator $T_{g,\varphi} : H_{\log_k} \to \mathscr{B}_\mu$ is bounded.

$(i) \Rightarrow (ii)$. This implication is obvious.

$(ii) \Rightarrow (iii)$. Assume that $T_{g,\varphi} : H_{\log_k,0} \to \mathscr{B}_\mu$ is bounded. For $a \in \mathbb{B}_n$, set

$$f_a(z) = \frac{1}{(1-\langle z,a\rangle)\left(\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|z|^2}\right)}. \tag{7}$$

It is easy to see that $f_a \in H_{\log_k,0}$ and $\sup_{a\in\mathbb{B}_n} \|f_a\|_{H_{\log_k}} < \infty$. For any $b \in \mathbb{B}_n$,

$$\begin{aligned}\infty > \|T_{g,\varphi}f_{\varphi(b)}\|_{\mathscr{B}_\mu} &= \sup_{z\in\mathbb{B}_n} \mu(|z|)|\Re(T_{g,\varphi}f_{\varphi(b)})(z)| \\ &\geqslant \frac{\mu(|b|)|\Re g(b)|}{(1-|\varphi(b)|^2)\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|\varphi(b)|^2}}, \end{aligned} \tag{8}$$

which implies (4). From the above proof, it is clear that (5) holds. The proof is completed. □

THEOREM 2. *Assume that μ is normal on $[0,1)$, $g \in H(\mathbb{B}_n)$ and φ is a holomorphic self-map of $\mathbb{B}_n$. Then the following statements are equivalent.*

(i) $T_{g,\varphi} : H_{\log_k} \to \mathscr{B}_\mu$ *is compact;*

(ii) $T_{g,\varphi} : H_{\log_k,0} \to \mathscr{B}_\mu$ *is compact;*

(iii) $g \in \mathscr{B}_\mu$ *and*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|\varphi(z)|^2}} = 0. \tag{9}$$

Proof. $(i) \Rightarrow (ii)$. This implication is clear.

$(ii) \Rightarrow (iii)$. Assume that $T_{g,\varphi} : H_{\log_k,0} \to \mathscr{B}_\mu$ is compact. Then it is clear that $T_{g,\varphi} : H_{\log_k,0} \to \mathscr{B}_\mu$ is bounded. Taking the function $f(z) \equiv 1 \in H_{\log_k,0}$, we get that $g \in \mathscr{B}_\mu$. Let $(\varphi(z_i))_{i\in\mathbb{N}}$ be a sequence in $\mathbb{B}_n$ such that $\lim_{i\to\infty} |\varphi(z_i)| = 1$. Set

$$f_i(z) = \frac{1-|\varphi(z_i)|^2}{(1-\langle z,\varphi(z_i)\rangle)^2 \prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|z|^2}}, \quad i \in \mathbb{N}, \; z \in \mathbb{B}_n. \tag{10}$$

It is easy to see that

$$f_i \in H_{\log_k,0} \quad \text{and} \quad \sup_{i\in\mathbb{N}} \|f_i\|_{H_{\log_k}} < \infty.$$

Moreover $f_i \to 0$ uniformly on compact subsets of $\mathbb{B}_n$ as $i \to \infty$. By Lemma 1,

$$\lim_{i\to\infty} \|T_{g,\varphi} f_i\|_{\mathscr{B}_\mu} = 0. \tag{11}$$

In addition,

$$\|T_{g,\varphi} f_i\|_{\mathscr{B}_\mu} = \sup_{z\in\mathbb{B}_n} \mu(|z|)|\Re g(z) f_i(\varphi(z))| \geqslant \frac{\mu(|z_i|)|\Re g(z_i)|}{(1-|\varphi(z_i)|^2)\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|\varphi(z_i)|^2}}, \tag{12}$$

which together with (11) implies that

$$\lim_{i\to\infty} \frac{\mu(|z_i|)|\Re g(z_i)|}{(1-|\varphi(z_i)|^2)\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|\varphi(z_i)|^2}} = 0.$$

The last equality implies that (9) holds.

