

## SOME BOUNDS FOR THE COMPLETE ELLIPTIC INTEGRALS OF THE FIRST AND SECOND KINDS

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*Abstract.* In the article, the complete elliptic integrals of the first and second kinds are bounded by using the power series expansions of some functions, the celebrated Wallis inequality, and an integral inequality due to R. P. Agarwal, P. Cerone, S. S. Dragomir and F. Qi.

### 1. Introduction

The complete elliptic integrals of the first and second kinds may be defined [1, pp. 590–592] as

$$E(t) = \int_0^{\pi/2} \sqrt{1 - t^2 \sin^2 \theta} \, d\theta \quad (1)$$

and

$$F(t) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - t^2 \sin^2 \theta}} \quad (2)$$

for  $0 < t < 1$ . They can also be defined by

$$E(a, b) = \int_0^{\pi/2} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} \, d\theta \quad (3)$$

and

$$F(a, b) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}} \quad (4)$$

for positive numbers  $a$  and  $b$ .

It is not difficult to see that if  $a > b > 0$  then

$$E(a, b) = aE\left(\sqrt{1 - \frac{b^2}{a^2}}\right) \quad \text{and} \quad F(a, b) = \frac{1}{a}F\left(\sqrt{1 - \frac{b^2}{a^2}}\right). \quad (5)$$

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Conversely, if  $0 < t < 1$ , then

$$E(t) = E(1, \sqrt{1-t^2}) \quad \text{and} \quad F(t) = F(1, \sqrt{1-t^2}). \tag{6}$$

For more information on the history, background, properties and applications, please refer to [3] and related references therein.

The aim of this paper is to establish several double inequalities for bounding the complete elliptic integrals  $E(t)$ ,  $F(t)$  and  $F(a, b)$ .

Our main results may be stated in the following theorems.

**THEOREM 1.** *For  $0 < t < 1$ , we have*

$$\frac{\pi}{2} - \frac{1}{2} \ln \frac{(1+t)^{1-t}}{(1-t)^{1+t}} < E(t) < \frac{\pi-1}{2} + \frac{1-t^2}{4t} \ln \frac{1+t}{1-t}. \tag{7}$$

**THEOREM 2.** *For  $b > a > 0$ , we have*

$$\frac{\pi}{2} \cdot \frac{\ln(\sqrt{b/a} + \sqrt{b/a-1})}{\sqrt{b(b-a)}} \leq F(a, b) \leq \frac{\pi}{2} \cdot \frac{\arctan \sqrt{b/a-1}}{\sqrt{a(b-a)}}. \tag{8}$$

**THEOREM 3.** *For  $0 < t < 1$ , we have*

$$\left| \frac{2}{\pi} E(t) - \frac{1 + \sqrt{1-t^2}}{2} \right| \leq \frac{1}{\pi} (1 - \sqrt{1-t^2}) \left[ 1 - \frac{2}{\pi} \cdot \frac{\sqrt{(1-t^2 + \sqrt{1-t^2})(1 + \sqrt{1-t^2})}}{(\sqrt{1-t^2} + 1)\sqrt[4]{1-t^2}} \right]. \tag{9}$$

**THEOREM 4.** *For  $0 < t < 1$ , we have*

$$\left| \frac{2}{\pi} F(t) - \frac{\sqrt{1-t^2} + 1}{2\sqrt{1-t^2}} \right| \leq \frac{1}{\pi} \cdot \frac{1 - \sqrt{1-t^2}}{\sqrt{1-t^2}} \times \left[ 1 - \frac{2}{\pi} \cdot \frac{(1 - \sqrt{1-t^2})(2 - t^2 - \sqrt{t^4 - t^2 + 1})^{3/2}}{\sqrt{(1-t^2)(\sqrt{t^4 - t^2 + 1} + t^2 - 1)(1 - \sqrt{t^4 - t^2 + 1})}} \right]. \tag{10}$$

### 2. Lemmas

In order to prove our main results, the following lemmas are necessary.

