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INEQUALITIES AND MAJORISATIONS FOR THE RIEMANN–STIELTJES INTEGRAL ON TIME SCALES

DOROTA MOZYRSKA, EWA PAWŁUSZEWICZ AND DELFIM F. M. TORRES

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Abstract. We prove dynamic inequalities of majorisation type for functions on time scales. The results are obtained using the notion of Riemann–Stieltjes delta integral and give a generalization of [App. Math. Let., **22**, 3 (2009), 416–421] to time scales.

1. Introduction

In the literature one can find many results known as *Majorisation Theorems*. In the recent papers [3, 9] inequalities of majorisation type for convex functions and Stieltjes integrals are given. The main goal of the present note is to unify and generalize such discrete-time and continuous-time inequalities by means of the notion of Riemann–Stieltjes integral on time scales [13, 14].

The theory and applications of delta derivatives and integrals on time scales is a relatively new area that is receiving an increase of interest and attention [7]. The concept of Riemann–Stieltjes integration on time scales was introduced in 1992 by S. Sailer [14] in a thesis under the direction of one of the founders of time scales calculus, B. Aulbach. Since 1992, several other works on the subject appeared — see, e.g., [2, 12, 13].

One important and very active subject being developed within the theory of time scales consists in the study of inequalities — see [1, 10, 11, 15, 16, 17] and references therein. To the best of our knowledge all the integral inequalities available in the literature of time scales are, however, formulated using the Riemann integral on time scales. Here we use the more general Riemann–Stieltjes integral on time scales [13, 14].

After some preliminaries on the Riemann–Stieltjes integral on time scales [13, 14] (Section 2), where we recall the main definitions and results necessary in the sequel, we begin by generalizing the notion of Riemann–Stieltjes delta integral for double integrals, proving its main properties (Section 3.1). The main contributions of the paper are the new dynamic inequalities for Riemann–Stieltjes delta integrals obtained in Section 3.2 that generalize the results of [3], and the two majorisation theorems of Section 3.3 that extend the results of [9] to the context of time scales.

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We are not aware of any paper in the literature about majorisation inequalities for Stieltjes integrals on time scales. Our results seem to be the first in this direction.

2. Preliminaries and Notation

Through the text $\mathbb{T}$, $\mathbb{T}_1$, and $\mathbb{T}_2$ denote time scales. Let $a,b \in \mathbb{T}$ and $a < b$. We distinguish $[a,b]$ as a real interval and we define $[a,b]_{\mathbb{T}} := [a,b] \cap \mathbb{T}$. In that sense $[a,b] = [a,b]_{\mathbb{R}}$. Thus, $[a,b]_{\mathbb{T}}$ is a nonempty and closed (bounded) set consisting of points from $\mathbb{T}$.

We recall the notion of Riemann–Stieltjes integral on a time scale. For more we refer the reader to [13]. A partition of $[a,b]_{\mathbb{T}}$ is any finite ordered subset

$$P = \{t_0, t_1, \ldots, t_n\} \subset [a,b]_{\mathbb{T}}, \text{ where } a = t_0 < t_1 < \ldots < t_n = b.$$

Each partition $P = \{t_0, t_1, \ldots, t_n\}$ of $[a,b]_{\mathbb{T}}$ decomposes it into subintervals $[t_{i-1}, t_i)_{\mathbb{T}}$, $i = 1,2,\ldots,n$, such that for $i \neq k$ one has $[t_{i-1}, t_i)_{\mathbb{T}} \cap [t_{k-1}, t_k)_{\mathbb{T}} = \emptyset$. Each such decomposition of $[a,b]_{\mathbb{T}}$ into subintervals is called a subdivision of $[a,b]_{\mathbb{T}}$. By $\Delta t_i = t_i - t_{i-1}$ we denote the length of the ith subinterval in the partition P. By $\mathscr{P}([a,b]_{\mathbb{T}})$ we denote the set of all partitions of $[a,b]_{\mathbb{T}}$. Let P_n, $P_m \in \mathscr{P}([a,b]_{\mathbb{T}})$. If $P_n \subset P_m$ we call P_m a *refinement of* P_n. If P_n, P_m are independently chosen, then the partition $P_n \cup P_m$ is a common refinement of P_n and P_m. This procedure is introduced in [7].

Let g be a real-valued non-decreasing function on $[a,b]_{\mathbb{T}}$. For the partition P we define the set

$$g(P) = \{g(a) = g(t_0), g(t_1), \ldots, g(t_{n-1}), g(t_n) = b\} \subset g([a,b]_{\mathbb{T}}).$$

Then, $\Delta g_i = g(t_i) - g(t_{i-1})$ is non negative and $\sum\limits_{i=1}^{n} \Delta g_i = g(b) - g(a)$. Note that $g(P)$ is a partition of $[g(a), g(b)]_{\mathbb{R}} = \bigcap\{J : g(P) \subset J\}$. It is clear that even for the class of rd-continuous functions defined on an arbitrary time scale, the image $g([a,b]_{\mathbb{T}})$ does not need to be a real interval (indeed, our interval $[a,b]_{\mathbb{T}}$ may contain scattered points).