$(iii) \Rightarrow (i)$. Assume that $g \in \mathscr{B}_\mu$ and (9) holds. From these it follows that (5) holds. Hence $T_{g,\varphi} : H_{\log_k} \to \mathscr{B}_\mu$ is bounded by Theorem 1. Let $(f_i)_{i\in\mathbb{N}}$ be a bounded sequence in $H_{\log_k}$ such that $f_i \to 0$ uniformly on compact subsets of $\mathbb{B}_n$ as $i \to \infty$. By (9) we have that for any $\varepsilon > 0$, there is a constant $\delta \in (0,1)$ such that

$$\frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|\varphi(z)|^2}} < \varepsilon \tag{13}$$

whenever $\delta < |\varphi(z)| < 1$. Let $K = \{w \in \mathbb{B}_n : |w| \leqslant \delta\}$. From (13) and $g \in \mathscr{B}_\mu$, we obtain

$$\begin{aligned}\|T_{g,\varphi}f_i\|_{\mathscr{B}_\mu} &\leqslant \Big(\sup_{|\varphi(z)|\leqslant\delta} + \sup_{\delta<|\varphi(z)|<1}\Big)\mu(|z|)|\Re g(z)||f_i(\varphi(z))| \\ &\leqslant \|g\|_{\mathscr{B}_\mu}\sup_{w\in K}|f_i(w)| + \sup_{\delta<|\varphi(z)|<1}\frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)\prod_{j=1}^k \ln^{[j]}\frac{e^{[k]}}{1-|\varphi(z)|^2}}\|f_i\|_{H_{\log_k}} \\ &\leqslant \|g\|_{\mathscr{B}_\mu}\sup_{w\in K}|f_i(w)| + C\varepsilon.\end{aligned}$$

Since K is a compact subset of $\mathbb{B}_n$, we get $\lim_{i\to\infty}\sup_{w\in K}|f_i(w)| = 0$. Hence $\lim_{i\to\infty}\|T_{g,\varphi}f_i\|_{\mathscr{B}_\mu} \leqslant C\varepsilon$. Since ε is an arbitrary positive number we see that $\lim_{i\to\infty}\|T_{g,\varphi}f_i\|_{\mathscr{B}_\mu} = 0$. Therefore, $T_{g,\varphi} : H_{\log_k} \to \mathscr{B}_\mu$ is compact. The proof is completed. □

THEOREM 3. *Assume that μ is normal on $[0,1)$, $g \in H(\mathbb{B}_n)$ and φ is a holomorphic self-map of $\mathbb{B}_n$. Then $T_{g,\varphi} : H_{\log_k,0} \to \mathscr{B}_{\mu,0}$ is bounded if and only if $T_{g,\varphi} : H_{\log_k,0} \to \mathscr{B}_\mu$ is bounded and $g \in \mathscr{B}_{\mu,0}$.*

Proof. Assume that $T_{g,\varphi} : H_{\log_k,0} \to \mathscr{B}_{\mu,0}$ is bounded, then it is clear that $T_{g,\varphi} : H_{\log_k,0} \to \mathscr{B}_\mu$ is bounded. In addition, taking $f(z) = 1$, we get that $g \in \mathscr{B}_{\mu,0}$.

Conversely, suppose that $T_{g,\varphi} : H_{\log_k,0} \to \mathscr{B}_\mu$ is bounded and $g \in \mathscr{B}_{\mu,0}$. For each polynomial $p(z)$, we get

$$\mu(|z|)|\Re(T_{g,\varphi}p)(z)| = \mu(|z|)|\Re g(z)|\|p\|_{H^\infty},$$

where $\|p\|_{H^\infty} := \sup_{z\in\mathbb{B}_n}|p(z)|$. Since $\|p\|_{H^\infty} < \infty$, we obtain $T_{g,\varphi}(p) \in \mathscr{B}_{\mu,0}$. Thus for every $f \in H_{\log_k,0}$, there is a sequence of polynomials $(p_i)_{i\in\mathbb{N}}$ such that $||p_i - f||_{H_{\log_k}} \to 0$ as $i \to \infty$ by the fact that the set of all polynomials is dense in $H_{\log_k,0}$. Since $T_{g,\varphi} : H_{\log_k,0} \to \mathscr{B}_\mu$ is bounded, we have

$$||T_{g,\varphi}p_i - T_{g,\varphi}f||_{\mathscr{B}_\mu} \leqslant ||T_{g,\varphi}||_{H_{\log_k,0}\to\mathscr{B}_\mu}||p_i - f||_{H_{\log_k}} \to 0, \quad \text{as } i \to \infty.$$

Hence we obtain $T_{g,\varphi}(H_{\log_k,0}) \subseteq \mathscr{B}_{\mu,0}$ by the fact that $\mathscr{B}_{\mu,0}$ is the closed subset of $\mathscr{B}_\mu$. This completes the proof. □