**LEMMA 1.** *For  $i \in \mathbb{N}$ , we have*

$$\int_0^{\pi/2} \sin^{2i} \theta \, d\theta = \frac{1}{4^i} \binom{2i}{i} \frac{\pi}{2}. \tag{11}$$

*Proof.* This follows easily from Wallis sine formula [46]:

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \begin{cases} \frac{\pi}{2} \cdot \frac{(n-1)!!}{n!!} & \text{for } n \text{ even,} \\ \frac{(n-1)!!}{n!!} & \text{for } n \text{ odd,} \end{cases} \quad (12)$$

where  $n!!$  denotes a double factorial.  $\square$

LEMMA 2. ([24, p. 98, (2.12)]) *For  $i \in \mathbb{N}$ , we have*

$$\frac{4^i}{\sqrt{\pi(i+1/2)}} < \binom{2i}{i} < \frac{4^i}{\sqrt{\pi(i+1/4)}}. \quad (13)$$

REMARK 1. Bounding Wallis formula (12) such as (13) has a long history. For more information, please refer to related contents in the books [24] and [25, pp. 192–193, p. 287]. In [9, 10, 11, 12, 13, 14, 35, 44] and [48, Theorem 2], the double inequality (13) and its sharpness were recovered, proved and refined once and again because of either without being aware of and finding out the original version of the paper [45], or making use of various approaches and subtle techniques, or repeating some existed routines. But, most of them were not devoted to improve the bounds in (13), except [7, 22, 49, 50, 51] and [48, Theorem 1]. Actually it was said early in [21] that it is unquestionable that inequalities similar to (13) can be improved indefinitely but at a sacrifice of simplicity. For more information, please refer to the survey and expository papers [26, pp. 69–72, Section 7.4] and [27, 28].

LEMMA 3. *For  $|s| < 1$ , we have*

$$\sum_{i=0}^{\infty} \frac{s^{2i}}{i+1/2} = \begin{cases} \frac{1}{s} \ln \frac{1+s}{1-s}, & s \neq 0; \\ 1, & s = 0. \end{cases} \quad (14)$$

*Proof.* This can be deduced readily from the power series expansions of the functions  $\ln(1 \pm s)$  at  $s = 0$ .  $\square$

LEMMA 4. ([2, 8, 33]) *Let  $f(x)$  be continuous on  $[a, b]$  and differentiable in  $(a, b)$ . Suppose that  $f(x)$  is not identically a constant, and that  $m \leq f'(x) \leq M$  in  $(a, b)$ . Then*

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{f(a)+f(b)}{2} \right| \leq -\frac{[M-S_0(a,b)][m-S_0(a,b)](b-a)}{2(M-m)}, \quad (15)$$

where

$$S_0(a, b) = \frac{f(b) - f(a)}{b - a}. \quad (16)$$

REMARK 2. For more information on the history, background, origin, refinements and generalizations of (15), please refer to [16, 17, 19, 29, 30, 31, 32, 34, 41, 42] and related references therein.

### 3. Proofs of main results

*Proof of Theorem 1.* It is general knowledge that

$$\sqrt{1-x} = 1 - \sum_{i=0}^{\infty} \frac{(2i)!}{2^{2i+1}(i+1)(i!)^2} x^{i+1} \tag{17}$$

for  $0 < x < 1$ . Hence, by replacing  $x$  by  $t^2 \sin^2 \theta$  in (17), integrating on both sides with respect to  $\theta \in [0, \frac{\pi}{2}]$  and making use of (11), we have

$$\begin{aligned} E(t) &= \int_0^{\pi/2} \left[ 1 - \sum_{i=0}^{\infty} \frac{(2i)!}{2^{2i+1}(i+1)(i!)^2} t^{2(i+1)} \sin^{2(i+1)} \theta \right] d\theta \\ &= \frac{\pi}{2} - \sum_{i=0}^{\infty} \frac{(2i)!}{2^{2i+1}(i+1)(i!)^2} t^{2(i+1)} \int_0^{\pi/2} \sin^{2(i+1)} \theta d\theta \\ &= \frac{\pi}{2} - \sum_{i=0}^{\infty} \frac{(2i)!}{2^{2i+1}(i+1)(i!)^2} \frac{1}{4^{i+1}} \binom{2i+2}{i+1} \frac{\pi}{2} t^{2(i+1)} \\ &= \frac{\pi}{2} \left[ 1 - \sum_{i=0}^{\infty} \frac{1}{4^{2i+2}(2i+1)} \binom{2i+2}{i+1}^2 t^{2(i+1)} \right] \\ &= \frac{\pi}{2} \left[ 1 - \sum_{i=1}^{\infty} \frac{1}{4^{2i}(2i-1)} \binom{2i}{i}^2 t^{2i} \right]. \end{aligned} \tag{18}$$

