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SIMULTANEOUS AND CONVERSE APPROXIMATION THEOREMS IN WEIGHTED LEBESGUE SPACES

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Abstract. In this paper we deal with the simultaneous and converse approximation by trigonometric polynomials of the functions in the Lebesgue spaces with weights satisfying so called Muckenhoupt's A_p condition.

1. Introduction and the main results

Let $\mathbf{T} := [-\pi,\pi]$. A positive almost everywhere (a.e.), integrable function $w : \mathbf{T} \to [0,\infty]$ is called as a weight function. With any given weight w we associate the w-weighted Lebesgue space $L_w^p(\mathbf{T})$ consisting of all measurable functions f on $\mathbf{T}$ such that

$$\|f\|_{L_w^p(\mathbf{T})} = \|fw\|_{L^p(\mathbf{T})} < \infty.$$

Let $1 < p < \infty$ and $1/p + 1/q = 1$. A weight function w belongs to the Muckenhoupt class $A_p(\mathbf{T})$ if

$$\left(\frac{1}{|I|}\int_I w^p(x)dx\right)^{1/p}\left(\frac{1}{|I|}\int_I w^{-q}(x)dx\right)^{1/q} \leqslant c$$

with a finite constant c independent of I, where I is any subinterval of $\mathbf{T}$ and $|I|$ denotes the length of I.

For formulation of the new results we will begin with some required informations. Let

$$f(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx} = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) \tag{1}$$

and

$$\tilde{f}(x) \sim \sum_{k=1}^{\infty}(a_k \sin kx - b_k \cos kx)$$

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be the Fourier and the conjugate Fourier series of $f \in L^1(\mathbf{T})$, respectively. In addition, we put

$$S_n(x,f) := \sum_{k=-n}^{n} c_k e^{ikx} = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx), \quad n = 1,2,....$$

By $L_0^1(\mathbf{T})$ we denote the class of $L^1(\mathbf{T})$ functions f for which the constant term c_0 in (1) equals zero. If $\alpha > 0$, then α*-th integral* of $f \in L_0^1(\mathbf{T})$ is defined as

$$I_\alpha(x,f) := \sum_{k \in \mathbb{Z}^*} c_k (ik)^{-\alpha} e^{ikx},$$

where $(ik)^{-\alpha} := |k|^{-\alpha} e^{(-1/2)\pi i\alpha \operatorname{sign} k}$ and $\mathbb{Z}^* := \{\pm 1, \pm 2, \pm 3, ...\}$.

For $\alpha \in (0,1)$ let

$$f^{(\alpha)}(x) := \frac{d}{dx} I_{1-\alpha}(x,f),$$

$$f^{(\alpha+r)}(x) := \left(f^{(\alpha)}(x)\right)^{(r)} = \frac{d^{r+1}}{dx^{r+1}} I_{1-\alpha}(x,f)$$

if the right hand sides exist, where $r \in \mathbb{Z}^+ := \{1,2,3,...\}$ [14, p. 347].

By c, $c(\alpha,...)$ we denote the absolute constants or the constants whose values depend only on the parameters given in their brackets.

Let $x,t \in \mathbb{R} := (-\infty,\infty)$, $r \in \mathbb{R}^+ := (0,\infty)$ and let

$$\triangle_t^r f(x) := \sum_{k=0}^{\infty} (-1)^k [C_k^r] f(x + (r-k)t), \quad f \in L^1(\mathbf{T}), \tag{2}$$

where $[C_k^r] := \frac{r(r-1)...(r-k+1)}{k!}$ for $k > 1$, $[C_k^r] := r$ for $k = 1$ and $[C_k^r] := 1$ for $k = 0$.

Since [14, p. 14]

$$|[C_k^r]| = \left|\frac{r(r-1)...(r-k+1)}{k!}\right| \leqslant \frac{c(r)}{k^{r+1}}, \quad k \in \mathbb{Z}^+$$

we have

$$C(r) := \sum_{k=0}^{\infty} |[C_k^r]| < \infty,$$

and therefore $\triangle_t^r f(x)$ is defined a.e. on $\mathbb{R}$. Furthermore, the series in (2) converges absolutely a.e. and $\triangle_t^r f(x)$ is measurable [16].

