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REAL AND COMPLEX OPERATOR NORMS BETWEEN QUASI–BANACH $L^p - L^q$ SPACES

LECH MALIGRANDA AND NATALIA SABOUROVA

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Abstract. Relations between the norms of an operator and its complexification as a mapping from L^p to L^q has been recognized as a serious problem in analysis after the publication of Marcel Riesz's work on convexity and bilinear forms in 1926. We summarize here what it is known about these relations in the case of normed Legesgue spaces and investigate the quasi-normed case, i. e. we consider all $0 < p,q \leqslant \infty$. In particular, in the lower triangle, that is, for $0 < p \leqslant q \leqslant \infty$ these norms are the same. In the upper triangle and the normed case, that is, when $1 \leqslant q < p \leqslant \infty$ the norm of the complexification of a real operator is obviously not bigger than 2 times its real norm. In 1977 Krivine proved that the constant 2 can be replaced by $\sqrt{2}$. On the other hand, it was suspected that in the case of quasi-normed Lebesgue spaces ($0 < q < p \leqslant \infty$) the corresponding constant could be arbitrarily large, but as we will see this is not the case. More precisely, we prove that this constant for quasi-normed Lebesgue spaces is between 1 and 2. Some additional properties and estimates of this constant with some results about the relation between complex and real norms of operators, including those between two-dimensional Orlicz spaces are presented in the first four chapters. Finally, in Chapter 5, we use the results on the estimates of the norms in the proof of the real Riesz-Thorin interpolation theorem valid in the first quadrant.

1. On the norm of operator between real and complex Lebesgue spaces

Any bounded linear operator $T : L^p(\mu) \to L^q(\nu)$ between real (quasi-)normed Lebesgue spaces (that is, when $0 < p,q \leqslant \infty$) has its natural extension to the complex-valued functions (also called *complexification*) $T_{\mathbb{C}} : L^p_{\mathbb{C}}(\mu) \to L^q_{\mathbb{C}}(\nu)$ given by

$$T_{\mathbb{C}}(f+ig) = T(f) + iT(g).$$

Operator $T_{\mathbb{C}}$ is linear and bounded. Moreover,

$$\begin{aligned} \|T_{\mathbb{C}}(f+ig)\|_q &= \|\,|Tf+iTg|\|_q = \|\sqrt{(Tf)^2+(Tg)^2}\,\|_q \\ &\leqslant \||Tf|+|Tg|\|_q \leqslant \max\{1,2^{1/q-1}\}\,[\|Tf\|_q + \|Tg\|_q] \\ &\leqslant \max\{1,2^{1/q-1}\}\|T\|_{L^p\to L^q}\,[\|f\|_p + \|g\|_p] \\ &\leqslant \max\{2,2^{1/q}\}\|T\|_{L^p\to L^q}\|\sqrt{f^2+g^2}\,\|_p \\ &= \max\{2,2^{1/q}\}\|T\|_{L^p\to L^q}\|\,|f+ig|\|_p \\ &= \max\{2,2^{1/q}\}\|T\|_{L^p\to L^q}\|f+ig\|_p, \end{aligned}$$

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that is,

$$\|T_{\mathbb{C}}\|_{L^p_{\mathbb{C}}\to L^q_{\mathbb{C}}} \leqslant C_{p,q}\|T\|_{L^p\to L^q}, \text{ where } C_{p,q} \leqslant \max\{2, 2^{1/q}\}. \tag{1}$$

Using these estimates we can see that the constant $C_{p,q}$ is the best constant in the inequality

$$\|\sqrt{(Tf)^2+(Tg)^2}\,\|_q \leqslant C_{p,q}\|T\|_{L^p\to L^q}\|\sqrt{f^2+g^2}\,\|_p \tag{2}$$

for all $f,g \in L^p(\mu)$. In the case of normed Lebesgue spaces (that is, when $1 \leqslant p,q \leqslant \infty$) we immediately obtain from the relation (1) that $C_{p,q} \leqslant 2$ and an even more precise estimate $C_{p,q} \leqslant C_{\infty,1} = \sqrt{2}$ was provided by Krivine in his remarkable paper [20]. On the other hand, relying on (1), it seems reasonable to suggest, that the constant $C_{p,q}$ can be very large when q is small, i.e. when $0 < q < 1$. However, it is not the case and we present here a simple proof of the statement (see Theorem 1)

$$1 \leqslant C_{p,q} \leqslant 2 \ \text{ for all } p,q \in (0,\infty]. \tag{3}$$

To obtain the upper estimate in (3) we need the following lemma, the first part of which was already mentioned in the classical book of Zygmund [44, p. 181].

LEMMA 1. *(a) Let $0 < p \leqslant \infty$. Then, for all $a,b \in \mathbb{R}$*

$$\left(\int_0^1 |a\cos 2\pi t + b\sin 2\pi t|^p\,dt\right)^{1/p} = d_p(a^2+b^2)^{1/2}, \tag{4}$$

where $d_p = \left(\int_0^1 |\cos 2\pi t|^p\,dt\right)^{1/p}$.

(b) The function $d : (0,\infty] \to (0,\infty)$ given by $d(p) := d_p$ is strictly increasing and $d_1 = \frac{2}{\pi}, d_2 = \frac{1}{\sqrt{2}}, d_\infty = 1$. Moreover, $d_0 := \lim_{p\to 0^+} d_p = \frac{1}{2}$ and so $d((0,\infty]) = (\frac{1}{2},1]$.

Proof. If we divide both sides of the equality (4) by $(a^2+b^2)^{1/2}$ (we can assume that $a^2+b^2 > 0$, since otherwise the equality obviously takes place) and note that

$$\begin{aligned}\frac{a}{(a^2+b^2)^{\frac{1}{2}}}\cos 2\pi t + \frac{b}{(a^2+b^2)^{\frac{1}{2}}}\sin 2\pi t &= \cos\theta\cos 2\pi t \pm \sin\theta\sin 2\pi t \\ &= \cos(2\pi t \mp \theta),\end{aligned}$$

then by 2π-periodicity of cosine we obtain

$$\begin{aligned}\int_0^1 |\cos(2\pi t \mp \theta)|^p\,dt &= \int_{\mp\theta}^{2\pi\mp\theta} |\cos s|^p\,\frac{ds}{2\pi} \\ &= \int_0^{2\pi} |\cos s|^p\,\frac{ds}{2\pi} = \int_0^1 |\cos 2\pi t|^p\,dt,\end{aligned}$$

that gives the required equality. Note that the constant d_p can be written in terms of gamma function as follows $d_p = \left(\int_0^1 |\cos 2\pi t|^p\,dt\right)^{1/p} = \left(\frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi}\,\Gamma(\frac{p+2}{2})}\right)^{1/p}$.

(b) The property that $d(p)$ is increasing in p can be easily obtained by the Hölder-Rogers inequality and since the equality in the Hölder-Rogers inequality applied to our case does not hold for any $p \neq q$ we have even that $d(p)$ is strictly increasing.

The values of $d(p)$ at $1,2$ and infinity can be straightforwardly calculated and it only remains to show the equality $d_0 = 1/2$.

By the F. Riesz lemma ([29]; see also Rudin [32, p. 71]) we have

$$d_0 = \lim_{p\to 0^+} \left(\int_0^1 |\cos 2\pi t|^p \, dt \right)^{1/p} = \exp\left(\int_0^1 \ln |\cos 2\pi t| \, dt \right).$$

Since $\int_0^1 \ln|\cos 2\pi t| dt = 4\int_0^{1/4} \ln(\cos 2\pi t)\, dt$, then denoting $I_1 := \int_0^{1/4} \ln(\cos 2\pi t)\, dt$ we obtain

$$I_1 = \int_0^{1/4} \ln\left[\cos\left(\frac{\pi}{2} - 2\pi t\right)\right] \, dt = \int_0^{1/4} \ln(\sin 2\pi t)\, dt$$

and so

$$\begin{aligned} 2I_1 &= \int_0^{1/4} \ln(\sin 2\pi t\ \cos 2\pi t)\, dt = \int_0^{1/4} \ln \frac{\sin 4\pi t}{2}\, dt \\ &= \int_0^{1/4} \ln(\sin 4\pi t)\, dt - \frac{1}{4}\ln 2 = \frac{1}{2}\int_0^{1/2} \ln(\sin 2\pi t)\, dt - \frac{1}{4}\ln 2 \\ &= I_1 - \frac{1}{4}\ln 2. \end{aligned}$$

Thus, $I_1 = -\frac{1}{4}\ln 2$ and $d_0 = \exp(4I_1) = \exp(-\ln 2) = \frac{1}{2}$.

Note, that $I_1 = \frac{1}{2\pi}\int_0^{\pi/2} \ln\cos t\, dt = \frac{1}{2\pi}(-\frac{\pi}{2}\ln 2) = -\frac{1}{4}\ln 2$ can also be proved by applying Taylor series expansion of the logarithmic function (see Russell [33]). □

Using Lemma 1 we now obtain the early mentioned statement about the constant $C_{p,q}$ for $0 < p,q \leqslant \infty$ or what is the same statement, Theorem 1, on the operator estimate. Note, Lemma 1 as the key ingredient in the proof of this theorem for values $p = q \geqslant 1$ was firstly used by Zygmund [44] and for $1 \leqslant p,q \leqslant \infty$ by Verbickiĭ-Sereda [41], Verbickiĭ [40]. Instead of using Lemma 1 Marcinkiewicz-Zygmund in an earlier paper from 1939 [25] based their proof of the restricted statement on the operator estimate for $0 < p \leqslant q \leqslant \infty$ on utilizing Gaussian variables.