We now recall the definitions of lower and upper sums and the notion of Darboux–Stieltjes sum (for more details see [13]). Let f be a real-valued and a bounded function on the interval $[a,b]_{\mathbb{T}}$. Let us take the partition $P = \{t_0, t_1, \ldots, t_n\}$ of $[a,b]_{\mathbb{T}}$. Let $m_i = \inf_{t \in [t_{i-1}, t_i)_{\mathbb{T}}} f(t)$ and $M_i = \sup_{t \in [t_{i-1}, t_i)_{\mathbb{T}}} f(t)$, $i = 1,2,\ldots,n$. The upper Darboux–Stieltjes sum of f with respect to the partition P, denoted by $U(P,f,g)$, is defined by $U(P,f,g) = \sum_{i=1}^{n} M_i \Delta g_i$ and the lower Darboux–Stieltjes sum of f with respect to the partition P, denoted by $L(P,f,g)$, is defined by $L(P,f,g) = \sum_{i=1}^{n} m_i \Delta g_i$.

DEFINITION 1. ([13]) The upper Darboux–Stieltjes Δ-integral from a to b with respect to function g is defined by $\overline{\int_a^b} f(t)\Delta g(t) = \inf_{P \in \mathscr{P}([a,b]_{\mathbb{T}})} U(P,f,g)$. The lower Darboux–Stieltjes Δ-integral from a to b with respect to function g is defined by $\underline{\int_a^b} f(t)\Delta g(t) = \sup_{P \in \mathscr{P}([a,b]_{\mathbb{T}})} L(P,f,g)$. If $\overline{\int_a^b} f(t)\Delta g(t) = \underline{\int_a^b} f(t)\Delta g(t)$, then we say that f is Δ-integrable with respect to g on $[a,b]_{\mathbb{T}}$, and the common value of the integrals is denoted by $\int_a^b f(t)\Delta g(t) = \int_a^b f \Delta g$ and it is called the Riemann–Stieltjes (or just Stieltjes) Δ-integral of f with respect to g on $[a,b]_{\mathbb{T}}$.

From now on we assume that f and g are arbitrary real-valued bounded functions on $[a,b]_{\mathbb{T}}$, where $a,b \in \mathbb{T}$ and g is non-decreasing on $[a,b]_{\mathbb{T}}$. Let us consider the partition $P = \{t_0, t_1, \ldots, t_n\}$ of $[a,b]_{\mathbb{T}}$ and let $X = \{x_1, \ldots, x_n\}$ denote an arbitrary selection of points from $[a,b]_{\mathbb{T}}$ with $x_i \in [t_{i-1}, t_i)_{\mathbb{T}}$, $i = 1, 2, \ldots, n$. We define

$$S_g(f,P,X) = \sum_{i=1}^{n} f(x_i)\,(g(t_i) - g(t_{i-1})) \tag{1}$$

as a *Riemann–Stieltjes* Δ*-sum for* f *with respect to* g.

DEFINITION 2. We say that f is Riemann–Stieltjes Δ-integrable with respect to g and write $f \in \mathscr{S}([a,b]_{\mathbb{T}}, g)$ if and only if there exists a number $\mathscr{I} \in \mathbb{R}$ such that for every $\varepsilon > 0$ there is a partition P^* for which $|S_g(f,P,X) - \mathscr{I}| < \varepsilon$ for all refinements $P \supset P^*$ and all possible selections of points X. If such a number exists, it is unique, and we define $\int_a^b f \Delta g = \mathscr{I}$.

Note that if g is non-decreasing, then $L(P,f,g) \leqslant S_g(f,P,X) \leqslant U(P,f,g)$ for any P and X. Let $\mathbb{T}_1$, $\mathbb{T}$ be time scales and $\psi : \mathbb{T}_1 \to \mathbb{T}$ be a rd-continuous non-decreasing map such that for $t_1 \in [\alpha, \beta]_{\mathbb{T}_1}$, $a = \psi(\alpha)$, $b = \psi(\beta)$. Then, because of the existing bijection between partitions of intervals $[a,b]_{\mathbb{T}}$ and $[\alpha,\beta]_{\mathbb{T}_1}$ and between selections of points from the respective intervals, the following holds:

$$\int_a^b f(t)\Delta t = \int_\alpha^\beta f(\psi(t_1))\Delta_1 g(\psi(t_1)).$$

The proof of Proposition 3 follows directly from (1) and Definition 2.

PROPOSITION 3. *Let* g *be non-decreasing on* $[a,b]_{\mathbb{T}}$ *and* f *be Riemann–Stieltjes* Δ*-integrable with respect to* g *on* $[a,b)_{\mathbb{T}}$*. Then,*

a) $\int_a^b \Delta g(t) = g(b) - g(a)$;

b) $\int_a^b f(t)\Delta g(t) = 0$ *for* g *constant;*

c) $\int_a^{\sigma(a)} f(t)\Delta g(t) = f(a)(g^\sigma(a) - g(a))$;

d) $\int_a^b \alpha f(t)\Delta(\beta g(t)) = \alpha\beta \int_a^b f(t)\Delta g(t)$, $\alpha,\ \beta \in \mathbb{R}$.