Substituting (13) into (18) gives

$$\frac{\pi}{2} - 2 \sum_{i=1}^{\infty} \frac{t^{2i}}{(2i-1)(4i+1)} < E(t) < \frac{\pi}{2} - \sum_{i=1}^{\infty} \frac{t^{2i}}{(2i-1)(2i+1)}. \tag{19}$$

Straightforward computation and utilization of (14) gives

$$\begin{aligned} \sum_{i=1}^{\infty} \frac{t^{2i}}{(2i-1)(2i+1)} &= \frac{1}{2} \sum_{i=1}^{\infty} \left( \frac{1}{2i-1} - \frac{1}{2i+1} \right) t^{2i} \\ &= \frac{1}{2} + \frac{t^2-1}{4} \sum_{i=0}^{\infty} \frac{t^{2i}}{i+1/2} = \frac{1}{2} + \frac{t^2-1}{4t} \ln \frac{1+t}{1-t}. \end{aligned}$$

This means the right-hand side inequality in (7).

By the similar argument as above, it follows that

$$\begin{aligned} \sum_{i=1}^{\infty} \frac{t^{2i}}{(2i-1)(4i+1)} &< \frac{1}{4} \sum_{i=1}^{\infty} \frac{t^{2i}}{i(2i-1)} = \frac{1}{2} \sum_{i=1}^{\infty} \left( \frac{1}{2i-1} - \frac{1}{2i} \right) t^{2i} \\ &= \frac{1}{2} \sum_{i=0}^{\infty} \frac{t^{2i+2}}{2i+1} - \frac{1}{4} \sum_{i=1}^{\infty} \frac{t^{2i}}{i} = \frac{t}{4} \ln \frac{1+t}{1-t} - \frac{1}{4} \ln(1-t^2) \\ &= \frac{1}{4} \ln \frac{(1+t)^{1-t}}{(1-t)^{1+t}}. \end{aligned} \tag{20}$$

The left-hand side inequality in (7) follows. The proof of Theorem 1 is complete.  $\square$

*Proof of Theorem 2.* In [20], by discussing

$$\sqrt{1+t^2 \cos^2 \theta} - \sqrt{1+t^2} + \frac{4}{\pi^2} (\sqrt{1+t^2} - 1) \theta^2 + \alpha \left( \frac{\pi}{2} - \theta \right)$$

or

$$\sqrt{1+t^2 \cos^2 \theta} - \sqrt{1+t^2} + \frac{2}{\pi} (\sqrt{1+t^2} - 1) \theta + \beta \left( \frac{\pi}{2} - \theta \right) \theta$$

on  $[0, \frac{\pi}{2}]$ , where  $\alpha$  and  $\beta$  are undetermined constants, the inequality

$$\begin{aligned} -\frac{8}{\pi^2} (\sqrt{1+t^2} - 1) \theta \left( \frac{\pi}{2} - \theta \right) \\ \leq \sqrt{1+t^2 \cos^2 \theta} - \left[ \sqrt{1+t^2} - \frac{4}{\pi^2} (\sqrt{1+t^2} - 1) \theta^2 \right] \leq 0 \end{aligned} \quad (21)$$

for  $\theta \in [0, \frac{\pi}{2}]$  was obtained, which is equivalent to

$$\begin{aligned} \frac{1}{\sqrt{1+t^2} - 4(\sqrt{1+t^2} - 1)\theta^2/\pi^2} &\leq \frac{1}{\sqrt{1+t^2 \cos^2 \theta}} \\ &\leq \frac{1}{\sqrt{1+t^2} - 4(\sqrt{1+t^2} - 1)\theta^2/\pi^2 - 8(\sqrt{1+t^2} - 1)\theta(\pi/2 - \theta)/\pi^2}. \end{aligned} \quad (22)$$