THEOREM 4. *Assume that μ is normal on $[0,1)$, $g \in H(\mathbb{B}_n)$ and φ is a holomorphic self-map of $\mathbb{B}_n$. Then the following statements are equivalent.*

(i) $T_{g,\varphi} : H_{\log_k} \to \mathscr{B}_{\mu,0}$ *is bounded;*

(ii) $T_{g,\varphi} : H_{\log_k} \to \mathscr{B}_{\mu,0}$ *is compact;*

(iii) $T_{g,\varphi} : H_{\log_k,0} \to \mathscr{B}_{\mu,0}$ *is compact;*

(iv) $g \in \mathscr{B}_{\mu,0}$ *and*

$$\lim_{|\varphi(z)|\to 1}\frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)\prod_{j=1}^k \ln^{[j]}\frac{e^{[k]}}{1-|\varphi(z)|^2}} = 0; \tag{14}$$

(v)

$$\lim_{|z|\to 1} \frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|\varphi(z)|^2}} = 0. \tag{15}$$

Proof. $(ii) \Rightarrow (iii)$. This implication is clear.

$(iii) \Rightarrow (iv)$. Assume that $T_{g,\varphi} : H_{\log_k,0} \to \mathscr{B}_{\mu,0}$ is compact. Then it is clear that $T_{g,\varphi} : H_{\log_k,0} \to \mathscr{B}_{\mu}$ is compact and hence (14) holds. In addition, taking $f(z) = 1$, we get $g \in \mathscr{B}_{\mu,0}$.

$(iv) \Rightarrow (v)$. Suppose that $g \in \mathscr{B}_{\mu,0}$ and (14) holds. From (14) it follows that for every $\varepsilon > 0$, there exists a $\delta \in (0,1)$, such that

$$\frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|\varphi(z)|^2}} < \varepsilon$$

when $\delta < |\varphi(z)| < 1$. From the assumption $g \in \mathscr{B}_{\mu,0}$, we have that for the above ε, there exists an $r \in (0,1)$, such that

$$\mu(|z|)|\Re g(z)| < \varepsilon(1-\delta^2)$$

when $r < |z| < 1$.

Therefore, if $r < |z| < 1$ and $\delta < |\varphi(z)| < 1$, we obtain

$$\frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|\varphi(z)|^2}} < \varepsilon. \tag{16}$$

If $r < |z| < 1$ and $|\varphi(z)| \leqslant \delta$, we have that

$$\begin{aligned} \frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|\varphi(z)|^2}} &\leqslant \frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1}} \\ &\leqslant \frac{\mu(|z|)|\Re g(z)|}{(1-\delta^2)} < \varepsilon. \end{aligned} \tag{17}$$

Combing (16) with (17) we get (15).

$(v) \Rightarrow (ii)$. Suppose that (15) holds. From (1) we get

$$\mu(|z|)|\Re(T_{g,\varphi} f)(z)| \leqslant \frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|\varphi(z)|^2}} \|f\|_{H_{\log_k}}. \tag{18}$$

Taking the supremum in the above inequality over all $f \in H_{\log_k}$ such that $\|f\|_{H_{\log_k}} \leqslant 1$, then letting $|z| \to 1$, by (15) it follows that

$$\lim_{|z|\to 1} \sup_{\|f\|_{H_{\log_k}} \leqslant 1} \mu(|z|)|\Re(T_{g,\varphi}(f))(z)| = 0.$$

Employing Lemma 3, we see that $T_{g,\varphi}: H_{\log_k} \to \mathscr{B}_{\mu,0}$ is compact.

$(ii) \Rightarrow (i)$. This implication is obvious.

$(i) \Rightarrow (v)$. From Lemma 2, there exist functions $f_1, \cdots, f_N \in H_{\log_k}$, such that

$$\sum_{m=1}^{N} |f_m(z)| \geqslant \frac{C}{(1-|z|^2)\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|z|^2}}, \quad z \in \mathbb{B}_n. \tag{19}$$

By the boundedness of $T_{g,\varphi}: H_{\log_k} \to \mathscr{B}_{\mu,0}$, we have $T_{g,\varphi} f_i \in \mathscr{B}_{\mu,0}, \quad i = 1, \cdots, N.$ Therefore

$$\begin{aligned} 0 &= \lim_{|z|\to 1} \sum_{m=1}^{N} \mu(|z|)|\Re g(z)||f_m(\varphi(z))| \\ &\geqslant \lim_{|z|\to 1} \frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|\varphi(z)|^2}}, \end{aligned} \tag{20}$$

which implies (15). The proof is completed. □

Finally, we consider the Volterra composition operator $T_{g,\varphi}$ from $H_{\log_k}$ to mixed norm spaces.