THEOREM 1. *Let $T : L^p(\mu) \to L^q(\nu)$ be an arbitrary bounded linear operator between real Lebesgue spaces, where μ and ν are arbitrary positive σ-finite measures.*

If $0 < p \leqslant q \leqslant \infty$, then

$$\|T_{\mathbb{C}}\|_{p,q} = \|T\|_{p,q}. \tag{5}$$

If $0 < q \leqslant p \leqslant \infty$, then

$$\|T_{\mathbb{C}}\|_{p,q} \leqslant \frac{d_p}{d_q}\|T\|_{p,q}. \tag{6}$$

In particular, in the relation $\|T_{\mathbb{C}}\|_{p,q} \leqslant C_{p,q}\|T\|_{p,q}$ *we always have* $1 \leqslant C_{p,q} \leqslant 2$*, and if, in addition,* $0 < p \leqslant q \leqslant \infty$*, then* $C_{p,q} = 1$.

Proof. First, we show that for all $f, g \in L^p(\mu)$ and $0 < p, q \leqslant \infty$ we have

$$\left(\int_0^1 \|f\cos 2\pi t + g\sin 2\pi t\|_p^q \, dt\right)^{1/q} \leqslant \max\{d_p, d_q\}\|f + ig\|_p. \tag{7}$$

If $0 < p \leqslant q < \infty$, then from the integral Minkowski inequality for $\|\cdot\|_{L^{q/p}[0,1]}$ (here we use the assumption that $q/p \geqslant 1$) and Lemma 1 it follows

$$\begin{aligned}
&\int_0^1 \|f\cos 2\pi t + g\sin 2\pi t\|_p^q \, dt \\
&= \int_0^1 \left(\int_\Omega |f(x)\cos 2\pi t + g(x)\sin 2\pi t|^p \, d\mu(x)\right)^{q/p} dt \\
&= \left\|\int_\Omega |f(x)\cos 2\pi t + g(x)\sin 2\pi t|^p \, d\mu(x)\right\|_{L^{q/p}[0,1]}^{q/p} \\
&\leqslant \left(\int_\Omega \||f(x)\cos 2\pi t + g(x)\sin 2\pi t|^p\|_{L^{q/p}[0,1]} \, d\mu(x)\right)^{q/p} \\
&= \left(\int_\Omega \left(\int_0^1 |f(x)\cos 2\pi t + g(x)\sin 2\pi t|^q \, dt\right)^{p/q} d\mu(x)\right)^{q/p} \\
&= d_q^q \left(\int_\Omega |f(x) + ig(x)|^p \, d\mu(x)\right)^{q/p} = d_q^q \, \|f + ig\|_p^q.
\end{aligned}$$

Note, that for $p = q$ the equality holds in each step of this derivation. A simple modification of this proof justifies the same statement for $q = \infty$ (recall $d_\infty = 1$).

If $0 < q \leqslant p \leqslant \infty$, then making use of the Hölder-Rogers inequality with $p/q \geqslant 1$ and just proved relation we obtain

$$\begin{aligned}
\left(\int_0^1 \|f\cos 2\pi t + g\sin 2\pi t\|_p^q \, dt\right)^{1/q} &\leqslant \left(\int_0^1 \|f\cos 2\pi t + g\sin 2\pi t\|_p^p \, dt\right)^{1/p} \\
&= d_p \, \|f + ig\|_p.
\end{aligned}$$

Now, from Lemma 1 and the estimate (7), it follows that

$$\begin{aligned}
\|T_{\mathbb{C}}(f + ig)\|_q &= \left\|\sqrt{(Tf)^2 + (Tg)^2}\right\|_q \\
&= \frac{1}{d_q}\left(\int_0^1 \|T(f\cos 2\pi t + g\sin 2\pi t)\|_q^q \, dt\right)^{1/q} \\
&\leqslant \frac{\|T\|_{p,q}}{d_q}\left(\int_0^1 \|f\cos 2\pi t + g\sin 2\pi t\|_p^q \, dt\right)^{1/q} \\
&\leqslant \max\{\frac{d_p}{d_q}, \frac{d_q}{d_q}\} \, \|T\|_{p,q} \, \|f + ig\|_p.
\end{aligned}$$

Thus, if $0 < p \leqslant q \leqslant \infty$, then $\|T_{\mathbb{C}}\|_{p,q} \leqslant \|T\|_{p,q}$ and since the reverse inequality obviously holds, then $\|T_{\mathbb{C}}\|_{p,q} = \|T\|_{p,q}$ for all such p and q. On the other hand, if $0 < q \leqslant p \leqslant \infty$, then $\|T_{\mathbb{C}}\|_{p,q} \leqslant \frac{d_p}{d_q}\|T\|_{p,q}$. In particular, from Lemma 1(b) it follows that

$$C_{p,q} \leqslant \max\left\{1, \frac{d_p}{d_q}\right\} \leqslant \frac{d_\infty}{\lim_{p\to 0^+} d_p} = \frac{d_\infty}{d_0} = 2. \qquad \square \tag{8}$$

Using Theorem 1, Lemma 1(b) and Krivine theorem we can now easily write the following estimates of the constant $C_{p,q}$, which is dependent on the points $(1/p, 1/q)$ as the elements of some regions in the first quadrant $Q_{++} = [0,\infty) \times [0,\infty)$ (see Fig. 1):

A: If $0 < p \leqslant q \leqslant \infty$, then $C_{p,q} = 1$.

B: If $1 \leqslant q < p \leqslant \infty$, then $C_{p,q} \leqslant C_{\infty,1} = \sqrt{2}$ (Krivine [20])

C: If $0 < q < p < 1$, then $C_{p,q} \leqslant d_p/d_q \leqslant d_1/d_0 = 4/\pi$.

D: If $0 < q \leqslant 1 < p \leqslant 2$, then $C_{p,q} \leqslant d_p/d_q \leqslant d_2/d_0 = \sqrt{2}$.

E: If $0 < q < 1$, $2 \leqslant p \leqslant \infty$, then $C_{p,q} \leqslant d_p/d_q \leqslant d_\infty/d_0 = 2$.

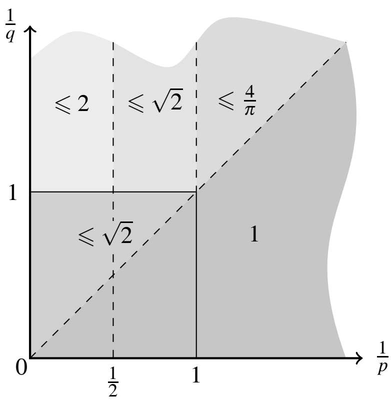
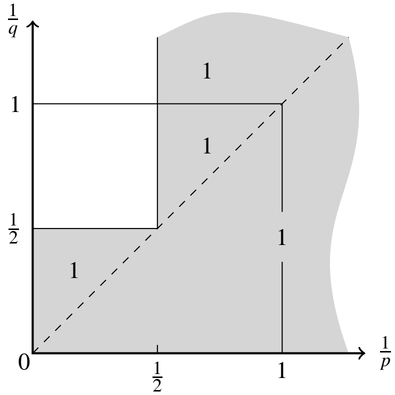


Fig. 1. The estimates of the constant $C_{p,q}$

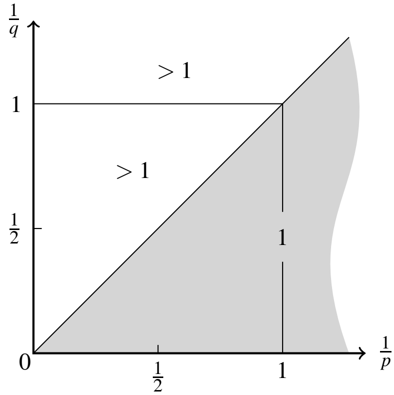
Let us mention that for the case $1 \leqslant p \leqslant q < \infty$ and n-dimensional spaces, i. e., for $T : l_n^p \to l_n^q$ the result that $C_{p,q}(\dim n) = 1$ was already obtained by Taylor [37] in 1958 by using differential calculus. He showed that the norm of T is attained on the real vector both in the real and in the complex case for $q \geqslant p \geqslant 1$.

In the case of operators between two-dimensional Banach spaces $T : l_2^p \to l_2^q$ the constant $C_{p,q}(\dim 2) = 1$ if and only if either $1 \leqslant p \leqslant 2$ or $2 \leqslant q \leqslant \infty$ (see Verbickiĭ-Sereda [41, Th. 2]). These conditions are equivalent to three cases, namely either $1 \leqslant p \leqslant q \leqslant \infty$ or $1 \leqslant q \leqslant p \leqslant 2$ or $2 \leqslant q \leqslant p \leqslant \infty$. Thus $C_{p,q}(\dim 2) > 1$ if and only if $1 \leqslant q < 2 < p \leqslant \infty$. On the other hand, if at least one of the spaces is quasi-Banach, namely, when $0 < q \leqslant p \leqslant 2$ we have $C_{p,q}(\dim 2) = 1$ (see Sabourova [34, Prop. 11]).

For three dimensional spaces already M. Riesz, in his paper on convexity and bilinear forms [30], found an operator with different real and complex norms (detailed proof of the Riesz example can be found in Gasch-Maligranda [16, pp. 112–113] and Vogt [42, pp. 6–8]). More precisely, if $p > q \geqslant 1$ then there exist $\varepsilon = \varepsilon_{p,q} > 0$ such that $T_\varepsilon : l_3^p \to l_3^q$ satisfies $\|T_\varepsilon\|_{p,q} < \|(T_\varepsilon)_{\mathbb{C}}\|_{p,q}$, where $T_\varepsilon = I - \varepsilon \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. The same operator is working even for $p > q > 0$.



(a) $C(l_2^p, l_2^q)$



(b) $C(l_3^p, l_3^q)$

Fig. 2. Constant $C_{p,q}$ for two and three dimensional spaces

2. On the norm of operator between some two-dimensional Orlicz spaces in the real and in the complex case

Having the equality $\|T_{\mathbb{C}}\|_{L^p \to L^p} = \|T\|_{L^p \to L^p}$ for all $0 < p \leqslant \infty$ it is an interesting question whether similar equality is always preserved for the operators between the same spaces but different from L^p. We give a negative answer to this question by showing the existence of operators between two-dimensional symmetric spaces and two-dimensional Orlicz spaces with different real and complex norms. The following example of such operator between two-dimensional symmetric spaces for $p = 1$ was firstly given by Sokolovski [35, Th. 1] and Gasch-Maligranda [16, Ex. 7].