Integrating on both sides of the above double inequality with respect to  $\theta \in [0, \frac{\pi}{2}]$  yields

$$\begin{aligned} \frac{\pi}{2} \cdot \frac{\ln \left( \frac{\sqrt{1+t^2} + \sqrt{\sqrt{1+t^2} - 1}}{\sqrt{1+t^2} - \sqrt{1+t^2}} \right)}{\sqrt{1+t^2} - \sqrt{1+t^2}} &\leq \int_0^{\pi/2} \frac{d\theta}{\sqrt{1+t^2 \cos^2 \theta}} \\ &\leq \int_0^{\pi/2} \frac{d\theta}{(\sqrt{t^2+1} - 1)(2\theta/\pi - 1)^2 + 1} \\ &= \frac{\pi}{2} \cdot \frac{\arctan \sqrt{\sqrt{1+t^2} - 1}}{\sqrt{\sqrt{t^2+1} - 1}}. \end{aligned} \quad (23)$$

Replacing  $t^2$  by  $\frac{b^2}{a^2} - 1$  for  $b > a > 0$  in (23) and simplifying give

$$\frac{\pi \ln(\sqrt{b/a} + \sqrt{b/a - 1})}{2 \sqrt{b(b-a)}} \leq \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}} \leq \frac{\pi \arctan \sqrt{b/a - 1}}{2 \sqrt{a(b-a)}}.$$

Since

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}}, \quad (24)$$

the proof of Theorem 2 is complete.  $\square$

*Proof of Theorem 3.* For  $0 < t < 1$  and  $\theta \in [0, \frac{\pi}{2}]$ , let  $f(\theta) = \sqrt{1 - t^2 \sin^2 \theta}$ . Direct calculation yields

$$\begin{aligned} f'(\theta) &= -\frac{t^2 \sin \theta \cos \theta}{\sqrt{1 - t^2 \sin^2 \theta}}, \\ f''(\theta) &= -\frac{t^2(t^2 \sin^4 \theta - \sin^2 \theta + \cos^2 \theta)}{(1 - t^2 \sin^2 \theta)^{3/2}} \\ &= -\frac{t^2 \sin^4 \theta (t^2 - 1 + \cot^4 \theta)}{(1 - t^2 \sin^2 \theta)^{3/2}}. \end{aligned}$$

Hence, the function  $f'(\theta)$  has a unique minimum

$$-\frac{t^2 \sqrt[4]{1 - t^2}}{\sqrt{(1 - t^2 + \sqrt{1 - t^2})(1 + \sqrt{1 - t^2})}} \quad (25)$$

at

$$\theta = \arctan \frac{1}{\sqrt[4]{1 - t^2}}. \quad (26)$$

Therefore, the maximum of  $f'(\theta)$  is

$$\lim_{\theta \rightarrow 0^+} f'(\theta) = \lim_{\theta \rightarrow (\pi/2)^-} f'(\theta) = 0. \quad (27)$$

Moreover, we have

$$f(0) = 1 \quad \text{and} \quad f\left(\frac{\pi}{2}\right) = \sqrt{1 - t^2}. \quad (28)$$

Substituting quantities above into (15) and simplifying lead to (9). The proof of Theorem 3 is complete.  $\square$

*Proof of Theorem 4.* For  $0 < t < 1$  and  $\theta \in [0, \frac{\pi}{2}]$ , let

$$h(\theta) = \frac{1}{\sqrt{1 - t^2 \sin^2 \theta}}. \quad (29)$$

Direct calculation yields

$$\begin{aligned} h'(\theta) &= \frac{t^2 \sin \theta \cos \theta}{(1 - t^2 \sin^2 \theta)^{3/2}}, \\ h''(\theta) &= -\frac{t^2(\sin^2 \theta - \cos^2 \theta - t^2 \sin^4 \theta - 2t^2 \cos^2 \theta \sin^2 \theta)}{(1 - t^2 \sin^2 \theta)^{5/2}} \\ &= -\frac{t^2[t^2 \sin^4 \theta + 2(1 - t^2) \sin^2 \theta - 1]}{(1 - t^2 \sin^2 \theta)^{5/2}}. \end{aligned}$$

Hence, the function  $h'(\theta)$  has a unique maximum

$$\frac{\sqrt{(\sqrt{t^4 - t^2 + 1} + t^2 - 1)(1 - \sqrt{t^4 - t^2 + 1})}}{(2 - t^2 - \sqrt{t^4 - t^2 + 1})^{3/2}} \tag{30}$$

at

$$\theta = \arcsin \frac{\sqrt{\sqrt{t^4 - t^2 + 1} + t^2 - 1}}{t}. \tag{31}$$

Therefore, the minimum of  $h'(\theta)$  is

$$\lim_{\theta \rightarrow 0^+} h'(\theta) = \lim_{\theta \rightarrow (\pi/2)^-} h'(\theta) = 0. \tag{32}$$