Note that if f is rd-continuous and g has its Δ-derivative also as a rd-continuous function, then we can write the approximating sum (1) for fg^Δ with respect to the constant function of value 1 in the form $S_1(fg^\Delta, P, X) = \sum_{i=1}^n f(x_i) g^\Delta(x_i)\Delta t_i$. Using the mean value theorem [7], we conclude with the following result:

THEOREM 4. ([13]) *Let* $a,\ b \in \mathbb{T}$*. Suppose that* g *is a non-decreasing function such that* g^Δ *is continuous on* $[a,b)_{\mathbb{T}}$ *and* f *is a real bounded function on* $[a,b]_{\mathbb{T}}$*. Then,* $f \in \mathscr{S}(g,[a,b]_{\mathbb{T}})$ *if and only if* $fg^\Delta \in \mathscr{S}(g,[a,b]_{\mathbb{T}})$*. Moreover,*

$$\int_a^b f(t)\Delta g(t) = \int_a^b f(t) g^\Delta(t)\Delta t.$$

3. Main Results

In order to generalize the results of [3] to an arbitrary time scale, one needs first to extend the Riemann–Stieltjes Δ-integral to functions of two-variables. Properties of the double Riemann Δ-integral and for multiple Lebesgue integrals on time scales were developed in [4, 5, 6].

3.1. The double Riemann–Stieltjes delta integral

Let $a,\ b \in \mathbb{T}_1,\ c,\ d \in \mathbb{T}_2$, where $a < b,\ c < d$, and $R = [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2} = \{(t,s) : t \in [a,b), s \in [c,d), t \in \mathbb{T}_1, s \in \mathbb{T}_2\}$. Let $g_i : \mathbb{T}_i \to \mathbb{R},\ i = 1,2$, be two non-decreasing functions on $[a,b]_{\mathbb{T}_1}$ and $[c,d]_{\mathbb{T}_2}$, respectively. Let $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ be bounded on R. Let us consider two partitions $P_1 = \{t_0, t_1, \ldots, t_n\}$ of $[a,b]_{\mathbb{T}_1}$ and $P_2 = \{s_0, s_1, \ldots, s_k\}$ of $[c,d]_{\mathbb{T}_2}$ and let $X_1 = \{x_1, \ldots, x_n\}$ denote an arbitrary selection of points from $[a,b]_{\mathbb{T}_1}$ with $x_i \in [t_{i-1}, t_i)_{\mathbb{T}_1},\ i = 1,2,\ldots,n$. Similarly, let $X_2 = \{y_1, \ldots, y_k\}$ denote an arbitrary selection of points from $[c,d]_{\mathbb{T}_2}$ with $y_j \in [s_{j-1}, s_j)_{\mathbb{T}_2}$, $j = 1,2,\ldots,k$. We define

$$\overline{S}_{g_1,g_2}(f,P_1,P_2,X_1,X_2) = \sum_{i=1}^{n}\sum_{j=1}^{k} f(x_i,y_j)\left(g_1(t_i) - g_1(t_{i-1})\right)\left(g_2(s_j) - g_2(s_{j-1})\right) \qquad (2)$$

as the Riemann–Stieltjes Δ-sum of f with respect to functions g_1 and g_2 and partitions $P_1 \in \mathscr{P}([a,b]_{\mathbb{T}_1})$ and $P_2 \in \mathscr{P}([c,d]_{\mathbb{T}_2})$.

DEFINITION 5. We say that f is Riemann–Stieltjes Δ-integrable with respect to g_1 and g_2 over R if there exists a number $\mathscr{I} \in \mathbb{R}$ such that for every $\varepsilon > 0$ there are partitions P_1^* and P_2^* for which $|\overline{S}_{g_1,g_2}(f,P_1,P_2,X_1,X_2) - \mathscr{I}| < \varepsilon$ for all refinements $P_1 \supset P_1^*$ and $P_2 \supset P_2^*$ and all possible selections of points X_1 and X_2 corresponding to P_1 and P_2, respectively. If such a number $\mathscr{I}$ exists, it is unique, and we define

$$\iint_R f(t,s)\Delta_{1,2}(g_1 \times g_2) = \mathscr{I}.$$

We can extend the properties of Proposition 3 using non-decreasing functions g_1 and g_2. The following proposition is obtained, *mutatis mutandis*, from the proofs of similar properties of the Riemann–Stieltjes Δ-integral [13].

PROPOSITION 6. *Let g_1 and g_2 be non-decreasing functions respectively on $[a,b]_{\mathbb{T}_1}$ and $[c,d]_{\mathbb{T}_2}$, and let f be Riemann–Stieltjes Δ-integrable with respect to g_1 and g_2 on $R = [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$. Then,*

a) $\iint_R A\Delta_{1,2}(g_1 \times g_2) = A\left(g_1(b) - g_1(a)\right)\left(g_2(d) - g_2(c)\right)$, A a constant;

b) $\iint_R f(t,s)\Delta_{1,2}(g_1 \times g_2) = 0$ when g_1 or g_2 are constant;

c) with $b = \sigma_1(a)$ and $d = \sigma_2(c)$ one has

$$\iint_R f(t,s)\Delta_{1,2}(g_1 \times g_2) = f(a,c)(g_1^{\sigma_1}(a) - g_1(a))(g_2^{\sigma_2}(c) - g_2(c));$$

d) $\iint_R \alpha f(t,s)\Delta_{1,2}[\beta(g_1\times g_2)] = \alpha\beta\iint_R f(t,s)\Delta_{1,2}(g_1\times g_2)$, α, β *constants.*

In the classical case, i.e., when $\mathbb{T}_1=\mathbb{T}_2=\mathbb{R}$, the Fubini theorem is the fundamental theorem that relates double and iterated integrals (see, e.g., [8]). The rule of iterated integration for double Riemann Δ-integrals on a rectangle was proved in [4, Theorem 3.10]. We extend here [4, Theorem 3.10] to the double Riemann–Stieltjes Δ-integral.