If $r \in \mathbb{Z}^+$, then the fractional difference $\triangle_t^r f(x)$ coincides with usual forward difference.

Now let

$$\sigma_\delta^r f(x) := \frac{1}{\delta} \int_0^\delta |\triangle_t^r f(x)| \, dt, \quad 1 < p < \infty$$

for $f \in L_w^p(\mathbf{T})$, $w \in A_p(\mathbf{T})$. Since the series in (2) converges absolutely a.e., we have $\sigma_\delta^r f(x) < \infty$ a.e. and using boundedness of the Hardy-Littlewood Maximal function [13] in $L_w^p(\mathbf{T})$, $w \in A_p(\mathbf{T})$, we get

$$\left\| \sigma_\delta^r f(x) \right\|_{L_w^p} \leqslant c(p,r) \|f\|_{L_w^p} < \infty. \tag{3}$$

Hence, if $r \in \mathbb{R}^+$ and $w \in A_p(\mathbf{T})$, $1 < p < \infty$, we can define the *r-th mean modulus of smoothness* of a function $f \in L_w^p(\mathbf{T})$ as

$$\Omega_r(f,h)_{L_w^p} := \sup_{|\delta| \leqslant h} \left\| \sigma_\delta^r f(x) \right\|_{L_w^p}.$$

If $r \in \mathbb{Z}^+$, then $\Omega_r(f,h)_{L_w^p}$ coincides with Ky's mean modulus of smoothness, defined in [9].

REMARK 1. Let $f,\ f_1,\ f_2 \in L_w^p(\mathbf{T})$, $w \in A_p(\mathbf{T})$, $1 < p < \infty$. The r-th mean modulus of smoothness $\Omega_r(f,h)_{L_w^p}$, $r \in \mathbb{R}^+$, has the following properties:
(i) $\Omega_r(f,h)_{L_w^p}$ is non-negative and non-decreasing function of $h \geqslant 0$.
(ii) $\Omega_r(f_1+f_2,\cdot)_{L_w^p} \leqslant \Omega_r(f_1,\cdot)_{L_w^p} + \Omega_r(f_2,\cdot)_{L_w^p}$.
(iii) $\lim\limits_{h\to 0} \Omega_r(f,h)_{L_w^p} = 0$.

The best approximation of $f \in L_w^p(\mathbf{T})$ in the class Π_n of trigonometric polynomials of degree not exceeding n is defined by

$$E_n(f)_{L_w^p} = \inf\left\{ \|f - T_n\|_{L_w^p} : T_n \in \Pi_n \right\}.$$

A polynomial $T_n(x,f) := T_n(x)$ of degree n is said to be a *near best approximant* of f if

$$\|f - T_n\|_{L_w^p} \leqslant c(p) E_n(f)_{L_w^p},\ n = 0,1,2,\ldots.$$

Let $W_{p,w}^\alpha(\mathbf{T})$, $\alpha > 0$, be the class of functions $f \in L_w^p(\mathbf{T})$ such that $f^{(\alpha)} \in L_w^p(\mathbf{T})$. $W_{p,w}^\alpha(\mathbf{T})$, $1 < p < \infty$, $\alpha > 0$, becomes a Banach space with the norm

$$\|f\|_{W_{p,w}^\alpha(\mathbf{T})} := \|f\|_{L_w^p} + \left\| f^{(\alpha)} \right\|_{L_w^p}.$$

In this paper we deal with the simultaneous and converse approximation by trigonometric polynomials of the functions in the Lebesgue spaces with weights satisfying Muckenhoupt's A_p condition.

Our new results are the following.

THEOREM 1. *Let* $f \in W_{p,w}^\alpha(\mathbf{T})$, $\alpha \in \mathbb{R}_0^+ := [0,\infty]$, $1 < p < \infty$, *and* $w \in A_p(\mathbf{T})$. *If* $T_n \in \Pi_n$ *is a near best approximant of* f, *then*

$$\left\| f^{(\alpha)} - T_n^{(\alpha)} \right\|_{L_w^p} \leqslant c E_n(f^{(\alpha)})_{L_w^p},\ n = 0,1,2,\ldots.$$

with a constant $c = c(p,\alpha) > 0$.