THEOREM 5. *Assume that* $0 < p, q < \infty$ *and* ϕ *is normal on* $[0,1)$. *Let* φ *be a holomorphic self-map of* $\mathbb{B}_n$ *and* $g \in H(\mathbb{B}_n)$. *Then the following statements are equivalent:*

(i) $T_{g,\varphi}: H_{\log_k} \to H(p,q,\phi)$ *is a bounded operator;*

(ii) $T_{g,\varphi}: H_{\log_k} \to H(p,q,\phi)$ *is a compact operator;*

(iii)

$$\int_0^1 \Big(\int_S \frac{|\Re g(r\xi)|^q d\sigma(\xi)}{(1-|\varphi(r\xi)|^2)\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|\varphi(r\xi)|^2}} \Big)^{p/q} \phi^p(r)(1-r)^{p-1} dr < \infty; \tag{21}$$

(iv)

$$\lim_{t\to 1} \int_0^1 \Big(\int_{|\varphi(r\xi)|>t} \frac{|\Re g(r\xi)|^q d\sigma(\xi)}{(1-|\varphi(r\xi)|^2)\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|\varphi(r\xi)|^2}} \Big)^{p/q} \phi^p(r)(1-r)^{p-1} dr = 0. \tag{22}$$

Proof. $(ii) \Rightarrow (i)$. It is obvious.

$(i) \Rightarrow (iii)$. Suppose that $T_{g,\varphi}: H_{\log_k} \to H(p,q,\phi)$ is bounded. From Lemma 2, we pick functions $f_1, \cdots, f_N \in H_{\log_k}$ such that

$$\sum_{m=1}^{N} |f_m(z)| \geqslant \frac{C}{(1-|z|^2)\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|z|^2}}, \quad z \in \mathbb{B}_n. \tag{23}$$

The assumption implies that

$$\int_0^1 \Big(\int_S |\Re g(r\xi)|^q |(f_i \circ \varphi)(r\xi)|^q d\sigma(\xi) \Big)^{p/q} \phi^p(r)(1-r)^{p-1} dr < \infty, \quad i = 1, \cdots, N. \tag{24}$$

Here we used the following asymptotic formula (see, e.g. [4])

$$\int_0^1 M_q^p(f,r)\frac{\phi^p(r)}{1-r}dr \asymp |f(0)|^q + \int_0^1 M_q^p(\Re f,r)\phi^p(r)(1-r)^{p-1}dr.$$

Using (23), (24) and the elementary inequality

$$(a+b)^p \leqslant \begin{cases} a^p+b^p & , p\in(0,1) \\ 2^p(a^p+b^p), & p\geqslant 1 \end{cases}, \quad a>0, \quad b>0,$$

we obtain

$$\begin{aligned}
&\int_0^1 \Big(\int_S \frac{C|\Re g(r\xi)|^q}{(1-|\varphi(r\xi)|^2)\prod_{j=1}^k \ln^{[j]}\frac{e^{[k]}}{1-|\varphi(r\xi)|^2}}d\sigma(\xi)\Big)^{p/q}\phi^p(r)(1-r)^{p-1}dr \\
&\leqslant \int_0^1 \Big(\int_S |\Re g(r\xi)|^q\Big(|f_1(\varphi(r\xi))|+\cdots+|f_N(\varphi(r\xi))|\Big)^q d\sigma(\xi)\Big)^{p/q}\phi^p(r)(1-r)^{p-1}dr \\
&\leqslant C\int_0^1 \Big(\int_S |\Re g(r\xi)|^q|(f_1\circ\varphi)(r\xi)|^q d\sigma(\xi)\Big)^{p/q}\phi^p(r)(1-r)^{p-1}dr+\cdots \\
&\quad +C\int_0^1 \Big(\int_S |\Re g(r\xi)|^q|(f_N\circ\varphi)(r\xi)|^q d\sigma(\xi)\Big)^{p/q}\phi^p(r)(1-r)^{p-1}dr \\
&\leqslant C\int_0^1 \Big(\int_S |\Re(T_{g,\varphi}f_1)(r\xi)|^q d\sigma(\xi)\Big)^{p/q}\phi^p(r)(1-r)^{p-1}dr+\cdots \\
&\quad +C\int_0^1 \Big(\int_S |\Re(T_{g,\varphi}f_N)(r\xi)|^q d\sigma(\xi)\Big)^{p/q}\phi^p(r)(1-r)^{p-1}dr \\
&<\infty, \qquad (25)
\end{aligned}$$

which implies that (21) holds.