Moreover, we have

$$h(0) = 1 \quad \text{and} \quad h\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{1 - t^2}}. \tag{33}$$

Substituting quantities above into (15) and simplifying lead to (10). The proof of Theorem 4 is complete.  $\square$

#### 4. Remarks

REMARK 3. In [43], it was posed that

$$\frac{\pi}{6} < \int_0^1 \frac{1}{\sqrt{4 - x^2 - x^3}} dx < \frac{\pi\sqrt{2}}{8}. \tag{34}$$

In [15], the inequality (34) was verified by using  $4 - x^2 > 4 - x^2 - x^3 > 4 - 2x^2$  on the unit interval  $[0, 1]$ .

In [37], by considering monotonicity and convexity of the function

$$\frac{1}{\sqrt{4 - x^2 - x^3}} - \frac{1}{2} + \frac{1 - \sqrt{2}}{2}x^4 + \alpha x^3(1 - x) \tag{35}$$

on  $[0, 1]$  for undetermined constant  $\alpha \geq 0$ , the inequality

$$\frac{1}{\sqrt{4 - x^2 - x^3}} \geq \frac{1}{2} + \frac{\sqrt{2} - 1}{2}x^4 + \left(\frac{11\sqrt{2}}{8} - 2\right)(1 - x)x^3 \tag{36}$$

for  $x \in [0, 1]$  was established, and then the lower bound in (34) was improved to

$$\int_0^1 \frac{1}{\sqrt{4 - x^2 - x^3}} dx > \frac{3}{10} + \frac{27\sqrt{2}}{160}. \tag{37}$$

It was also remarked in [37] that if discussing the auxiliary functions

$$\frac{1}{\sqrt{4 - x^2 - x^3}} - \frac{1}{2} + \frac{1 - \sqrt{2}}{2}x^2 + \beta(1 - x)x^2 \tag{38}$$

and

$$\frac{1}{\sqrt{4-x^2-x^3}} - \frac{1}{2} + \frac{1-\sqrt{2}}{2}x^4 + \theta(1-x^3)x \quad (39)$$

on  $[0, 1]$ , then inequalities

$$\frac{1}{\sqrt{4-x^2-x^3}} \geq \frac{1}{2} + \frac{\sqrt{2}-1}{2}x^2 + \left(\frac{3\sqrt{2}}{8} - 1\right)(1-x)x^2 \quad (40)$$

and

$$\frac{1}{\sqrt{4-x^2-x^3}} \geq \frac{1}{2} + \frac{\sqrt{2}-1}{2}x^4 + \left(\frac{2}{3} - \frac{11\sqrt{2}}{24}\right)(x^3-1)x \quad (41)$$

can be obtained, and then, by integrating on both sides of above two inequalities, the lower bound in (34) may be improved to

$$\int_0^1 \frac{1}{\sqrt{4-x^2-x^3}} dx > \frac{1}{4} + \frac{19\sqrt{2}}{96} \quad (42)$$

and

$$\int_0^1 \frac{1}{\sqrt{4-x^2-x^3}} dx > \frac{1}{5} + \frac{19\sqrt{2}}{80}. \quad (43)$$

Numerical computation shows that the lower bound in (37) is better than those in (42) and (43).

In [47], by directly proving the inequality (36) and

$$\frac{1}{\sqrt{4-x^2-x^3}} \leq \frac{1}{2} + \frac{\sqrt{2}-1}{2}x^2 + \frac{5-4\sqrt{2}}{8}x^2(1-x) \left(\frac{8\sqrt{2}-9}{8\sqrt{2}-10} + x\right), \quad (44)$$

the inequality (37) and an improved upper bound in (34),

$$\int_0^1 \frac{1}{\sqrt{4-x^2-x^3}} dx < \frac{79}{192} + \frac{\sqrt{2}}{10}, \quad (45)$$

were obtained.