This simultaneous approximation theorem in case of $\alpha \in \mathbb{Z}^+$ for Lebesgue spaces $L^p(\mathbf{T})$, $1 \leqslant p \leqslant \infty$, was proved in [3]. Detailed information on simultaneous weighted approximation can be found in the book [4].

THEOREM 2. *If* $f \in W^r_{p,w}(\mathbf{T})$, $r \in \mathbb{R}^+$, $1 < p < \infty$, *and* $w \in A_p(\mathbf{T})$, *then*

$$\Omega_r(f,h)_{L^p_w} \leqslant ch^r \left\| f^{(r)} \right\|_{L^p_w}, \quad 0 < h \leqslant \pi$$

with a constant $c = c(p,r) > 0$.

In case of $r \in \mathbb{Z}^+$, for the usual nonweighted modulus of smoothness defined in the Lebesgue spaces $L^p(\mathbf{T})$, $1 \leqslant p \leqslant \infty$, this inequality was proved in [11] and for the general case $r \in \mathbb{R}^+$ was obtained in [2] (See also [16]). In case of $r \in \mathbb{Z}^+$, $w \in A_p(\mathbf{T})$, $1 < p < \infty$, this inequality in the weighted Lebesgue spaces $L^p_w(\mathbf{T})$ was proved in [9].

THEOREM 3. *Let* $f \in L^p_w(\mathbf{T})$, $1 < p < \infty$, *and* $w \in A_p(\mathbf{T})$. *Then for a given* $r \in \mathbb{R}^+$, *and* $\gamma = \min\{2,p\}$

$$\Omega_r(f,\pi/(n+1))_{L^p_w} \leqslant \frac{c}{(n+1)^r} \left(\sum_{k=0}^{n} (k+1)^{r\gamma-1} E_k^\gamma(f)_{L^p_w} \right)^{1/\gamma}$$

with a constant c *independent of* n *and* f.

In the space $L^p(\mathbf{T})$, $1 \leqslant p \leqslant \infty$, this inequality was proved in [16] without γ. In case of $r \in \mathbb{Z}^+$ in the spaces $L^p_w(\mathbf{T})$, $w \in A_p(\mathbf{T})$, $1 < p < \infty$, this theorem was proved in [9] without γ. For the positive and even integer r this theorem in spaces $L^p_w(\mathbf{T})$, $w \in A_p(\mathbf{T})$, by using Butzer-Wehrens's type modulus of smoothness was obtained in [5]. The analogues of some classical theorems for best polynomial approximation in weighted spaces with doubling weights were proved in [12].

THEOREM 4. *Let* $f \in L^p_w(\mathbf{T})$, $1 < p < \infty$, *and* $w \in A_p(\mathbf{T})$. *If*

$$\sum_{k=1}^{\infty} k^{\alpha\gamma-1} E_k^\gamma(f)_{L^p_w} < \infty$$

for $\alpha \in (0,\infty)$ *and* $\gamma = \min\{2,p\}$, *then* $f \in W^\alpha_{p,w}(\mathbf{T})$ *and the estimate*

$$E_n(f^{(\alpha)})_{L^p_w} \leqslant c \left\{ n^\alpha E_n(f)_{L^p_w} + \left(\sum_{k=n+1}^{\infty} k^{\alpha\gamma-1} E_k^\gamma(f)_{L^p_w} \right)^{1/\gamma} \right\} \tag{4}$$

holds with a constant c *independent of* n *and* f.

In the space $L^p(\mathbf{T})$, $1 \leqslant p \leqslant \infty$, this inequality for $\alpha \in \mathbb{Z}^+$ was proved without γ in [15]. In case of $\alpha \in \mathbb{Z}^+$, in $L^p_w(\mathbf{T})$, $w \in A_p(\mathbf{T})$, $1 < p < \infty$, an inequality of type (4) was proved in [7].