PROPOSITION 7. *Let $g_i:\mathbb{T}_i\to\mathbb{R}$, $i=1,2$, be two non-decreasing functions on $[a,b]_{\mathbb{T}_1}$ and $[c,d]_{\mathbb{T}_2}$, respectively. Let us assume that function $f:\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{R}$ is bounded on the set $R=[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}$. Then, the existence of the integral*

$$\iint_R |f|\Delta_{1,2}(g_1\times g_2)$$

implies the existence and the equality of the iterated integrals:

$$\begin{aligned}\iint_R f\Delta_{1,2}(g_1\times g_2) &= \int_a^b\left(\int_c^d f(t,s)\Delta_2 g_2(s)\right)\Delta_1 g_1(t)\\ &= \int_c^d\left(\int_a^b f(t,s)\Delta_1 g_1(t)\right)\Delta_2 g_2(s).\end{aligned}\tag{3}$$

Proof. Let us begin noticing that if one of the functions g_1 or g_2 is constant, then relation (3) gives the truism zero equals zero. Assume now that none of the functions g_1 and g_2 is constant. As it is usually done in the classical double integral calculus, the evaluation of a double Stieltjes integral can be reduced to the successive evaluation of two simple Stieltjes integrals. Let $P_1\in\mathscr{P}([a,b]_{\mathbb{T}_1})$ and $P_2\in\mathscr{P}([c,d]_{\mathbb{T}_2})$ where, as in the introduction to this section, we use $P_1=\{t_0,t_1,\ldots,t_n\}$, $P_2=\{s_0,s_1,\ldots,s_k\}$, $X_1=\{x_1,\ldots,x_n\}$, $X_2=\{y_1,\ldots,y_k\}$, with $x_i\in[t_{i-1},t_i)_{\mathbb{T}_1}$, $i=1,2,\ldots,n$, and $y_j\in[s_{j-1},s_j)_{\mathbb{T}_2}$, $j=1,2,\ldots,k$. We can assume that P_1 is such that $\sum_{i=1}^n(g_1(t_i)-g_1(t_{i-1}))>0$, as g_1 is not constant. According to definition (2) of Riemann–Stieltjes Δ-sum we can write

$$\overline{S}_{g_1,g_2}(f,P_1,P_2,X_1,X_2)=\sum_{i=1}^n(g_1(t_i)-g_1(t_{i-1}))\sum_{j=1}^k f(x_i,y_j)\left(g_2(s_j)-g_2(s_{j-1})\right).$$

Let us now denote by $\Phi(x_{i-1})=\int_c^d f(x_{i-1},s)\Delta_2 g_2(s)$ the simple Stieltjes integral of the function $f(x_{i-1},\cdot)$ with respect to g_2 on the interval $[c,d]_{\mathbb{T}_2}$. Using Definition 2 we can write that for every

$$\overline{\varepsilon}=\frac{\varepsilon}{2\sum_{i=1}^n(g_1(t_i)-g_1(t_{i-1}))}>0,$$

$\varepsilon>0$, there is a partition P_2^* such that for all refinement $P_2\supset P_2^*$ with a selection X_2 we have that

$$\left|S_{g_2}(f(x_{i-1},\cdot),P_2,X_2)-\Phi(x_{i-1})\right|<\overline{\varepsilon}.$$

For any partition P_1 of $[a,b]_{\mathbb{T}_1}$ with some selection X_1 the following holds:

$$\left|\overline{S}_{g_1,g_2}(f,P_1,P_2,X_1,X_2) - \sum_{i=1}^{n}\left(g_1(t_i)-g_1(t_{i-1})\right)\Phi(x_{i-1})\right| < \frac{\varepsilon}{2}.$$

It is easy to notice that the sum $\sum_{i=1}^{n}\left(g_1(t_i)-g_1(t_{i-1})\right)\Phi(x_{i-1})$ represents a Riemann–Stieltjes Δ-sum for the integral $\int_a^b \Phi(t)\Delta_1 g_1(t)$. Let $\mathscr{I} = \iint_R f\Delta_{1,2}(g_1\times g_2)$. Using again the definition in [13] of the simple Stieltjes delta integral on $[a,b]_{\mathbb{T}_1}$, we see that for all $\varepsilon/2>0$ there is a partition P_1^* such that for all refinements $P_1\supset P_1^*$ together with all possible selections X_1 the following holds:

$$\left|\overline{S}_{g_1,g_2}(f,P_1,P_2,X_1,X_2) - \mathscr{I}\right| < \frac{\varepsilon}{2}.$$

Hence,

$$\left|\mathscr{I} - \sum_{i=1}^{n}\left(g_1(t_i)-g_1(t_{i-1})\right)\Phi(x_{i-1})\right| < \varepsilon$$

and $\iint_R f\Delta_{1,2}(g_1\times g_2) = \int_a^b\left(\int_c^d f(t,s)\Delta_2 g_2(s)\right)\Delta_1 g_1(t)$. Similarly, if we proceed in the reverse order we get the analogous formula

$$\iint_R f\Delta_{1,2}(g_1\times g_2) = \int_c^d\left(\int_a^b f(t,s)\Delta_1 g_1(t)\right)\Delta_2 g_2(s). \quad \square$$

3.2. Inequalities for Riemann–Stieltjes delta integrals

In what follows $g:\mathbb{T}\to\mathbb{R}$ is a non-decreasing function on the interval $[a,b]_{\mathbb{T}}$.