EXAMPLE 1. *For $1 \leqslant p < 2$ let us consider real two-dimensional symmetric spaces $X_p = (\mathbb{R}^2, \|\cdot\|_{\frac{1}{p}})$ equipped with the norms*

$$\|x\|_{\frac{1}{p}} = \frac{(|x_1|^p + |x_2|^p)^{1/p} + 2^{\frac{1}{2p}} \max\{|x_1|, |x_2|\}}{1 + 2^{\frac{1}{2p}}}, \; x = (x_1, x_2) \in \mathbb{R}^2. \tag{9}$$

For the operator $T : X_p \to X_p$ given by the formula $T(x,y) = (x+y, x-y)$ we have

$$\|T_{\mathbb{C}}\| > \|T\| = 2^{1-\frac{1}{2p}}.$$

In fact, using the generalized real Clarkson inequality twice (cf. [24], Th. 2.1), we find the real norm of this operator:

$$
\begin{aligned}
\|T(x_1,x_2)\|_p^1 &= \frac{(|x_1+x_2|^p+|x_1-x_2|^p)^{1/p}+2^{\frac{1}{2p}}\max\{|x_1+x_2|,|x_1-x_2|\}}{1+2^{\frac{1}{2p}}} \\
&\leqslant \frac{2\max\{|x_1|,|x_2|\}+2^{\frac{1}{2p}}2^{1-\frac{1}{p}}(|x_1|^p+|x_2|^p)^{1/p}}{1+2^{\frac{1}{2p}}} \\
&= 2^{1-\frac{1}{2p}}\frac{(|x_1|^p+|x_2|^p)^{1/p}+2^{\frac{1}{2p}}\max\{|x_1|,|x_2|\}}{1+2^{\frac{1}{2p}}} \\
&= 2^{1-\frac{1}{2p}}\ \|(x_1,x_2)\|_p^1.
\end{aligned}
$$

The equality holds for $x_1=x_2=1$ and therefore $\|T\|=2^{1-\frac{1}{2p}}$. To show that the complex norm of this operator is strictly bigger than $2^{1-\frac{1}{2p}}$ we consider two cases. Firstly, if $p=1$, then taking $z_1=1$, $z_2=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$ we obtain

$$
\frac{\|T_{\mathbb{C}}(z_1,z_2)\|_1^1}{\|(z_1,z_2)\|_1^1}=\frac{(1+\sqrt{2})(2+\sqrt{2})^{1/2}+(2-\sqrt{2})^{1/2}}{2+\sqrt{2}}>\sqrt{2}=\|T\|.
$$

Secondly, in the case when $1<p<2$ we consider the function

$$
\begin{aligned}
\psi(\theta)&=\frac{\left\|T_{\mathbb{C}}(1,e^{i\theta})\right\|_p^1}{\|(1,e^{i\theta})\|_p^1}=\frac{\left((2+2\cos\theta)^{\frac{p}{2}}+(2-2\cos\theta)^{\frac{p}{2}}\right)^{\frac{1}{p}}+2^{\frac{1}{2p}}\max\left\{(2\pm 2\cos\theta)^{\frac{1}{2}}\right\}}{2^{\frac{1}{p}}+2^{\frac{1}{2p}}} \\
&=\frac{\sqrt{2}}{2^{\frac{1}{p}}+2^{\frac{1}{2p}}}\varphi(\theta), \varphi(\theta)=\left((1+\cos\theta)^{\frac{p}{2}}+(1-\cos\theta)^{\frac{p}{2}}\right)^{\frac{1}{p}}+2^{\frac{1}{2p}}\max\left\{(1\pm\cos\theta)^{\frac{1}{2}}\right\}.
\end{aligned}
$$

For $0<\theta\leqslant\pi/2$ we have

$$
\begin{aligned}
\varphi'(\theta)&=-\frac{\sin\theta}{2}[(1+\cos\theta)^{\frac{p}{2}}+(1-\cos\theta)^{\frac{p}{2}}]^{\frac{1}{p}-1}[(1+\cos\theta)^{\frac{p}{2}-1}-(1-\cos\theta)^{\frac{p}{2}-1}] \\
&\quad -2^{\frac{1}{2p}}\frac{\sin\theta}{2}(1+\cos\theta)^{-1/2},
\end{aligned}
$$

where

$$
\varphi'(0^+)=\lim_{\theta\to 0^+}\frac{\sin\theta}{2}\frac{2^{1/2-p/2}}{(1-\cos\theta)^{1-p/2}}=\frac{1}{\sqrt{2}}\lim_{\theta\to 0^+}\sin^{p-1}\frac{\theta}{2}\cos\frac{\theta}{2}=0.
$$

Moreover,

$$
\varphi''(0^+)=-\frac{1}{2\sqrt{2}}-2^{\frac{1}{2p}-\frac{3}{2}}+\frac{1}{2^{\frac{p+1}{2}}}\lim_{\theta\to 0^+}\frac{1-(2-p)\cos^2\frac{\theta}{2}}{(1-\cos\theta)^{1-\frac{p}{2}}}=+\infty.
$$

Therefore there exists $\theta_0 \in (0, \frac{\pi}{2})$ such that $\varphi(\theta_0) > \varphi(0) = \sqrt{2} + 2^{\frac{1}{2}+\frac{1}{2p}}$ and consequently

$$\psi(\theta_0) > \psi(0) = \frac{\sqrt{2}}{2^{\frac{1}{p}} + 2^{\frac{1}{2p}}} \left(\sqrt{2} + 2^{\frac{1}{2}+\frac{1}{2p}}\right) = 2^{1-\frac{1}{2p}}.$$

Thus, we showed that $\|T_{\mathbb{C}}\| > \|T\|$ in both cases.

EXAMPLE 2. *For* $1 \leqslant p < 2$ *let us consider the* p*-convexification* $X^{(p)}$ *of the space* $X = X_1$ *given by the norm of* $x = (x_1, x_2) \in \mathbb{R}^2$

$$\|x\|_{X^{(p)}} = \||x|^p\|_X^{1/p} = \left(\frac{|x_1|^p + |x_2|^p + \sqrt{2}\max\{|x_1|, |x_2|\}^p}{1+\sqrt{2}}\right)^{1/p}.$$

For the operator $T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ *we can prove similarly as in Example* 1 *that*

$$\|T\|_{X^{(p)} \to X^{(p)}} = 2^{1-\frac{1}{2p}} \text{ and } \|T_{\mathbb{C}}\| > 2^{1-\frac{1}{2p}}.$$

THEOREM 2. *There exists a real two-dimensional Orlicz space* l_2^{φ} *such that for the linear operator* $T : l_2^{\varphi} \to l_2^{\varphi}$ *given by* $T(x,y) = (x+y, x-y)$ *we have* $\|T_{\mathbb{C}}\| > \|T\|$*, where* l_2^{φ} *is equipped with the Luxemburg-Nakano norm. Moreover, for the same operator* $T : l_2^{\varphi^*} \to l_2^{\varphi^*}$*, where* φ^* *is a complementary function to* φ *and the Orlicz space* $l_2^{\varphi^*}$ *is equipped with the Orlicz norm we have* $\|T_{\mathbb{C}}\| > \|T\|$.

Proof. Let us consider two-dimensional space $X = X_1$ from Example 1. Clearly, the unit ball defined by the norm (9) is a compact symmetric convex non-empty subset of $\mathbb{R}^2$. Using the result of Grząślewicz (cf. [17]) saying that every compact symmetric convex subset of $\mathbb{R}^2$ with non-empty interior is a unit ball of some Orlicz space l_2^{φ} we can construct a two-dimensional Orlicz space isometric to X.

Let us take real two-dimensional Orlicz space l_2^{φ}, generated by the convex Orlicz function on $[0, \infty)$ given by

$$\varphi(u) = \begin{cases} \frac{u}{\sqrt{2}}, & \text{if } 0 \leqslant u \leqslant \frac{1}{\sqrt{2}}, \\ u(1+\frac{1}{\sqrt{2}}) - \frac{1}{\sqrt{2}}, & \text{if } u \geqslant \frac{1}{\sqrt{2}}. \end{cases} \tag{10}$$

It is simple to check that the Luxemburg-Nakano norm on l_2^{φ} which is given by

$$\|(x_1, x_2)\|_{\varphi} = \inf\{k > 0 : \varphi\left(\left|\frac{x_1}{k}\right|\right) + \varphi\left(\left|\frac{x_2}{k}\right|\right) \leqslant 1\} \tag{11}$$

coincides with the original norm (9) on X_1. Indeed, without loss of generality, let us fix $0 \leqslant x_1 \leqslant x_2$. Then, $\|(x_1, x_2)\|_1^1 = x_2 + \frac{x_1}{1+\sqrt{2}}$. To calculate (11) we consider the following three cases obtained naturally from the representation of φ, namely

$\|(x_1,x_2)\|_\varphi = \min\{\inf A,\ \inf B,\ \inf C\}$, where

$$A = \left\{ k>0 : k \leqslant \sqrt{2}x_1 \text{ and } \varphi\left(\frac{x_1}{k}\right)+\varphi\left(\frac{x_2}{k}\right) = \left(\frac{x_1+x_2}{k}\right)\left(1+\frac{1}{\sqrt{2}}\right)-\sqrt{2} \leqslant 1 \right\},$$

$$B = \left\{ k>0 : \sqrt{2}x_1 \leqslant k \leqslant \sqrt{2}x_2 \text{ and } \varphi\left(\frac{x_1}{k}\right)+\varphi\left(\frac{x_2}{k}\right) = \frac{x_1}{\sqrt{2}k}+\frac{x_2}{k}\frac{1+\sqrt{2}}{\sqrt{2}}-\frac{1}{\sqrt{2}} \leqslant 1 \right\},$$

$$C = \left\{ k>0 : k \geqslant \sqrt{2}x_2 \text{ and } \varphi\left(\frac{x_1}{k}\right)+\varphi\left(\frac{x_2}{k}\right) = \frac{x_1+x_2}{k\sqrt{2}} \leqslant 1 \right\}.$$

The set A consists only of $\sqrt{2}x_1$ since $\frac{x_1+x_2}{k} \leqslant \frac{1+\sqrt{2}}{1+1/\sqrt{2}} = \sqrt{2}$ and to satisfy $\frac{x_1+x_2}{\sqrt{2}} \leqslant k \leqslant \sqrt{2}x_1 \leqslant \frac{x_1+x_2}{\sqrt{2}}$ we should have $x_1 = x_2$ and therefore $k = \sqrt{2}x_1$. Since $\inf C \geqslant \sqrt{2}x_2$ and in the set B we have $\sqrt{2}x_1 \leqslant k \leqslant \sqrt{2}x_2$ and $k \geqslant x_2 + \frac{x_1}{1+\sqrt{2}}$, and also since $\sqrt{2}x_1 \leqslant x_2 + \frac{x_1}{1+\sqrt{2}} \leqslant \sqrt{2}x_2$ and $x_1 + \frac{x_1}{1+\sqrt{2}} = \sqrt{2}x_1$ it follows that $\|(x_1,x_2)\|_\varphi = \inf B = x_2 + \frac{x_1}{1+\sqrt{2}}$, and the equality $\|(x_1,x_2)\|_\varphi = \|(x_1,x_2)\|_1^1$ is proved.