COROLLARY 1. *Let* $f \in L_w^p(\mathbf{T}),\ 1 < p < \infty$, *and* $w \in A_p(\mathbf{T})$ *and* $r > 0$. *If*

$$\sum_{k=1}^{\infty} k^{\alpha\gamma-1} E_k^{\gamma}(f)_{L_w^p} < \infty$$

for $\alpha \in (0,\infty)$ *and* $\gamma = \min\{2,p\}$, *then* $f \in W_{p,w}^{\alpha}$ *and for* $n = 0,1,2,...$

$$\Omega_r(f^{(\alpha)}, \pi/(n+1))_{L_w^p} \leqslant \frac{c}{(n+1)^r}\left\{\left(\sum_{k=1}^{n} k^{(\alpha+r)\gamma-1} E_{k-1}^{\gamma}(f)_{L_w^p}\right)^{1/\gamma} + \left(\sum_{k=n+1}^{\infty} k^{\alpha\gamma-1} E_k^{\gamma}(f)_{L_w^p}\right)^{1/\gamma}\right\}$$

with a constant c *independent of* n *and* f.

In cases of $\alpha,\ r \in \mathbb{Z}^+$ and $\alpha,\ r \in \mathbb{R}^+$, this corollary in the spaces $L^p(\mathbf{T}),\ 1 \leqslant p \leqslant \infty$, was proved without γ in [18] (See also [15]) and in [17], respectively. For the weighted Lebesgue spaces $L_w^p(\mathbf{T}),\ 1 < p < \infty$, when $w \in A_p(\mathbf{T})$, and $\alpha,\ r \in \mathbb{Z}^+$, similar type inequality was obtained using generalized modulus of continuity for the derivatives of $f \in L_w^p(\mathbf{T})$ in [7].

2. Auxiliary results

LEMMA 1. *Let* $w \in A_p(\mathbf{T})$ *and* $r \in \mathbb{R}^+,\ 1 < p < \infty$. *If* $T_n \in \Pi_n,\ n \geqslant 1$, *then there exists a constant* $c > 0$ *depends only on* r *and* p *such that*

$$\Omega_r(T_n,h)_{L_w^p} \leqslant ch^r \left\| T_n^{(r)} \right\|_{L_w^p}, \quad 0 < h \leqslant \pi/n.$$

Proof. Since

$$\triangle_t^r T_n\left(x - \frac{[r]}{2}t\right) = \sum_{\nu \in \mathbb{Z}_n^*} \left(2i\sin\frac{t}{2}\nu\right)^r c_\nu e^{i\nu x},$$

$$\triangle_t^{[r]} T_n^{(r-[r])}\left(x - \frac{[r]}{2}t\right) = \sum_{\nu \in \mathbb{Z}_n^*} \left(2i\sin\frac{t}{2}\nu\right)^{[r]} (i\nu)^{r-[r]} c_\nu e^{i\nu x}$$

with $\mathbb{Z}_n^* := \{\pm 1, \pm 2, \pm 3, ...\}$, $[r] \equiv$ integer part of r, putting

$$\varphi(z) := \left(2i\sin\frac{t}{2}z\right)^{[r]} (iz)^{r-[r]},\ g(z) := \left(\frac{2}{z}\sin\frac{t}{2}z\right)^{r-[r]}, \quad -n \leqslant z \leqslant n,\ g(0) := t^{r-[r]},$$

we get

$$\triangle_t^{[r]} T_n^{(r-[r])}\left(x - \frac{[r]}{2}t\right) = \sum_{\nu \in \mathbb{Z}_n^*} \varphi(\nu) c_\nu e^{i\nu x},\ \ \triangle_t^r T_n\left(x - \frac{[r]}{2}t\right) = \sum_{\nu \in \mathbb{Z}_n^*} \varphi(\nu) g(\nu) c_\nu e^{i\nu x}.$$

Taking into account the fact that [16]

$$g(z)=\sum_{k=-\infty}^{\infty} d_k e^{ik\pi z/n}$$

uniformly in $[-n,n]$, with $d_0>0,\ (-1)^{k+1}d_k\geqslant 0,\ d_{-k}=d_k\ (k=1,2,...)$, we have

$$\triangle_t^r T_n(\cdot)=\sum_{k=-\infty}^{\infty} d_k \triangle_t^{[r]} T_n^{(r-[r])}\left(\cdot+\frac{k\pi}{n}+\frac{r-[r]}{2}t\right).$$