PROPOSITION 8. *Let $f:\mathbb{T}\to\mathbb{R}$ be Riemann–Stieltjes Δ-integrable on $[a,b]_{\mathbb{T}}$ with respect to a non-decreasing function g. If f is nonnegative on $[a,b]_{\mathbb{T}}$, then*

$$\int_a^b f(t)\Delta g(t) \geqslant 0.$$

Proof. If f is a nonnegative function, then for any partition $P\in\mathscr{P}([a,b]_{\mathbb{T}})$ we have $\int_a^b f(t)\Delta g(t)\geqslant L(P,f,g)\geqslant 0$. $\square$

COROLLARY 9. *Let $f_1,f_2:\mathbb{T}\to\mathbb{R}$ be Riemann–Stieltjes delta integrable on $[a,b]_{\mathbb{T}}$ with respect to a non-decreasing function g. Suppose that $f_1(t)\geqslant f_2(t)$ for all $t\in[a,b]_{\mathbb{T}}$. Then,*

$$\int_a^b f_1(t)\Delta g(t) \geqslant \int_a^b f_2(t)\Delta g(t).$$

Proof. The result follows immediately from Proposition 8 and the nonnegativity of function $f(t)=f_1(t)-f_2(t)$. $\square$

Similarly, we can also show the following:

PROPOSITION 10. *Let $R=[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}$ and f, f_1, and f_2 be bounded functions on R satisfying the inequality $f_1(t,s)\geqslant f_2(t,s)$ for all $(t,s)\in R$. Then,*

$$\iint_R f_1\Delta_{1,2}(g_1\times g_2)\geqslant\iint_R f_2\Delta_{1,2}(g_1\times g_2)$$

and

$$\left|\iint_R f(t,s)\Delta_{1,2}(g_1\times g_2)\right|\leqslant\iint_R |f(t,s)|\Delta_{1,2}(g_1\times g_2).$$

PROPOSITION 11. *Let $f:\mathbb{T}\to\mathbb{R}$ be Riemann–Stieltjes Δ-integrable on $[a,b]_{\mathbb{T}}$ with respect to a non-decreasing function g. If f is nonnegative on $[a,b]_{\mathbb{T}}$, then*

$$F(t)=\int_a^t f(\tau)\Delta g(\tau)$$

is a non-decreasing function on $[a,b]_{\mathbb{T}}$.

Proof. If g is Δ-differentiable on $[a,b)_{\mathbb{T}}$, then Theorem 4 states that

$$\int_a^t f(\tau)\Delta g(\tau)=\int_a^t f(\tau)g^{\Delta}(\tau)\Delta\tau.$$

Thus, $F^{\Delta}(t)=f(t)g^{\Delta}(t)\geqslant 0$ and F is a non-decreasing function on $[a,b)_{\mathbb{T}}$. On the other hand, we can use the property that

$$\int_a^{\sigma(t)} f\Delta g=\int_a^t f\Delta g+f(t)(g^{\sigma}(t)-g(t)).$$

This means that in the case when t is right-scattered then

$$F^{\Delta}(t)=\frac{f(t)(g^{\sigma}(t)-g(t))}{\mu(t)}\geqslant 0;$$

in the case when t is right-dense then $F^{\Delta}(t)=\lim_{s\to t}\left|\frac{\int_s^t f\Delta g}{t-s}\right|\geqslant 0$. Hence, F is non-decreasing. □

Let I be an interval of real numbers and $F:I\to\mathbb{R}$ be a convex function on I. Then F is continuous on $int(I)$ (the interior of I) and has finite left and right derivatives (F'_+ and F'_-) at each point of $int(I)$. For a convex function $F:I\to\mathbb{R}$ the subdifferential of F is defined as the set ∂F of all extended functions $\varphi:I\to\mathbb{R}\cup\{\pm\infty\}$ such that $\varphi(int(I))\subset\mathbb{R}$ and

$$F(x)\geqslant F(y)+(x-y)\varphi(y),\ \text{for } x,y\in I. \tag{4}$$

When F is convex, then the set ∂F is nonempty because at least F'_+, $F'_-\in\partial F$. Moreover, if $\varphi\in\partial F$ then $F'_-(x)\leqslant\varphi(x)\leqslant F'_+(x)$ for $x\in int(I)$, and φ is a non-decreasing function. If $x:\mathbb{T}\to I\subset\mathbb{R}$, then the composition $F\circ x:\mathbb{T}\to\mathbb{R}$ is a function on $\mathbb{T}$.

The following result is a generalization of [3, Theorem 5].