In [36], by considering an auxiliary function

$$\frac{1}{\sqrt{4-x^2-x^3}} - \frac{1}{2} + \frac{1-\sqrt{2}}{2}x^2 + \alpha x^2(1-x) \left(\frac{8\sqrt{2}-9}{8\sqrt{2}-10} + x\right) \quad (46)$$

on  $[0, 1]$ , the sharpness of the inequality (44) and the following sharp inequality

$$\frac{1}{\sqrt{4-x^2-x^3}} \geq \frac{1}{2} + \frac{\sqrt{2}-1}{2}x^2 - \frac{1137(4\sqrt{2}-5)}{64(64-39\sqrt{2})}(1-x) \left(\frac{8\sqrt{2}-9}{8\sqrt{2}-10} + x\right) \quad (47)$$

were presented, and then the inequality (45) was obtained by integrating on both sides of (44).



REMARK 4. Integrating on both sides of (21) with respect to  $\theta$  from 0 to  $\frac{\pi}{2}$  yields

$$\frac{\pi}{6}(2a+b) < \int_0^{\pi/2} \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \, d\theta \leq \frac{\pi}{6}(a+2b). \quad (48)$$

When  $b \geq 7a$ , the right-hand side of the inequality (48) is stronger than

$$\frac{\pi}{4}(a+b) \leq \int_0^{\pi/2} \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \, d\theta \leq \frac{\pi}{4} \sqrt{2(a^2 + b^2)} \quad (49)$$

which can be obtained by using some properties of definite integral.

REMARK 5. The complete elliptic integral of the third kind may be defined for  $0 < t < 1$  as

$$H(t, h) = \int_0^{\pi/2} \frac{d\theta}{(1 + h \sin^2 \theta) \sqrt{1 - t^2 \sin^2 \theta}}. \quad (50)$$

By the way, some other estimates for complete elliptic integrals, obtained by using Tchebycheff's integral inequality in [38], are mentioned below:

$$\frac{\pi \arcsin t}{2t} < F(t) < \frac{\pi}{4t} \ln \frac{1+t}{1-t}; \quad (51)$$

$$E(t) < \frac{16 - 4t^2 - 3t^4}{4(4 + t^2)} F(t); \quad (52)$$

$$F(t) < \left(1 + \frac{h}{2}\right) H(t, h), \quad -1 < h < 0 \quad \text{or} \quad h > \frac{t^2}{2 - 3t^2} > 0; \quad (53)$$

$$H(t, h)E(t) > \frac{\pi^2}{4\sqrt{1+h}}, \quad -2 < 2h < t^2; \quad (54)$$

$$E(t) \geq \frac{16 - 28t^2 + 9t^4}{4(4 - 5t^2)} F(t), \quad t^2 \leq \frac{2}{3}. \quad (55)$$

For  $0 < 2h < t^2$ , the inequality (53) is reversed. For  $h > \frac{t^2}{2 - 3t^2} > 0$ , the inequality (54) is reversed.

As concrete examples, the following estimates of the complete elliptic integrals are also deduced in [38]:

$$\frac{\pi^2}{4\sqrt{2}} < \int_0^{\pi/2} \left(1 - \frac{\sin^2 x}{2}\right)^{-1/2} dx < \frac{\pi \ln(1 + \sqrt{2})}{\sqrt{2}}, \quad (56)$$

$$\int_0^{\pi/2} \left(1 + \frac{\cos x}{2}\right)^{-1} dx < \frac{\pi(\ln 3 - \ln 2)}{2}, \quad (57)$$

$$\int_0^{\pi/2} \left(1 - \frac{\sin x}{2}\right)^{-1} dx = \int_{\pi/2}^{\pi} \left(1 + \frac{\cos x}{2}\right)^{-1} dx > \frac{\pi \ln 2}{2}. \quad (58)$$

These results are better than those in [23, p. 607].

REMARK 6. It is noted that some new results on complete elliptic integrals are obtained in [5] recently. It was pointed in [5] that the right-hand side inequality in (51) is a recovery of [4, Theorem 3.10]. In [5], the inequality (51) was also generalized to the case of generalized complete elliptic integrals by the same method as in [35, 38].

In [6], some of the results in [5] were further improved.

For more information on recent development of elliptic integrals, please refer to related texts in [39, 40] and closely-related references therein.

REMARK 7. This is a modified version of the preprint [18].

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