Consequently we get

$$\begin{aligned}\left\|\frac{1}{\delta}\int_0^\delta |\triangle_t^r T_n(\cdot)|\,dt\right\|_{L_w^p} &= \left\|\frac{1}{\delta}\int_0^\delta \left|\sum_{k=-\infty}^{\infty} d_k \triangle_t^{[r]} T_n^{(r-[r])}\left(\cdot+\frac{k\pi}{n}+\frac{r-[r]}{2}t\right)\right| dt\right\|_{L_w^p}\\ &\leqslant \sum_{k=-\infty}^{\infty}|d_k|\left\|\frac{1}{\delta}\int_0^\delta \left|\triangle_t^{[r]} T_n^{(r-[r])}\left(\cdot+\frac{k\pi}{n}+\frac{r-[r]}{2}t\right)\right| dt\right\|_{L_w^p}\end{aligned}$$

and since [19, p. 103]

$$\triangle_t^{[r]} T_n^{(r-[r])}(\cdot)=\int_0^t ... \int_0^t T_n^{(r)}(\cdot+t_1+...t_{[r]})dt_1...dt_{[r]}$$

we find

$$\begin{aligned}&\Omega_r(T_n,h)_{L_w^p}\leqslant \sup_{|\delta|\leqslant h}\sum_{k=-\infty}^{\infty}|d_k|\left\|\frac{1}{\delta}\int_0^\delta \left|\triangle_t^{[r]} T_n^{(r-[r])}\left(\cdot+\frac{k\pi}{n}+\frac{r-[r]}{2}t\right)\right| dt\right\|_{L_w^p}\\ &=\sup_{|\delta|\leqslant h}\sum_{k=-\infty}^{\infty}|d_k|\left\|\frac{1}{\delta}\int_0^\delta \left|\int_0^t ... \int_0^t T_n^{(r)}\left(\cdot+\frac{k\pi}{n}+\frac{r-[r]}{2}t+t_1+...t_{[r]}\right)dt_1...dt_{[r]}\right| dt\right\|_{L_w^p}\\ &\leqslant h^{[r]}\sup_{|\delta|\leqslant h}\sum_{k=-\infty}^{\infty}|d_k|\left\|\frac{1}{\delta}\int_0^\delta \frac{1}{\delta^{[r]}}\int_0^\delta ... \int_0^\delta \left|T_n^{(r)}\left(\cdot+\frac{k\pi}{n}+\frac{r-[r]}{2}t+t_1+...t_{[r]}\right)\right| dt_1...dt_{[r]}dt\right\|_{L_w^p}\\ &\leqslant h^{[r]}\sup_{|\delta|\leqslant h}\sum_{k=-\infty}^{\infty}|d_k|\left\|\frac{1}{\delta^{[r]}}\int_0^\delta ... \int_0^\delta \left\{\frac{1}{\delta}\int_0^\delta \left|T_n^{(r)}\left(\cdot+\frac{k\pi}{n}+\frac{r-[r]}{2}t+t_1+...t_{[r]}\right)\right| dt\right\} dt_1...dt_{[r]}\right\|_{L_w^p}\\ &\leqslant c(p,r)h^{[r]}\sup_{|\delta|\leqslant h}\sum_{k=-\infty}^{\infty}|d_k|\left\|\frac{1}{\delta}\int_0^\delta \left|T_n^{(r)}\left(\cdot+\frac{k\pi}{n}+\frac{r-[r]}{2}t\right)\right| dt\right\|_{L_w^p}\\ &\leqslant c(p,r)h^{[r]}\sup_{|\delta|\leqslant h}\sum_{k=-\infty}^{\infty}|d_k|\left\|\frac{1}{\frac{r-[r]}{2}\delta}\int_{\cdot+\frac{k\pi}{n}}^{\cdot+\frac{k\pi}{n}+\frac{r-[r]}{2}\delta}\left|T_n^{(r)}(u)\right| du\right\|_{L_w^p}.\end{aligned}$$

On the other hand [16]

$$\sum_{k=-\infty}^{\infty} |d_k| < 2g(0) = 2t^{r-[r]}, \quad 0 < t \leqslant \pi/n$$

and for $0 < t < \delta < h \leqslant \pi/n$ we have

$$\sum_{k=-\infty}^{\infty} |d_k| < 2g(0) = 2h^{r-[r]}.$$

Therefore the boundedness of Hardy-Littlewood maximal function in $L_w^p(\mathbf{T})$ implies that

$$\Omega_r(T_n, h)_{L_w^p} \leqslant c(p,r)h^r \left\| T_n^{(r)} \right\|_{L_w^p}.$$