$(iii)\Rightarrow(iv)$. This implication follows from the dominated convergence Theorem.

$(iv)\Rightarrow(ii)$. Assume that (22) holds. Let $(f_i)_{i\in\mathbb{N}}$ be a bounded sequence in $H_{\log_k}$ such that $\{f_i\}$ converges to zero uniformly on compact subset of $\mathbb{B}_n$, we have

$$\begin{aligned}
&\int_0^1 \Big(\int_{|\varphi(r\xi)|>t} |\Re g(r\xi)|^q|(f_i\circ\varphi)(r\xi)|^q d\sigma(\xi)\Big)^{p/q}\phi^p(r)(1-r)^{p-1}dr \\
&\leqslant \|f_i\|^p_{H_{\log_k}}\int_0^1 \Big(\int_{|\varphi(r\xi)|>t} \frac{|\Re g(r\xi)|^q d\sigma(\xi)}{(1-|\varphi(r\xi)|^2)\prod_{j=1}^k \ln^{[j]}\frac{e^{[k]}}{1-|\varphi(r\xi)|^2}}\Big)^{p/q}\phi^p(r)(1-r)^{p-1}dr \\
&\leqslant C\int_0^1 \Big(\int_{|\varphi(r\xi)|>t} \frac{|\Re g(r\xi)|^q d\sigma(\xi)}{(1-|\varphi(r\xi)|^2)\prod_{j=1}^k \ln^{[j]}\frac{e^{[k]}}{1-|\varphi(r\xi)|^2}}\Big)^{p/q}\phi^p(r)(1-r)^{p-1}dr, \qquad (26)
\end{aligned}$$

for all i. Take $\varepsilon>0$. (22) and (26) imply that there exists $t_0\in(0,1)$ such that

$$\int_0^1 \Big(\int_{|\varphi(r\xi)|>t_0} |\Re g(r\xi)|^q F|(f_i\circ\varphi)(r\xi)|^q d\sigma(\xi)\Big)^{p/q}\phi^p(r)(1-r)^{p-1}dr<\varepsilon, \qquad (27)$$

for all i. For the above ε, since $\{f_i\}$ converges to 0 on any compact subset of $\mathbb{B}_n$, there exists a i_0 such that

$$\int_0^1 \Big(\int_{|\varphi(r\xi)|\leqslant t_0} |\Re g(r\xi)|^q|(f_i\circ\varphi)(r\xi)|^q d\sigma(\xi)\Big)^{p/q}\phi^p(r)(1-r)^{p-1}dr<\varepsilon, \qquad (28)$$

for all $i > i_0$. Hence by (27) and (28) we have

$$\begin{aligned}\|T_{g,\varphi}f_i\|_{H(p,q,\phi)} &= \int_0^1 \Big(\int_S |\Re g(r\xi)|^q |(f_i\circ\varphi)(r\xi)|^q d\sigma(\xi)\Big)^{p/q}\phi^p(r)(1-r)^{p-1}dr\\ &= \int_0^1 \Big(\int_{|\varphi(r\xi)|>t_0} |\Re g(r\xi)|^q |(f_i\circ\varphi)(r\xi)|^q d\sigma(\xi)\Big)^{p/q}\phi^p(r)(1-r)^{p-1}dr\\ &\quad + \int_0^1 \Big(\int_{|\varphi(r\xi)|\leqslant t_0} |\Re g(r\xi)|^q |(f_i\circ\varphi)(r\xi)|^q d\sigma(\xi)\Big)^{p/q}\phi^p(r)(1-r)^{p-1}dr\\ &\leqslant 2\varepsilon, \text{ as } i > i_0,\end{aligned}$$

from which we obtain

$$\lim_{i\to\infty}\|T_{g,\varphi}f_i\|_{H(p,q,\phi)} = 0.$$

Thus $T_{g,\varphi} : H_{\log_k} \to H(p,q,\phi)$ is compact by Lemma 1. The proof of this theorem is completed. □

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