THEOREM 12. *Let $\mathbb{T}$ be a time scale with $a,b \in \mathbb{T}$, $F : I \to \mathbb{R}$ be a convex function on the real interval I, and $x,y,p : [a,b]_{\mathbb{T}} \to I$ with $p(\cdot)$ nonnegative on $[a,b]_{\mathbb{T}}$. If $\varphi \in \partial F$ and $g : [a,b]_{\mathbb{T}} \to I$ is a non-decreasing function on $[a,\rho(b)]_{\mathbb{T}}$, then the inequality*

$$\int_a^b p(t)F(x(t))\Delta g(t) - \int_a^b p(t)F(y(t))\Delta g(t) \geqslant \int_a^b p(t)x(t)\varphi(y(t))\Delta g(t) - \int_a^b p(t)y(t)\varphi(y(t))\Delta g(t) \quad (5)$$

holds assuming that the Riemann–Stieltjes Δ-integrals in (5) *exist.*

Proof. For all $t \in [a,b]_{\mathbb{T}}$ we have $x(t)$, $y(t) \in I$. From inequality (4) we conclude that $F(x(t)) - F(y(t)) \geqslant (x(t)-y(t))\varphi(y(t))$. Multiplying by nonnegative values $p(t)$ and integrating with respect to the non-decreasing function g, we arrive to (5) with the help of Corollary 9. □

We can use inequality (5) of Theorem 12 to prove a new Jensen's type inequality on time scales [15] for Riemann–Stieltjes integrals.

COROLLARY 13. *Let $\mathbb{T}$ be a time scale with $a,b \in \mathbb{T}$; $F : I \to \mathbb{R}$ be a convex function on I; $x,p : [a,b]_{\mathbb{T}} \to I$ be rd-continuous with $p(\cdot)$ nonnegative on $[a,b]_{\mathbb{T}}$; and $g : [a,b]_{\mathbb{T}} \to I$ be non-decreasing on $[a,\rho(b)]_{\mathbb{T}}$. Define $A := \int_a^b p(t)\Delta g(t) > 0$. Then,*

$$\frac{1}{A}\int_a^b p(t)F(x(t))\Delta g(t) \geqslant F\left(\frac{1}{A}\int_a^b p(t)x(t)\Delta g(t)\right)$$

provided both integrals exist.

Proof. It is enough to take the constant function $y(s) \equiv \frac{1}{A}\int_a^b p(t)x(t)\Delta g(t)$ for each $s \in [a,b]_{\mathbb{T}}$, and see that $y(s) \in I$. We do the proof for $I = [c,d]$. For $x : [a,b]_{\mathbb{T}} \to I$ we have $cp(t) \leqslant p(t)x(t) \leqslant dp(t)$. Integrating both sides with respect to the non-decreasing function g we obtain: $Ac \leqslant \int_a^b p(t)x(t)\Delta g(t) \leqslant Ad$. Hence, $c \leqslant y(s) \leqslant d$. Taking into account inequality (5) of Theorem 12 we get:

$$\frac{1}{A}\int_a^b p(t)F(x(t))\Delta g(t) \geqslant F(y(s)) + \varphi(y(s))\left(\frac{1}{A}\int_a^b p(t)x(t)\Delta g(t) - y(s)\right),$$

where the right-hand side is equal to $F(y(s))$. □

Similarly, one can obtain a Riemann–Stieltjes Jensen's reverse integral inequality on time scales:

COROLLARY 14. *Let $\mathbb{T}$ be a time scale with $a,b\in\mathbb{T}$; $F: I\to\mathbb{R}$ be a continuous convex function on I; $x,p:[a,b]_{\mathbb{T}}\to I$ be rd-continuous with $p(\cdot)$ nonnegative on $[a,b]_{\mathbb{T}}$; and $g:[a,b]_{\mathbb{T}}\to I$ be non-decreasing on $[a,\rho(b)]_{\mathbb{T}}$ with $A=\int_a^b p(t)\Delta g(t)>0$. If $\varphi\in\partial F$ and the Riemann–Stieltjes Δ-integrals $\int_a^b p(t)y(t)\varphi(y(t))\Delta g(t)$ and $\int_a^b p(t)\varphi(y(t))\Delta g(t)$ exist, then*

$$\begin{aligned} 0 &\leqslant \frac{1}{A}\int_a^b p(t)F(y(t))\Delta g(t) - F\left(\frac{1}{A}\int_a^b p(t)y(t)\Delta g(t)\right) \\ &\leqslant \frac{1}{A}\left(\int_a^b p(t)y(t)\varphi(y(t))\Delta g(t) - \frac{1}{A}\int_a^b p(t)y(t)\Delta g(t)\cdot\int_a^b p(t)\varphi(y(t))\Delta g(t)\right). \end{aligned}$$

REMARK 15. Corollary 14 coincides with [3, Corollary 2] in the particular case when $\mathbb{T}=\mathbb{R}$.

Using the Riemann–Stieltjes double integral we can prove an inequality of Čebyšev's type on time scales. The inequality (7) of Proposition 17 is motivated by the Čebyšev's inequality on time scales proved in [17].