From Example 1 we have that $\|T_{\mathbb{C}}\| > \|T\|$ for the Orlicz space with the Luxemburg-Nakano norm. To show that the same is true for the Orlicz norm we need some duality results which we present below. □

From the theory of Orlicz spaces it is well-known that the dual norm to the Luxemburg-Nakano $\|\cdot\|_\varphi$ is an Orlicz norm $\|\cdot\|_\varphi^o$ (cf. Maligranda [23]). Let us then calculate it.

PROPOSITION 1. *The dual space $X^* = (X^*, \|\cdot\|^*)$ to $X = X_1$ has the norm*

$$\|y\|^* = \sup_{x\neq 0} \frac{|x_1y_1+x_2y_2|}{\|x\|} = \max\left(\frac{|y_1|+|y_2|}{\sqrt{2}}, |y_1|, |y_2|\right).$$

Proof. Without loss of generality we can assume that $0 \leqslant y_2 \leqslant y_1$. Then

$$\|y\|^* = \sup_{x_1 \geqslant x_2 \geqslant 0} \frac{x_1y_1+x_2y_2}{\frac{x_1+x_2+\sqrt{2}x_1}{1+\sqrt{2}}} = \sup_{0\leqslant x\leqslant 1} \frac{y_1+xy_2}{1+\frac{x}{1+\sqrt{2}}} = \max\left(\frac{y_1+y_2}{\sqrt{2}}, y_1\right).$$

We can also calculate this dual norm in another way using the theory of Orlicz spaces (cf. Maligranda [23]). The complementary function φ^* to φ is

$$\begin{aligned}\varphi^*(v) &= \sup_{u>0}[uv-\varphi(u)] \\ &= \max\left[\sup_{0<u\leqslant 1/\sqrt{2}}\left(uv-u/\sqrt{2}\right), \sup_{u\geqslant 1/\sqrt{2}}\left(uv-u(1+1/\sqrt{2})+1/\sqrt{2}\right)\right],\end{aligned}$$

and so

$$\varphi^*(v) = \begin{cases} 0, & \text{if } 0 \leqslant v \leqslant \frac{1}{\sqrt{2}}, \\ \frac{v}{\sqrt{2}}-\frac{1}{2}, & \text{if } \frac{1}{\sqrt{2}} \leqslant v \leqslant 1+\frac{1}{\sqrt{2}}, \\ \infty, & \text{if } v \geqslant 1+\frac{1}{\sqrt{2}}, \end{cases} \tag{12}$$

The dual norm to the Luxemburg-Nakano norm is the Orlicz norm, which can be written in the Amemiya form as follows

$$\|y\|_{\varphi}^{*} = \|y\|_{\varphi^*}^{OA} = \inf_{k>0} \frac{1+\varphi^*(k|y_1|)+\varphi^*(k|y_2|)}{k} := N.$$

Assume that $|y_1| \geqslant |y_2|$ and consider three cases. If $k\max(|y_1|,|y_2|) \leqslant 1/\sqrt{2}$, then

$$N = \inf_{k>0} \frac{1}{k} = \sqrt{2}\max(|y_1|,|y_2|).$$

If $1/\sqrt{2} \leqslant k|y_1| \leqslant 1+1/\sqrt{2}$ and $k|y_2| \leqslant 1/\sqrt{2}$, then

$$\begin{aligned} N &= \inf_{k>0} \frac{1+k|y_1|/\sqrt{2}-1/2}{k} = \inf_{k>0}\left(\frac{1}{2k}+\frac{|y_1|}{\sqrt{2}}\right) \\ &= \max\left(\frac{|y_1|}{2(1+1/\sqrt{2})}+\frac{|y_1|}{\sqrt{2}}, \frac{|y_2|}{\sqrt{2}}+\frac{|y_1|}{\sqrt{2}}\right) \\ &= \max\left(|y_1|, \frac{|y_1|+|y_2|}{\sqrt{2}}\right). \end{aligned}$$

If $1/\sqrt{2} \leqslant k|y_1| \leqslant 1+1/\sqrt{2}$ and $1/\sqrt{2} \leqslant k|y_2| \leqslant 1+1/\sqrt{2}$, then

$$N = \inf_{k>0} \frac{1+k|y_1|/\sqrt{2}-1/2+k|y_2|/\sqrt{2}-1/2}{k} = \frac{|y_1|+|y_2|}{\sqrt{2}}.$$

Finally,

$$\begin{aligned} N &= \min\left[\sqrt{2}\max(|y_1|,|y_2|), \max\left(|y_1|, \frac{|y_1|+|y_2|}{\sqrt{2}}\right)\right] \\ &= \max\left(|y_1|, \frac{|y_1|+|y_2|}{\sqrt{2}}\right). \qquad \square \end{aligned}$$

To finish the proof of Theorem 2 note that the operator $T = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ in the dual space has the norm $\|T\| = \sqrt{2}$ that can be obtained similarly as in Example 1 (with $p=1$) or it follows from the fact that T is self-adjoint, that is, $T^* = T$. Moreover,

$$\|T_{\mathbb{C}}\|_{X^*\to X^*} = \|T_{\mathbb{C}}\|_{X\to X} > \|T\|_{X\to X} = \|T\|_{X^*\to X^*}.$$

Therefore, for the operator T between two-dimensional Orlicz space $l_2^{\varphi^*}$ with the Orlicz-Amemiya norm we also have $\|T_{\mathbb{C}}\| > \|T\|$.

REMARK 1. In Theorem 2 we showed one example of Orlicz function (Orlicz space), but we can also construct a number of examples of two-dimensional Orlicz spaces having the same property as those in Theorem 2 by considering family of Orlicz functions φ_p given by $\varphi_p(u) = \varphi(u^p)$ with φ from (10) and $1 \leqslant p < 2$. The two-dimensional real Orlicz spaces $l_2^{\varphi_p}$ with the Luxemburg-Nakano norms are p-convexifications of l_2^{φ} and for $1 \leqslant p < 2$ the operator $T = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ gives $\|T_{\mathbb{C}}\| > \|T\|$ by Example 2. The dual space $\left(l_2^{\varphi_p}, \|\cdot\|_{\varphi_p}\right)^* = \left(l_2^{\varphi_p^*}, \|\cdot\|_{\varphi_p^*}^o\right)$, where

$$\varphi_p^*(v) = \begin{cases} \frac{v^{p'}}{p'}\left(\frac{\sqrt{2}}{p}\right)^{\frac{1}{p-1}}, & \text{if } 0 \leqslant v \leqslant \frac{p}{2^{1-\frac{1}{2p}}}, \\ \frac{v}{2^{\frac{1}{2p}}} - \frac{1}{2}, & \text{if } \frac{p}{2^{1-\frac{1}{2p}}} \leqslant v \leqslant \frac{p(1+\sqrt{2})}{2^{1-\frac{1}{2p}}}, \\ \frac{v^{p'}}{p'}\left(\frac{\sqrt{2}}{p(1+\sqrt{2})}\right)^{\frac{1}{p-1}} + \frac{1}{\sqrt{2}}, & \text{if } v \geqslant \frac{p(1+\sqrt{2})}{2^{1-\frac{1}{2p}}}, \end{cases}$$

and

$$\|y\|_{\varphi_p^*}^0 = \max\left\{\frac{|y_1| + |y_2|}{2^{\frac{1}{2p}}}, \left(|y_1|^{p'} + |y_2|^{p'}(1+\sqrt{2})^{\frac{1}{p-1}}\right)^{\frac{1}{p'}}\right\}.$$

3. Vector-valued constants and the monotonicity property

More general estimates than those given by the inequality (2) are called *vector-valued estimates* or *Marcinkiewicz-Zygmund estimates* and they are defined as follows: for $0 < p, q, r \leqslant \infty$ and natural number $n \geqslant 2$ let $K_{p,q}^{(n)}(r)$ be the best constant $C \geqslant 1$ in the inequality

$$\|(\sum_{k=1}^{n} |Tf_k|^r)^{1/r}\|_q \leqslant C\|T\|_{L^p \to L^q} \|(\sum_{k=1}^{n} |f_k|^r)^{1/r}\|_p \tag{13}$$

for all $f_1, f_2, \ldots, f_n \in L^p(\mu)$ and any bounded linear operator $T : L^p(\mu) \to L^q(\nu)$ between real (quasi-) normed Lebesgue spaces with arbitrary σ-finite measures μ and ν. Of course, $K_{p,q}^{(2)}(2) = C_{p,q}$.

Properties of the constants $K_{p,q}^{(n)}(r)$ and $K_{p,q}(r) = \sup_{n\geqslant 2} K_{p,q}^{(n)}(r)$ for $1 \leqslant p, q, r \leqslant \infty$ were investigated by Marcinkiewicz-Zygmund [25], Herz [18], Krivine [21], [22], Gasch-Maligranda [16], Defant-Floret [12], Vogt [42] and Defant-Junge [13].