By similar way for $0 < -h \leqslant \pi/n$, the same inequality also holds and the proof of Lemma 1 is completed. □

3. Proof of the main results

Proof of Theorem 1. We set

$$W_n(f) := W_n(x,f) := \frac{1}{n+1} \sum_{\nu=n}^{2n} S_\nu(x,f), \; n = 0,1,2,....$$

Since

$$W_n(\cdot, f^{(\alpha)}) = W_n^{(\alpha)}(\cdot, f),$$

we have

$$\begin{aligned} \left\| f^{(\alpha)}(\cdot) - T_n^{(\alpha)}(\cdot, f) \right\|_{L_w^p} &\leqslant \left\| f^{(\alpha)}(\cdot) - W_n(\cdot, f^{(\alpha)}) \right\|_{L_w^p} + \left\| T_n^{(\alpha)}(\cdot, W_n(f)) - T_n^{(\alpha)}(\cdot, f) \right\|_{L_w^p} \\ &+ \left\| W_n^{(\alpha)}(\cdot, f) - T_n^{(\alpha)}(\cdot, W_n(f)) \right\|_{L_w^p} =: I_1 + I_2 + I_3. \end{aligned}$$

We denote by $T_n^*(x,f)$ the best approximating polynomial of degree at most n to f in $L_w^p(\mathbf{T})$. In this case, from the boundedness of in $L_w^p(\mathbf{T})$ we have

$$\begin{aligned} I_1 &\leqslant \left\| f^{(\alpha)}(\cdot) - T_n^*(\cdot, f^{(\alpha)}) \right\|_{L_w^p} + \left\| T_n^*(\cdot, f^{(\alpha)}) - W_n(\cdot, f^{(\alpha)}) \right\|_{L_w^p} \\ &\leqslant c(p)E_n(f^{(\alpha)})_{L_w^p} + \left\| W_n(\cdot, T_n^*(f^{(\alpha)}) - f^{(\alpha)}) \right\|_{L_w^p} \leqslant c(p,\alpha)E_n(f^{(\alpha)})_{L_w^p}. \end{aligned}$$

By [10, Theorem 1]

$$I_2 \leqslant c(p,\alpha)n^\alpha \left\| T_n(\cdot, W_n(f)) - T_n(\cdot, f) \right\|_{L_w^p}$$

and

$$\begin{aligned} I_3 &\leqslant c(p,\alpha)(2n)^{\alpha}\left\|W_n(\cdot,f)-T_n(\cdot,W_n(f))\right\|_{L_w^p} \\ &\leqslant c(p,\alpha)(2n)^{\alpha}E_n(W_n(f))_{L_w^p}. \end{aligned}$$

Now we have

$$\begin{aligned} &\left\|T_n(\cdot,W_n(f))-T_n(\cdot,f)\right\|_{L_w^p} \\ &\leqslant \left\|T_n(\cdot,W_n(f))-W_n(\cdot,f)\right\|_{L_w^p}+\left\|W_n(\cdot,f)-f(\cdot)\right\|_{L_w^p}+\left\|f(\cdot)-T_n(\cdot,f)\right\|_{L_w^p} \\ &\leqslant c(p)E_n(W_n(f))_{L_w^p}+c(p)E_n(f)_{L_w^p}+c(p)E_n(f)_{L_w^p}. \end{aligned}$$

Since

$$E_n(W_n(f))_{L_w^p}\leqslant c(p)E_n(f)_{L_w^p},$$

we get

$$\begin{aligned} &\left\|f^{(\alpha)}(\cdot)-T_n^{(\alpha)}(\cdot,f)\right\|_{L_w^p} \\ &\leqslant c(p,\alpha)E_n(f^{(\alpha)})_{L_w^p}+c(p)n^{\alpha}E_n(W_n(f))_{L_w^p}+c(p)n^{\alpha}E_n(f)_{L_w^p} \\ &\quad +c(p,\alpha)(2n)^{\alpha}E_n(W_n(f))_{L_w^p} \\ &\leqslant c(p,\alpha)E_n(f^{(\alpha)})_{L_w^p}+c(p,\alpha)n^{\alpha}E_n(f)_{L_w^p}. \end{aligned}$$