PROPOSITION 16. *Suppose that $p\in C_{rd}([a,b]_{\mathbb{T}})$ with $p(t)\geqslant 0$ for all $t\in[a,b]_{\mathbb{T}}$, and let $g:[a,b]_{\mathbb{T}}\to\mathbb{R}$ be non-decreasing on $[a,\rho(b)]_{\mathbb{T}}$. Let $f_1,\ f_2\in C_{rd}([a,b]_{\mathbb{T}})$ be similarly (oppositely) ordered, that is, for all $t,s\in[a,b]_{\mathbb{T}}$*

$$(f_1(t)-f_1(s))(f_2(t)-f_2(s))\geqslant 0\,(\leqslant 0).$$

Then,

$$\int_a^b\int_a^b p(t)p(s)(f_1(t)-f_1(s))(f_2(t)-f_2(s))\Delta g(t)\Delta g(s)\geqslant 0\,(\leqslant 0). \tag{6}$$

Proof. Follows from Proposition 8. □

PROPOSITION 17. *Suppose that $p\in C_{rd}([a,b]_{\mathbb{T}})$ with $p(t)\geqslant 0$ for all $t\in[a,b]_{\mathbb{T}}$, and let $g:[a,b]_{\mathbb{T}}\to\mathbb{R}$ be non-decreasing on $[a,\rho(b)]_{\mathbb{T}}$. Let $f_1,\ f_2\in C_{rd}([a,b]_{\mathbb{T}})$ be similarly (oppositely) ordered. Then,*

$$\int_a^b p(t)\Delta g(t)\int_a^b p(t)f_1(t)f_2(t)\Delta g(t)\geqslant(\leqslant)\int_a^b p(t)f_1(t)\Delta g(t)\int_a^b p(t)f_2(t)\Delta g(t). \tag{7}$$

Proof. We need to rewrite inequality (6) as (7). Because p is a rd-continuous function on the interval $[a,b]_{\mathbb{T}}$ and g is non-decreasing on $[a,\rho(b)]_{\mathbb{T}}$ (see [13]), function p

is Riemann–Stiejtles Δ-integrable with respect to g. Then,

$$\begin{aligned}
&\int_a^b \int_a^b p(t)p(s)\,(f_1(t)-f_1(s))\,(f_2(t)-f_2(s))\,\Delta g(t)\Delta g(s)\\
&\quad= \int_a^b p(s) \int_a^b (p(t)f_1(t)f_2(t) - p(t)f_1(t)f_2(s) - p(t)f_1(s)f_2(t)\\
&\qquad\qquad + p(t)f_1(s)f_2(s))\,\Delta g(t)\Delta g(s)\\
&\quad= 2\left(\int_a^b p(t)\Delta g(s) \int_a^b p(t)f_1(t)f_2(t)\Delta g(t)\right.\\
&\qquad\qquad \left. - \int_a^b p(t)f_1(t)\Delta g(s) \int_a^b p(t)f_2(t)\Delta g(t)\right) \geqslant 0
\end{aligned}$$

and the result is proved. □

Corollary 18 gives a Winckler-type formula for the delta Riemann–Stieltjes integral on time scales. In the particular case $g(t)=t$ one obtains the result in [17]; in the case $g(t)=t$ and $\mathbb{T}=\mathbb{N}$ we can easily obtain the classical Winckler formula: if $a=(a_1,\dots,a_n)$ and $b=(b_1,\dots,b_n)$ are similarly (oppositely) ordered, then

$$\sum_{i=1}^{n} p_i \sum_{i=1}^{n} a_i b_i \geqslant (\leqslant) \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i\,.$$

COROLLARY 18. *Let* $p\in C_{rd}([a,b]_{\mathbb{T}})$ *with* $p(t)\geqslant 0$ *for all* $t\in[a,b]_{\mathbb{T}}$ *and let* $g:[a,b]_{\mathbb{T}}\to\mathbb{R}$ *be non-decreasing on* $[a,\rho(b)]_{\mathbb{T}}$. *If* f *and* $1/f\in C_{rd}([a,b]_{\mathbb{T}})$, *then*

$$\left(\int_a^b p(t)\Delta g(t)\right)^2 \geqslant \int_a^b p(t)f(t)\Delta g(t) \int_a^b \frac{p(t)\Delta g(t)}{f(t)}\,. \tag{8}$$

Proof. It is enough to take $f_1=f$ and $f_2=\pm 1/f$ in Proposition 17. Indeed, from the assumption that f_1, $f_2\in C_{rd}([a,b]_{\mathbb{T}})$ it follows that $f_1(t)f_2(t)=\pm 1$ for each $t\in[a,b]_{\mathbb{T}}$. Since f_1 and f_2 are obviously similarly or oppositely ordered, we end up with inequality (8). □

From Corollary 18 we can obtain other Winckler formulas by choosing different time scales and different non-decreasing functions g on $\mathbb{T}$:

EXAMPLE 19. Let $\mathbb{T}=\overline{q^{\mathbb{Z}}}$, $q>1$, and $g(t)=t^2$. Choose $a=0\in\mathbb{T}$ and $b=1\in\mathbb{T}$. We consider the integral $\int_0^1 p(t)f_1(t)f_2(t)\Delta g(t)$ on this time scale. The q-scale integral is in this case represented by an infinite series:

$$\int_0^1 p(t)\Delta g(t) = \sum_{k=1}^{+\infty} p(q^{-k})q^{-k}(q+1)\,.$$

Let us take $p(t)=t$ and, analogously as in Corollary 18, consider similarly ordered functions f_1 and f_2 on $[0,1]_{\mathbb{T}}$ with $f_1(t)f_2(t)=1$. It follows that $\left(\int_0^1 p(t)\Delta g(t)\right)^2=$