The assertions that $K_{p,p}(2) = 1$ and $K_{p,p}(r) = 1$, where $0 < p < r \leqslant 2$, were proved by Marcinkiewicz-Zygmund with the help of Gaussian and r-stable Gaussian variables [25, Thm 1 and Thm 2] (see also Edwards-Gaudry [14, pp. 203–204] and Andersen [1]). Some similar results on the tensor products of operators from L^p to L^q, from which one can obtain the equality $K_{p,q}(2) = 1$ for $1 \leqslant p \leqslant q \leqslant \infty$, were established by Beckner [2], Figiel-Iwaniec-Pełczyński [15] and Rosenthal-Szarek [31] (cf. also Defant-Floret [12, p. 87], Reinov [28]).

Besides obvious fact that the constant $K_{p,q}^{(n)}(r)$ is increasing in n it is also increasing in p and decreasing in q. These properties of $K_{p,q}^{(n)}(r)$ for $1 \leqslant p, q, r \leqslant \infty$ follow directly, for example, from their tensor product description (see Krivine [21], [22] for $r = 2$ and Gasch-Maligranda [16] for $1 \leqslant r \leqslant \infty$). However, for the values $0 < p, q, r \leqslant \infty$ it is a problem if such a description is possible and therefore another method of the proof of the monotonicity of $K_{p,q}^{(n)}(r)$ is needed. One such simple proof for the case when $1 \leqslant p,q \leqslant \infty$ and $r=2$ was obtained by Vogt in [42, Thm 1.9]. It turned out that this proof also provides the monotonicity property of $K_{p,q}^{(n)}(r)$ for p,q in the entire first quadrant and $0<r\leqslant\infty$. It is a weighty reason to present it here. The idea of the proof is based on the generalized Hölder-Rogers inequality and the equality case in this inequality.

THEOREM 3. *Let $0<r\leqslant\infty$. If $0<p\leqslant s\leqslant\infty$ and $0<t\leqslant q\leqslant\infty$, then $K_{p,q}^{(n)}(r)\leqslant K_{s,t}^{(n)}(r)$.*

Proof. To prove the statement of this theorem is equivalent to prove that for $T\in\mathscr{L}(L^p(\mu),L^q(\nu))$ and $f_1,\dots,f_n\in L^p(\mu)$

$$\|(\sum_{k=1}^n |Tf_k|^r)^{1/r}\|_q \leqslant K_{s,t}^{(n)}(r)\|T\|\|(\sum_{k=1}^n |f_k|^r)^{1/r}\|_p.$$

Let us denote $F:=(\sum_{k=1}^n|f_k|^r)^{1/r}\in L^p(\mu)$ and $G:=(\sum_{k=1}^n|Tf_k|^r)^{1/r}\in L^q(\nu)$.

If $q<\infty$, then from the equality in the generalized Hölder-Rogers inequality it follows that for any $G\in L^q(\nu)$ there exists $\psi\in L^\alpha(\nu)$, where $\frac{1}{t}=\frac{1}{q}+\frac{1}{\alpha}$, such that $\|\psi\|_\alpha=1$ and $\|\psi G\|_t=\|G\|_q$. We can take, for example, $\psi:=\left(\frac{|G|}{\|G\|_q}\right)^{q/\alpha}$.

If $q=\infty$ (and $\alpha=t$), then for any $G\in L^\infty(\nu)$ and any $\varepsilon>0$ there exists $\psi\in L^\alpha(\nu)$ such that $\|\psi\|_\alpha=1$ and $\|\psi G\|_t\geqslant(1-\varepsilon)\|G\|_\infty$. In fact, let $A=\{x\in\Omega_2 : |G(x)|>\|G\|_\infty(1-\varepsilon)\}$. Then $\nu(A)>0$ and therefore since a σ-finite measure ν has the *finite subset property* it follows that there exists a set $B\subset A, B\in\Sigma_2$ such that $0<\nu(B)<\infty$. If we take $\psi:=\frac{\chi_B}{\nu(B)^{1/\alpha}}$, then $\|\psi\|_\alpha=1$ and

$$\|\psi G\|_t=\|\psi G\|_\alpha=\frac{1}{\nu(B)^{1/\alpha}}\left(\int_B|G(x)|^\alpha\,d\nu\right)^{1/\alpha}\geqslant(1-\varepsilon)\|G\|_\infty.$$

Moreover, from the generalized Hölder-Rogers inequality, in both above cases, it follows that the norm of the multiplication linear operator $M_\psi : L^q(\nu)\to L^t(\nu)$ is $\|M_\psi\|_{q\to t}\leqslant\|\psi\|_\alpha=1$.

At the same time, for any $F\in L^p(\mu), 0<p<\infty$ there exists $\varphi\in L^\beta(\mu)$, where $\frac{1}{p}=\frac{1}{s}+\frac{1}{\beta}$, such that $\|\frac{F}{\varphi}\|_s=\|F\|_p$ and $\|\varphi\|_\beta=1$. We can take, for example, $\varphi:=\left(\frac{|F|}{\|F\|_p}\right)^{1-p/s}$. By the generalized Hölder-Rogers inequality we have that the norm of the multiplication operator $M_\varphi : L^s(\mu)\to L^p(\mu)$ is $\|M_\varphi\|_{s\to p}\leqslant\|\varphi\|_\beta=1$.

Now, let consider the factorization of operator $\widetilde{T} : L^s(\mu)\to L^t(\nu)$ in the form $\widetilde{T}=M_\psi\circ T\circ M_\varphi$ as it is shown on Fig. 3. Set $g_k:=\frac{1}{\varphi}f_k$. By convention $\frac{0}{0}=0$ and if $\varphi(x)=0$, then $F(x)=0$ and therefore each $f_k(x)=0$. Thus, clearly $M_\varphi g_k=f_k$ and consequently

$$\left(\sum_{k=1}^n|\widetilde{T}g_k|^r\right)^{1/r}=\left(\sum_{k=1}^n|\psi Tf_k|^r\right)^{1/r}=|\psi|G$$

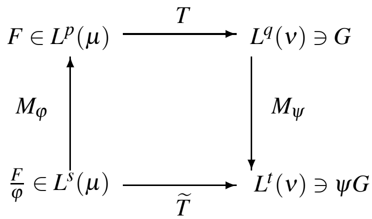


Fig. 3. Operator diagram

and $(\sum_{k=1}^{n}|g_k|^r)^{1/r} = \frac{1}{|\varphi|}F$. Putting all together, in the case when $0 < q < \infty$, we obtain

$$
\begin{aligned}
\|G\|_q = \|\psi G\|_t &= \|(\sum_{k=1}^{n} |\widetilde{T} g_k|^r)^{1/r}\|_t \\
&\leqslant K_{s,t}^{(n)}(r)\,\|\widetilde{T}\|_{s\to t}\,\|(\sum_{k=1}^{n} |g_k|^r)^{1/r}\|_s \\
&\leqslant K_{s,t}^{(n)}(r)\,\|M_\psi\|_{q\to t}\,\|T\|_{p\to q}\,\|M_\varphi\|_{s\to p}\,\|\frac{1}{\varphi}F\|_s \\
&\leqslant K_{s,t}^{(n)}(r)\,\|T\|_{p\to q}\,\|\frac{1}{\varphi}F\|_s = K_{s,t}^{(n)}(r)\,\|T\|_{p\to q}\,\|F\|_p,
\end{aligned}
$$

whence $K_{p,q}^{(n)}(r) \leqslant K_{s,t}^{(n)}(r)$. If $q = \infty$, then we similarly can obtain that

$$(1-\varepsilon)\|G\|_\infty \leqslant K_{s,t}^{(n)}(r)\,\|T\|_{p\to\infty}\,\|F\|_p,$$

and so $(1-\varepsilon)K_{p,\infty}^{(n)}(r) \leqslant K_{s,t}^{(n)}(r)$. Since $\varepsilon > 0$ was taken arbitrary we have the required estimate $K_{p,\infty}^{(n)}(r) \leqslant K_{s,t}^{(n)}(r)$ and the proof is complete. □

REMARK 2. Theorem 3 together with the fact that $C_{p,p} = K_{p,p}^{(2)}(2) = 1$ which simply follows from the Fubini theorem and Lemma 1 also shows that $C_{p,q} = K_{p,q}^{(2)}(2) = 1$ for $1 \leqslant p \leqslant q \leqslant \infty$ (cf. Theorem 1 and also Verbickiĭ-Sereda [41], Verbickiĭ [40]). Moreover, the same Theorem 3 and the equality $K_{p,p}^{(n)}(2) = 1$, which can be easily obtained by using the relation for the spherical coordinates (see Stechkin [36]) and the Fubini theorem, proves the more general equality $K_{p,q}^{(n)}(2) = 1$ for $0 < p \leqslant q \leqslant \infty$.

REMARK 3. For any natural $n \geqslant 2$ we have $K_{p,q}^{(n)}(r) = 1$ if either $r = 2$ and $0 < p \leqslant q < \infty$ or $0 < p < r < 2$ and $0 < p \leqslant q < \infty$ or $1 < p \leqslant q < \infty$ and $\min(p,2) \leqslant r \leqslant \max(p,2)$.

In fact, by Theorem 3, Marcinkiewicz-Zygmund and Herz results we obtain

$$K_{p,q}^{(n)}(r) \leqslant K_{p,p}^{(n)}(r) \leqslant K_{p,p}(r) = 1.$$

The last equality we are getting from the Marcinkiewicz-Zygmund result, which tells us that $K_{p,p}(r) = 1$ if either $r = 2$ and $0 < p < \infty$ or $0 < p < r < 2$ (cf. [25] or [16]). The case when $1 < p < \infty$ and $\min(p,2) \leqslant r \leqslant \max(p,2)$, which gives that $K_{p,p}(r) = 1$ was proved by Hertz (cf. [18] or [16]). There are some more cases where we can have the equality $K_{p,p}(r) = 1$ and this was investigated by Gasch-Maligranda [16, Th. 2], but this is not our main here therefore we don't describe all of them.