Since [1]

$$E_n(f)_{L_w^p}\leqslant \frac{c(p,\alpha)}{(n+1)^{\alpha}}E_n(f^{(\alpha)})_{L_w^p}, \tag{5}$$

we obtain

$$\left\|f^{(\alpha)}(\cdot)-T_n^{(\alpha)}(\cdot)\right\|_{L_w^p}\leqslant cE_n(f^{(\alpha)})_{L_w^p}$$

and the proof is completed. □

Proof of Theorem 2. Let $T_n\in\Pi_n$ be the trigonometric polynomial of the best approximation of f in $L_w^p(\mathbf{T})$ metric. By Remark 1 (*ii*), Lemma 1 and (3) we get

$$\begin{aligned} \Omega_r(f,h)_{L_w^p} &\leqslant \Omega_r(T_n,h)_{L_w^p}+\Omega_r(f-T_n,h)_{L_w^p} \\ &\leqslant c(p,r)h^r\left\|T_n^{(r)}\right\|_{L_w^p}+c(p,r)E_n(f)_{L_w^p}, \quad 0<h<\pi/n. \end{aligned}$$

Using (5), the direct inequality in [9, Theorem 2] and the inequality

$$\Omega_l(f,h)_{L_w^p}\leqslant ch^l\left\|f^{(l)}\right\|_{L_w^p}, \ \ f\in W_{p,w}^l(\mathbf{T}), l=1,2,3,...,$$

given in [9, Theorem 1], we have

$$\begin{aligned} E_n(f)_{L_w^p} &\leqslant \frac{c(p,r)}{(n+1)^{r-[r]}}E_n(f^{(r-[r])})_{L_w^p}\leqslant \frac{c(p,r)}{(n+1)^{r-[r]}}\Omega_{[r]}\left(f^{(r-[r])},\frac{2\pi}{n+1}\right)_{L_w^p} \\ &\leqslant \frac{c(p,r)}{(n+1)^{r-[r]}}\left(\frac{2\pi}{n+1}\right)^{[r]}\left\|f^{(r)}\right\|_{L_w^p}. \end{aligned}$$

On the other hand, by Theorem 1 we find

$$\left\|T_n^{(r)}\right\|_{L_w^p} \leqslant \left\|T_n^{(r)} - f^{(r)}\right\|_{L_w^p} + \left\|f^{(r)}\right\|_{L_w^p}$$
$$\leqslant c(p,r)E_n(f^{(r)})_{L_w^p} + \left\|f^{(r)}\right\|_{L_w^p} \leqslant c(p,r)\left\|f^{(r)}\right\|_{L_w^p}.$$

Choosing h with $\pi/(n+1) < h \leqslant \pi/n,\ (n = 1,2,3,...)$, we obtain

$$\Omega_r(f,h)_{L_w^p} \leqslant c(p,r)h^r\left\|f^{(r)}\right\|_{L_w^p}$$

and we are done. □

Proof of Theorem 3. Let S_n be the $n-th$ partial sum of the Fourier series of $f \in L_w^p(\mathbf{T}),\ w \in A_p(\mathbf{T})$ and let $m \in \mathbb{Z}^+$. Thanks to the Theorem of Hunt-Muckenhoupt-Wheeden [6], we obtain that the best approximation by trigonometric polynomials in $L_w^p(\mathbf{T})$ with $w \in A_p(\mathbf{T})$ has the same order as deviation by the partial sum of the Fourier series. It means that for $\varphi \in L_w^p$

$$\|\varphi - S_n(\varphi)\|_{L_w^p} \leqslant cE_n(\varphi)_{L_w^p}$$

with a positive constant c independent on φ and n.

By Remark 1 (ii) and (3)

$$\Omega_r(f,\pi/(n+1))_{L_w^p} \leqslant \Omega_r(f - S_{2^m},\pi/(n+1))_{L_w^p} + \Omega_r(S_{2^m},\pi/(n+1))_{L_w^p}$$
$$\leqslant c(p,r)E_{2^m}(f)_{L_w^p} + \Omega_r(S_{2^m},\pi/(n+1))_{L_w^p}$$

and by Lemma 1,

$$\Omega_r(S_{2^m},\pi/(n+1))_{L_w^p} \leqslant c(p,r)\left(\frac{\pi}{n+1}\right)^r\left\|S_{2^m}^{(r)}\right\|_{L_w^p}, \quad n+1 \geqslant 2^m.$$