$\frac{1}{(q-1)^2}$ while

$$\int_0^1 p(t)f_1(t)\Delta g(t)\int_0^1 p(t)f_2(t)\Delta g(t) = (q+1)^2\sum_{k=1}^{+\infty} q^{-2k}f_1(q^{-2k})\sum_{k=1}^{+\infty} q^{-2k}f_2(q^{-2k}).$$

Hence,

$$\sum_{k=1}^{+\infty} q^{-2k}f(q^{-2k})\sum_{k=1}^{+\infty}\frac{q^{-2k}}{f(q^{-2k})} \leqslant \frac{1}{(q^2-1)^2},$$

where $f = f_1$.

3.3. Majorisation theorems

We now extend some majorisation type results from [3, 9].

THEOREM 20. *Let $\mathbb{T}$ be a time scale with $a,b\in\mathbb{T}$; functions $x,y,p,g:[a,b]_{\mathbb{T}}\to I\subset\mathbb{R}$ be rd-continuous on $[a,b]_{\mathbb{T}}$ with g non-decreasing and p bounded. Additionally, let $F: I\to\mathbb{R}$ be a continuous convex function on I. If y and $x-y$ are both non-decreasing or non-increasing and*

$$\int_a^b p(t)y(t)\Delta g(t) = \int_a^b p(t)x(t)\Delta g(t), \tag{9}$$

then

$$\int_a^b p(t)F(y(t))\Delta g(t) \leqslant \int_a^b p(t)F(x(t))\Delta g(t). \tag{10}$$

Proof. The rd-continuity assumptions imply the existence of all integrals in (9) and (10). Moreover, if $\varphi\in\partial F$, then both φ and $\varphi\circ y$ are non-decreasing on $[a,b]_{\mathbb{T}}$. Since $p(\cdot)$ is bounded on $[a,b]_{\mathbb{T}}$, the rd-continuity of g implies the existence of the Riemann–Stieltjes Δ-integral $\int_a^b p(t)[x(t)-y(t)]\varphi(y(t))\Delta g(t)$. Since g is non-decreasing on $[a,b]_{\mathbb{T}}$, then (5) implies that

$$\int_a^b p(t)F(x(t))\Delta g(t) - \int_a^b p(t)F(y(t))\Delta g(t) \geqslant \int_a^b p(t)[x(t)-y(t)]\varphi(y(t))\Delta g(t). \tag{11}$$

Taking $f_1(t) = x(t)-y(t)$ and $f_2(t) = \varphi(y(t))$ in inequality (7) and noting that f_1 and f_2 are similarly ordered, we obtain:

$$\int_a^b p(t)\Delta g(t)\int_a^b p(t)[x(t)-y(t)]\varphi(y(t))\Delta g(t)$$
$$\geqslant \int_a^b p(t)[x(t)-y(t)]\Delta g(t)\int_a^b p(t)\varphi(y(t))\Delta g(t).$$

Equality (9) implies that $p(t)[x(t)-y(t)] = 0$, so

$$\int_a^b p(t)\Delta g(t)\int_a^b p(t)[x(t)-y(t)]\varphi(y(t))\Delta g(t) \geqslant 0.$$

From Proposition 8 it follows that $\int_a^b p(t)\Delta g(t) \geqslant 0$. Thus,

$$\int_a^b p(t)[x(t)-y(t)]\varphi(y(t))\Delta g(t) \geqslant 0.$$

Inequality (10) follows from (11). □

THEOREM 21. *Let $\mathbb{T}$ be a time scale with $a,b \in \mathbb{T}$; functions $x,y,p,g : [a,b]_{\mathbb{T}} \to I \subset \mathbb{R}$ be rd-continuous on $[a,b]_{\mathbb{T}}$ with g non-decreasing and p bounded and nonnegative. Additionally, let $F : I \to \mathbb{R}$ be a non-decreasing continuous and convex function on I. If y and $x-y$ are both non-decreasing or non-increasing and*

$$\int_a^b p(t)y(t)\Delta g(t) \leqslant \int_a^b p(t)x(t)\Delta g(t), \tag{12}$$

then (10) *holds true.*

Proof. The integrals $\int_a^b p(t)[x(t)-y(t)]\Delta g(t)$ and $\int_a^b p(t)\varphi(y(t))\Delta g(t)$ that appear in the proof of Theorem 20 are nonnegative because of (12) and the monotonicity of F and nonnegativeness of p. Thus,

$$\int_a^b p(t)[x(t)-y(t)]\varphi(y(t))\Delta g(t) \geqslant 0. \quad \square$$

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Dorota Mozyrska
Faculty of Computer Science
Białystok University of Technology
15-351 Białystok
Poland
e-mail: d.mozyrska@pb.edu.pl

Ewa Pawłuszewicz
Faculty of Computer Science
Białystok University of Technology
15-351 Białystok
Poland
and
Department of Mathematics
University of Aveiro
3810-193 Aveiro
Portugal
e-mail: ewa@ua.pt

Delfim F. M. Torres
Department of Mathematics
University of Aveiro
3810-193 Aveiro
Portugal
e-mail: delfim@ua.pt