REMARK 4. We can see from the proof that the monotonicity property, as in Theorem 3, will also work for constant $K^{(n)}_{p,q}(r,\mu,\nu)$ with two fixed σ-finite measures μ and ν.

4. On one exact description in the upper triangle

We have discussed so far the exact description of the constant $K^{(n)}_{p,q}(r)$ for $0 < p,q,r \leqslant \infty$ only in the lower triangle. In regard to such a description in the upper triangle Verbickiĭ [40, Th. 3] proved that $C_{2,q} = \frac{d_2}{d_q}$ for $1 \leqslant q < 2$ and later on Defant [11] obtained the equality $C_{p,q} = \frac{d_p}{d_q}$ for $1 \leqslant q \leqslant p \leqslant 2$. Of course, from this description and the duality principle we have $C_{p,q} = C_{q',p'} = \frac{d_{q'}}{d_{p'}}$ for $2 < q \leqslant p \leqslant \infty$. Note, that in the latter case $\frac{d_{q'}}{d_{p'}} < \frac{d_p}{d_q}$ (see Verbickiĭ [40, Remark I]).

In this chapter we extend the result of Defant [11] obtained for $1 \leqslant q \leqslant p \leqslant 2$ to the larger set in the first quadrant. In fact, the proof of Defant is also working in the case $0 < q \leqslant p \leqslant 2$ but it is necessary to control carefully each step of the proof. It is interesting to note that the method of the proof establishes the connection between the constant $K^{(n)}_{p,q}(r)$ and the theories of p-summing operators, (p,q)-mixing operators and the probability theory in Banach spaces.

THEOREM 4. *If* $0 < q \leqslant p \leqslant 2$, *then* $C_{p,q} = \frac{d_p}{d_q}$.

Proof. The proof is based on the following arguments. First is the Maurey theory of (p,q)-mixing operators [26] (see also [27, Chapter 20] and [10]). More precisely, on the estimating of q-summing norm via a quotient space. Secondly, on the Lévy theorem that states the fundamental stability property of r-stable random variables. At last, on the expression of p-summing norm of the identity map on l^2_n in terms of Gaussian measures and on Theorem 1. All these statements can equally well be applied even when some of the positive indices p and q is less than one and their proofs can be found in the books of Pietsch [27] and Defant-Floret [12] (see also Carl-Defant [10] and Maurey [26]).

In what follows assume that E is a Banach space, B_E is its unit ball and E^* is its dual space. We use the standard notations from [27] and [12]. By definition, a bounded linear operator $T: E \to F$ is *absolutely* p*-summing* $(0 < p < \infty, F$ Banach space) if

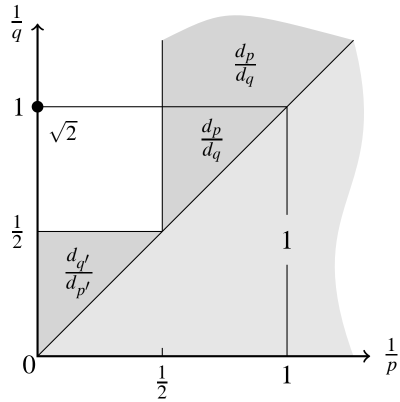


Fig. 4. The best constant in the upper triangle

there exists a constant $C > 0$ such that for all $x_1,...,x_n \in E$ and $n = 1,2,...$ we have

$$(\sum_{i=1}^{n} \|T(x_i)\|_F^p)^{1/p} \leqslant C\,\omega_p(x_i),$$

where $\omega_p(x_i) = \sup\{(\sum_{i=1}^{n} |\langle x_i, a\rangle|^p)^{\frac{1}{p}} : a \in B_{E^*}\}$. The infimum over all possible C is denoted by $\pi_p(T)$. Also, a bounded linear operator $T : E \to F$ is (p,q)-*mixing* ($0 < q \leqslant p < \infty$, F Banach space) if there exists $C > 0$ such that for all $x_1,...,x_n \in E$ and the functionals $b_1,..,b_m \in F^*$, $m,n = 1,2,...$ we have

$$\left(\sum_{i=1}^{n} \left(\sum_{k=1}^{m} |\langle Tx_i, b_k\rangle|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} \leqslant C\,\omega_q(x_i) l_p(b_k),$$

where $l_p(b_k) = (\sum_{k=1}^{m} \|b_k\|^p)^{1/p}$. The infimum over all possible C is the norm of the operator T which is denoted by $\mu_{p,q}(T)$. By [27, Sec. 20.2.1], for $0 < q \leqslant p \leqslant \infty$ the product $S \circ T$ of an absolutely p-summing operator S and the (p,q)-mixing operator T is an absolutely q-summing operator, moreover,

$$\pi_q(S \circ T) \leqslant \mu_{p,q}(T)\pi_p(S).$$

The Lévy theorem states: for $0 < q < p \leqslant 2$ there is a constant $a_{p,q}$ such that for every $n \in \mathbb{N}$ there is a probability measure μ_p^n on $\mathbb{R}^n$ satisfying

$$\frac{1}{a_{p,q}} \left(\int_{\mathbb{R}^n} |\langle x, \omega\rangle|^q \mu_p^n(d\omega)\right)^{1/q} = \|x\|_p$$

for all $x \in \mathbb{R}^n$ or in other words if $\{\omega_k\}$ is an p-stable sequence for any finite sequence of real numbers $\{x_k\}$, then $\sum_{k=1}^{n} x_k \omega_k$ has the same distribution as $(\sum_{k=1}^{n} |x_k|^p)^{1/p} \omega_1$. Using the Gauss-Khinchine equality and the Grothendieck-Pietsch domination theorem the p-summing norm of the identity map on l_n^2 can be calculated (Gordon result – see, for example, Pietsch [27, p. 300] for $0 < p < \infty$ and Defant-Floret [12, p. 137] for $1 \leqslant p < \infty$):

$$\pi_p(id_{\mathbb{R}_2^n}) = c_{2,p}^{(n)}/c_{2,p}^{(1)},$$

where $c_{2,p}^{(n)} = \left(\int_{\mathbb{K}^n} \|\omega\|_2^p \gamma_n(d\omega)\right)^{1/p}$ and $c_{2,p,\mathbb{R}}^{(n)} = \sqrt{2}\left(\frac{\Gamma(\frac{p+n}{2})}{\Gamma(\frac{n}{2})}\right)^{1/p}$. Thus, taking the expression for d_p from Lemma 1 we have

$$\pi_p(\mathrm{id}_{\mathbb{R}_2^2}) = \left(\frac{\sqrt{\pi}\,\Gamma\left(\frac{p+2}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)}\right)^{1/p} = 1/d_p.$$

Now, clearly

$$\frac{d_p}{d_q} = \frac{\pi_q(\mathrm{id}_{\mathbb{R}_2^2})}{\pi_p(\mathrm{id}_{\mathbb{R}_2^2})} \leqslant \mu_{p,q}(\mathrm{id}_{\mathbb{R}_2^2}). \tag{14}$$

To show that $\mu_{p,q}(\mathrm{id}_{\mathbb{R}_2^2}) \leqslant K_{p,q}^{(2)}(2)$, let us consider the operator $F^n : l_n^p \hookrightarrow L^q(\mu_p^n)$ given by the formula $F^n(x) = \frac{1}{a_{p,q}}\langle \cdot, x\rangle$. Then, for $x_1, \ldots, x_n \in \mathbb{R}^2$ and $b_1, .., b_m \in \mathbb{R}^2$

$$\begin{aligned}
&\left(\sum_{i=1}^n \left(\sum_{k=1}^m |\langle b_k, x_i\rangle|^p\right)^{q/p}\right)^{1/q} \\
&= \frac{1}{a_{p,q}}\left(\sum_{i=1}^n \int_{\mathbb{R}^n} |\sum_{k=1}^m \omega_k \langle b_k, x_i\rangle|^q \mu_p^n(d\omega)\right)^{1/q} \\
&= \frac{1}{a_{p,q}}\left(\int_{\mathbb{R}^n} \sum_{i=1}^n |\langle \sum_{k=1}^m \omega_k b_k, x_i\rangle|^q \mu_p^n(d\omega)\right)^{1/q} \\
&= \frac{1}{a_{p,q}}\left(\int_{\mathbb{R}^n} \|\sum_{k=1}^m \omega_k b_k\|_2^q \sum_{i=1}^n |\langle \|\sum_{k=1}^m \omega_k b_k\|_2^{-q} \sum_{k=1}^m \omega_k b_k, x_i\rangle|^q \mu_p^n(d\omega)\right)^{1/q} \\
&\leqslant \frac{1}{a_{p,q}}\left(\int_{\mathbb{R}^n} \|\sum_{k=1}^m \omega_k b_k\|_2^q \mu_p^n(d\omega)\right)^{1/q} \left(\sup_{\|y\|_2 \leqslant 1} \sum_{i=1}^n |\langle y, x_i\rangle|^q\right)^{1/q} \\
&\leqslant \|F_{\mathbb{C}}^n\| \left(\sum_{k=1}^m \|b_k\|_2^p\right)^{1/p} \left(\sup_{\|y\|_2 \leqslant 1} \sum_{i=1}^n |\langle y, x_i\rangle|^q\right)^{1/q} \\
&\leqslant K_{p,q}^{(n)}(2)\|F^n\| \left(\sum_{k=1}^m \|b_k\|_2^p\right)^{1/p} \left(\sup_{\|y\|_2 \leqslant 1} \sum_{i=1}^n |\langle y, x_i\rangle|^q\right)^{1/q}.
\end{aligned}$$

Taking into account that F^n is an isometry, that is $\|F^n\| = 1$ and by the definition of (p,q)-mixing norm we get $\mu_{p,q}(\mathrm{id}_{\mathbb{R}_2^2}) \leqslant K_{p,q}^{(2)}(2)$. Putting this inequality together with the relation (14) we conclude that $K_{p,q}^{(2)}(2) \geqslant d_p/d_q$. Since $K_{p,q}^{(2)}(2) \leqslant d_p/d_q$ for $0 < q \leqslant p \leqslant 2$ holds by Theorem 1 we obtain equality $K_{p,q}^{(2)}(2) = d_p/d_q$. By the same proof we even get that $K_{p,q}^{(n)}(2) = d_p^{(n)}/d_q^{(n)}$, where $d_p^{(n)} = \left(\frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{p+n}{2})}\right)^{1/p}$. $\square$

Observe that $K_{p,q}^{(2)}(2) = d_p/d_q > 1$ for $0 < q < p \leqslant 2$ (and $K_{p,q}^{(2)}(2) = d_{q'}/d_{p'} > 1$ for $2 \leqslant q < p \leqslant \infty$, cf. Defant [11]) and because of the strictly increasing property of the constant d_p and the monotonicity property of $K_{p,q}^{(2)}(2)$ established in Theorem 3 we obtain a strong inequality $C_{p,q} = K_{p,q}^{(2)}(2) = d_p/d_q > 1$ for the extended region $0 < q < p \leqslant \infty$.

In the following two examples we make use of the Fourier transform and first produce a simpler, in comparison with the construction used in Theorem 4, operator for which $K_{2,q}^{(2)}(2) = d_2/d_q$, where $0 < q \leqslant 2$. Secondly, we show that there exists a couple of p and q connected by $0 < q < p \leqslant \infty$ and $p \geqslant 1$ such that $K_{p,q}^{(2)}(2) \geqslant \frac{\pi}{2} > \sqrt{2}$.

EXAMPLE 3. *Let $0 < q \leqslant 2$. Consider an operator $T : l_2^2 \to L^q[0,1]$ that maps the basis vectors e_1 and e_2 into values $T(e_i) = \alpha_i$, where $\alpha_1(t) = \cos 2\pi t$ and $\alpha_2(t) = \sin 2\pi t$. Then,*

$$\|T\|_{l_2^2 \to L^q[0,1]} = d_q \quad \text{and} \quad \|T_{\mathbb{C}}\|_{l_2^2 \to L^q[0,1]} = d_2. \tag{15}$$

Thus, $K_{2,q}^{(2)}(2) \geqslant \|T_{\mathbb{C}}\|/\|T\| = d_2/d_q$. On the other hand, Theorem 1 gives for $0 < q \leqslant 2$ the upper bound $K_{2,q}^{(2)}(2) \leqslant d_2/d_q$ and hence

$$K_{2,q}^{(2)}(2) = \frac{d_2}{d_q}. \tag{16}$$

Proof. To show (15) we note, that for arbitrary $f \in l_2^2$ there exist $c_1, c_2 \in \mathbb{R}$ such that $f = c_1e_1 + c_2e_2$. Then, by Lemma 1 we have

$$\begin{aligned}\|Tf\|_q &= \|c_1Te_1 + c_2Te_2\|_q = \|c_1 \cos 2\pi t + c_2 \sin 2\pi t\|_q \\ &= d_q \left(c_1^2 + c_2^2\right)^{1/2} = d_q \|f\|_2 .\end{aligned}$$

Hence, $\|T\|_{l_2^2 \to L^q[0,1]} = d_q$. Let show that $\|T_{\mathbb{C}}\|_{l_2^2 \to L^q[0,1]} = d_2$. For this purpose let us take arbitrary $f, g \in l_2^2$, where $f = a_1e_1 + a_2e_2$ and $g = b_1e_1 + b_2e_2$. Using the Hölder-Rogers inequality with $2/q \geqslant 1$ we obtain

$$\begin{aligned}\|T(f+ig)\|_q &= (\int_0^1 [(a_1 \cos 2\pi t + a_2 \sin 2\pi t)^2 + (b_1 \cos 2\pi t + b_2 \sin 2\pi t)^2]^{q/2} dt)^{1/q} \\ &\leqslant (\int_0^1 (a_1 \cos 2\pi t + a_2 \sin 2\pi t)^2 + (b_1 \cos 2\pi t + b_2 \sin 2\pi t)^2 dt)^{1/2} \\ &= \left(d_2^2(a_1^2 + a_2^2) + d_2^2(b_1^2 + b_2^2)\right)^{1/2} \\ &= d_2 \left[(a_1^2 + b_1^2) + (a_2^2 + b_2^2)\right]^{1/2} = d_2 \|f + ig\|_2\end{aligned}$$

and therefore $\|T(f+ig)\|_q \leqslant d_2\|f+ig\|_2$. The equality holds when $f=e_1$ and $g=e_2$ since in this case $\|f+ig\|_2=\sqrt{2}$, $d_2=1/\sqrt{2}$ and $\|T(f+ig)\|_q=1$. □

If, in the above example, we took the Rademacher functions instead of sin and cos, then the relation between complex and real operator norms would be equal to one (the proof is by the Riesz-Thorin interpolation theorem). In other words, the operator which maps an element from l_2^2 into the linear combination of Rademacher functions is not enough to attain the best constant $K_{2,q}^{(2)}(2)$, for this purpose we need, for example, the orthogonal set of trigonometric functions.

EXAMPLE 4. *If* $0<q<p\leqslant\infty$ *and* $p\geqslant 1$, *then there exists an operator* $F\in\mathscr{L}(L^p(0,1),L^q(0,1))$ *such as*

$$\|F_{\mathbb{C}}\|_{p,q} \geqslant \frac{d_2^2}{d_{p'}d_q}\|F\|_{p,q}.$$

For $p=\infty$ *and* $q\to 0^+$ *by continuity we get* $\|F_{\mathbb{C}}\|_{p,q}/\|F\|_{p,q}\to\pi/2>\sqrt{2}$.

Proof. The idea of this proof is due to Verbickiĭ [40] but we extend it to the case when $0<q<1$. Consider the operator $F: L^p(0,1)\to L^q(0,1)$ defined by

$$Ff(x)=a(f)\cos 2\pi x+b(f)\sin 2\pi x, \tag{17}$$

with $a(f)=2\int_0^1 f(t)\cos 2\pi t\,dt$, $b(f)=2\int_0^1 f(t)\sin 2\pi t\,dt$ and $f\in L^p(0,1)$.

Applying Lemma 1 twice we get

$$\|Ff\|_q=d_q\left(a^2(f)+b^2(f)\right)^{1/2}=\frac{d_q}{d_2}\|Ff\|_2.$$

Consequently,

$$\|F\|_{p,q}=\frac{d_q}{d_2}\|F\|_{p,2}. \tag{18}$$

To find $\|F\|_{p,2}$ we note that by duality $\|F\|_{p,2}=\|F^*\|_{2,p'}$ and F^* is defined by

$$F^*\varphi(x)=a(\varphi)\cos 2\pi x+b(\varphi)\sin 2\pi x,$$

where as before $a(\varphi)=2\int_0^1\varphi(t)\cos 2\pi t\,dt$ and $b(\varphi)=2\int_0^1\varphi(t)\sin 2\pi t\,dt$. Applying Lemma 1 it yields

$$\|F^*\varphi\|_{p'}=d_{p'}[a^2(\varphi)+b^2(\varphi)]^{1/2}.$$

By the Parseval identity the norm $\|F\|_{p',2}$ is attained on the function having form $\varphi(x)=a(\varphi)\cos 2\pi x+b(\varphi)\sin 2\pi x$. This function is in $L^2(0,1)$ and has the norm that can be derived by using Lemma 1 again

$$\begin{aligned}\|\varphi\|_2 &= \left(\int_0^1 |a(\varphi)\cos 2\pi t+b(\varphi)\sin 2\pi t|^2\,dt\right)^{1/2}\\ &= d_2\left[a^2(\varphi)+b^2(\varphi)\right]^{1/2}.\end{aligned}$$

Then,

$$\|F\|_{p,2} = \|F^*\|_{2,p'} = \|F^*\varphi\|_{p'}/\|\varphi\|_2 = d_{p'}/d_2.$$

Substituting this expression into the equality (18) we obtain

$$\|F\|_{p,q} = \frac{d_{p'}d_q}{d_2^2}. \tag{19}$$

Let $F_{\mathbb{C}}$ be the natural complexification of the operator F, then by (17) we find

$$F_{\mathbb{C}}(e^{2\pi ix}) = F(\cos 2\pi x) + iF(\sin 2\pi x) = \cos 2\pi x + i\sin 2\pi x = e^{2\pi ix},$$

that implies

$$\|F_{\mathbb{C}}\|_{p,q} \geqslant \frac{\|F_{\mathbb{C}}(e^{2\pi ix})\|_q}{\|e^{2\pi ix}\|_p} = \frac{\|e^{2\pi ix}\|_q}{\|e^{2\pi ix}\|_p} = 1.$$

Therefore,

$$\|F_{\mathbb{C}}\|_{p,q} \geqslant \frac{d_2^2}{d_{p'}d_q}\|F\|_{p,q}.$$

If we take $p = \infty$ and $q \to 0^+$, then for the operator F given by (17) we have

$$\frac{\|F_{\mathbb{C}}\|_{p,q}}{\|F\|_{p,q}} \to \frac{\pi}{2} > \sqrt{2}. \qquad \square$$

PROBLEM 1. *Is it true that* $C_{p,q} \leqslant \frac{\pi}{2}$ *for* $0 < p,q \leqslant \infty$*?*

5. The complex and the real Riesz-Thorin interpolation theorem in the first quadrant

Let $0 < p,q \leqslant \infty$. A linear operator T defined for all simple *complex-valued* functions f on (Ω_1,μ) is of *strong type* (p,q) if Tf is ν-measurable and

$$\|Tf\|_q \leqslant M\|f\|_p$$

with constant $M > 0$ independent of f. The smallest constant M, called the norm of T, we denote by $\|T\|_{p,q}$.

The complex Riesz-Thorin interpolation theorem, proved by Marcel Riesz in 1926 and Thorin in 1939, holds with constant 1 (cf. [30], [38], [39] and also [5]): let (Ω_1,μ) and (Ω_2,ν) be two measure spaces with σ-finite measures and T be a linear operator defined for all simple complex-valued functions f on (Ω_1,μ). Let $1 \leqslant p_0,p_1,q_0,q_1 \leqslant \infty$, $0 \leqslant \theta \leqslant 1$ and

$$\left(\frac{1}{p},\frac{1}{q}\right) = (1-\theta)\left(\frac{1}{p_0},\frac{1}{q_0}\right) + \theta\left(\frac{1}{p_1},\frac{1}{q_1}\right)$$

If T is simultaneously of strong types (p_0,q_0) and (p_1,q_1), then T is also of strong type (p,q) and its norm satisfies

$$\|T\|_{p,q} \leqslant \|T\|_{p_0,q_0}^{1-\theta}\|T\|_{p_1,q_1}^{\theta}.$$

Geometrically we can say that the set of points

$$\left\{ \left(\frac{1}{p}, \frac{1}{q} \right) \in [0,1] \times [0,1] : \ T \text{ is of strong type } (p,q) \right\}$$

is a convex subset of $[0,1] \times [0,1]$.

In 1950 Calderón-Zygmund [8, Th. E] (cf. also [9, Th. A_1]) extended the complex Riesz-Thorin convexity theorem to the strip $[0,1] \times [0,\infty)$ and their proof works equally well for the entire first quadrant $[0,\infty) \times [0,\infty)$. This result seems to be a complete version of the complex Riesz-Thorin interpolation theorem.

THEOREM 5. *(Complex Riesz-Thorin interpolation theorem in the first quadrant). Let (Ω_1, μ) and (Ω_2, ν) be two measure spaces with σ-finite measures and T be a linear operator defined for all simple complex-valued functions f on (Ω_1, μ). Let $0 < p_0, p_1, q_0, q_1 \leqslant \infty$, $0 < \theta < 1$ and*

$$\left(\frac{1}{p}, \frac{1}{q} \right) = (1-\theta) \left(\frac{1}{p_0}, \frac{1}{q_0} \right) + \theta \left(\frac{1}{p_1}, \frac{1}{q_1} \right) \tag{20}$$

If T is simultaneously of strong types (p_0, q_0) and (p_1, q_1), then T is also of strong type (p,q) and its strong type norm satisfies

$$\|T\|_{p,q} \leqslant \|T\|_{p_0,q_0}^{1-\theta} \|T\|_{p_1,q_1}^{\theta}. \tag{21}$$

Note that in the case of quasi-normed Lebesgue spaces, that is, when $0 < p, q \leqslant \infty$ it can happen that the operators $T \in \mathscr{L}(L^p(\mu), L^q(\nu))$ can be only trivial ones. For example, if $T : L^p(\mu) \to L^q(\nu)$ is a bounded linear operator, where μ is a non-atomic measure and $0 < p < \min\{1, q\}$, then $T = 0$ (see e.g. Brudnyĭ-Krugljak [6, p. 89] and Maligranda [23, p. 150]). Of course, if the measure μ is atomic Theorem 5 is no longer trivial (cf. Bennett [3], [4]).

In the real case, as it was already noticed by Riesz [30], the estimate (21) can be not true in the upper triangle (cf. Riesz [30, pp. 493–494] and [6, pp. 16–17]). For the operator $T = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ we have $\|T\|_{\infty,1} = 2, \|T\|_{2,2} = \sqrt{2}$ and $\|T\|_{4,4/3} = \sqrt{3}$. Taking $\theta = 1/2$ in (20) we obtain $(\frac{1}{4}, \frac{3}{4}) = (1 - \frac{1}{2})\ (0,1) + \frac{1}{2}\ (\frac{1}{2}, \frac{1}{2})$ and

$$\|T\|_{4,\frac{4}{3}} = \sqrt{3} > \|T\|_{\infty,1}^{1/2} \|T\|_{2,2}^{1/2} = 2^{1/2} (\sqrt{2})^{1/2} = 2^{3/4}.$$

Using the relation between the norms of an operator and its complexification, that were considered in the previous chapters of this paper, we obtain the real case of the Riesz-Thorin theorem from its complex version with, in general, some additional constant on the right hand side of the estimate (21), that is,

$$\|T\|_{p,q} \leqslant C \|T\|_{p_0,q_0}^{1-\theta} \|T\|_{p_1,q_1}^{\theta}. \tag{22}$$

The real Riesz-Thorin theorem for the normed Lebesgue spaces has, obviously, constant 2 in the estimate (22) but, as it follows from the Krivine result this estimate is even true with constant $\sqrt{2}$. More precisely, in the real case we have two different cases:

(a) If we are in the lower triangle of the square $[0,1]\times[0,1]$, *that is* $1 \leqslant p_i \leqslant q_i \leqslant \infty, i = 0,1$, *then estimate (22) holds with constant* $C = 1$.

(b) If we are in the upper triangle of the square $[0,1]\times[0,1]$, *that is* $1 \leqslant q_i < p_i \leqslant \infty, i = 0,1$, *then estimate (22) holds with constant* $C \leqslant C_{p_0,q_0}^{1-\theta} C_{p_1,q_1}^{\theta} \leqslant C_{\infty,1} = \sqrt{2}$.

Brudnyĭ-Krugljak [6, p. 87] asked the question if in the real case we still have constant 1 in the estimate (22), if the conditions $1 \leqslant p_i \leqslant q_i \leqslant \infty$, $i = 0,1$ will be replaced by the weaker one $1 \leqslant p \leqslant q \leqslant \infty$. The answer is “NO” and the counterexample was given by Vogt [43]: for the operator $T = \begin{pmatrix} 1 & 1.1 \\ 1 & -1 \end{pmatrix}$ we have $\|T\|_{\infty,1} = 2.1$, $\|T\|_{3/2,3} = \sqrt[3]{7/3}$ and $\|T\|_{2,2} = 1.5$. If we take $\theta = 3/4$, then $(1-\frac{3}{4})\,(0,1) + \frac{3}{4}\,(\frac{2}{3},\frac{1}{3}) = (\frac{1}{2},\frac{1}{2})$ and

$$\|T\|_{2,2} = 1.5 > 2.1^{1/4}\sqrt[3]{7/3}^{3/4} = \|T\|_{\infty,1}^{1-3/4}\|T\|_{3/2,3}^{3/4}.$$

In the real case for the quasi-normed Lebesgue spaces, that is, when $0 < p_i, q_i \leqslant \infty, i = 0,1$ Krasnosel’skiĭ-Zabreĭko-Pustyl’nik-Sobolevskiĭ [19, Th. 2.4] just mentioned about some constant $C = C(q_0, q_1, \theta)$ in (22), but gave no information about it. Burenkov [7, pp. 76–77] was more precise (using the relation (1) between real norm and its complexification) and obtained that

$$C \leqslant (\max\{2, 2^{1/q_0}\})^{1-\theta}(\max\{2, 2^{1/q_1}\})^{\theta} = 2^{(1-\theta)\max\{1,\frac{1}{q_0}\}+\theta\max\{1,\frac{1}{q_1}\}}$$

in the estimate (22).

The last estimate suggests that the constant can be large, but this is not the case. Using our Theorem 1 for any bounded linear operator between quasi-normed Lebesgue spaces and taking a “good” estimate of the complex norm of this operator by the norm of its real part we obtain the following result:

THEOREM 6. *The real Riesz-Thorin interpolation theorem in the entire first quadrant holds with the estimate (22), where the constant* $C \leqslant C_{p_0,q_0}^{1-\theta} C_{p_1,q_1}^{\theta}$ *and more precisely:*

A: $C = 1$ *if* $0 < p_i \leqslant q_i \leqslant \infty, i = 0,1$.

B: $C \leqslant C_{\infty,1} = \sqrt{2}$ *if* $1 \leqslant q_i < p_i \leqslant \infty, i = 0,1$.

C: $C \leqslant C_{1,0^+} \leqslant 4/\pi$ *if* $0 < q_i < p_i < 1, i = 0,1$.

D: $C \leqslant C_{2,0^+} = \sqrt{2}$ *if* $0 < q_i \leqslant 1 < p_i \leqslant 2, i = 0,1$.

E: $C \leqslant C_{\infty,0^+} \leqslant 2$ *if* $0 < q_i < 1,\ 2 \leqslant p_i \leqslant \infty, i = 0,1$.

F: *If one point* $\left(\frac{1}{p_0}, \frac{1}{q_0}\right)$ *is in one of the above regions and the other one* $\left(\frac{1}{p_1}, \frac{1}{q_1}\right)$ *is in the second one, then the constant C is estimated by the larger of the corresponding estimates.*

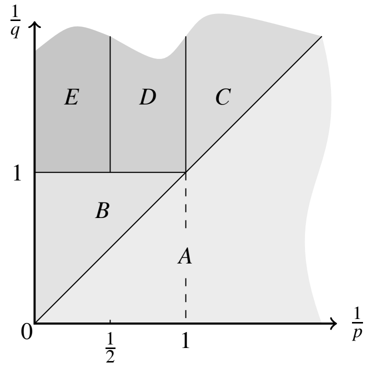


Fig. 5. The areas in Theorem 6

From the above discussion we have that in the complex Riesz-Thorin interpolation theorem the estimate (22) holds with constant $C = 1$ in the entire first quadrant $Q_{++} = [0,\infty) \times [0,\infty)$ while in the real case the estimate (22) holds with constant $C = 1$ only in the lower triangle of the square Q_{++}, i.e., for $0 < p_i \leqslant q_i \leqslant \infty, i = 0,1$. In the upper triangle of the square Q_{++} the estimate (22) holds with constant 2.

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Lech Maligranda
Department of Mathematics
Luleå University of Technology
SE-971 87 Luleå
Sweden
e-mail: lech@sm.luth.se

Natalia Sabourova
Department of Civil, Mining and Environmental Engineering
Luleå University of Technology
SE-971 87 Luleå
Sweden
e-mail: natalia.sabourova@ltu.se