Since

$$S_{2^m}^{(r)}(x) = S_1^{(r)}(x) + \sum_{\nu=0}^{m-1}\left\{S_{2^{\nu+1}}^{(r)}(x) - S_{2^\nu}^{(r)}(x)\right\},$$

we have

$$\Omega_r(S_{2^m},\pi/(n+1))_{L_w^p} \leqslant c(p,r)\left(\frac{\pi}{n+1}\right)^r\left\{\left\|S_1^{(r)}\right\|_{L_w^p} + \left\|\sum_{\nu=0}^{m-1}\left[S_{2^{\nu+1}}^{(r)} - S_{2^\nu}^{(r)}\right]\right\|_{L_w^p}\right\}. \quad (6)$$

Applying the weighted version of Littlewood-Paley's theorem [8] and following the method used in [7], we obtain for $1 < p \leqslant 2$

$$\left\|\sum_{\nu=0}^{m-1}\left[S_{2^{\nu+1}}^{(r)}(x) - S_{2^\nu}^{(r)}(x)\right]\right\|_{L_w^p} = \left\|\sum_{\nu=0}^{m-1}\sum_{k=2^\nu+1}^{2^{\nu+1}} B_{k,r}(x)\right\|_{L_w^p}$$
$$\leqslant c\left(\sum_{\nu=0}^{m-1}\left|\sum_{k=2^\nu+1}^{2^{\nu+1}} k^r B_{k,r}(x)\right|_{L_w^p}^2\right)^{\frac{1}{2}}$$

$$\leqslant c\left(\sum_{\nu=0}^{m-1}\left\|\sum_{k=2^{\nu}+1}^{2^{\nu+1}} k^{r}B_{k,r}(x)\right\|_{L_w^p}^{p}\right)^{\frac{1}{p}}$$

$$= c\left(\sum_{\nu=0}^{m-1}\left\|S_{2^{\nu+1}}^{(r)}(x)-S_{2^{\nu}}^{(r)}(x)\right\|_{L_w^p}^{p}\right)^{\frac{1}{p}}$$

where $B_{k,r}(x)$ is the $r-th$ derivative of $(a_k\cos kx + b_k \sin kx)$, and for $p>2$

$$\begin{aligned}\left\|\sum_{\nu=0}^{m-1}\left[S_{2^{\nu+1}}^{(r)}(x)-S_{2^{\nu}}^{(r)}(x)\right]\right\|_{L_w^p} &= \left\|\sum_{\nu=0}^{m-1}\sum_{k=2^{\nu}+1}^{2^{\nu+1}} k^{r}B_{k,r}(x)\right\|_{L_w^p}\\ &\leqslant c\left(\sum_{\nu=0}^{m-1}\left|\sum_{k=2^{\nu}+1}^{2^{\nu+1}} k^{r}B_{k,r}(x)\right|_{L_w^p}^{2}\right)^{\frac{1}{2}}\\ &\leqslant c\left(\sum_{\nu=0}^{m-1}\left\|\sum_{k=2^{\nu}+1}^{2^{\nu+1}} k^{r}B_{k,r}(x)\right\|_{L_w^p}^{2}\right)^{\frac{1}{2}}\\ &= c\left(\sum_{\nu=0}^{m-1}\left\|S_{2^{\nu+1}}^{(r)}(x)-S_{2^{\nu}}^{(r)}(x)\right\|_{L_w^p}^{2}\right)^{\frac{1}{2}}.\end{aligned}$$

Consequently, we have

$$\left\|\sum_{\nu=0}^{m-1}\left[S_{2^{\nu+1}}^{(r)}(x)-S_{2^{\nu}}^{(r)}(x)\right]\right\|_{L_w^p} \leqslant c\left(\sum_{\nu=0}^{m-1}\left\|S_{2^{\nu+1}}^{(r)}(x)-S_{2^{\nu}}^{(r)}(x)\right\|_{L_w^p}^{\gamma}\right)^{\frac{1}{\gamma}}, \quad \gamma=\min\{p,2\}.$$

Hence, by [10, Theorem 1], we get

$$\left\|S_{2^{\nu+1}}^{(r)}(x)-S_{2^{\nu}}^{(r)}(x)\right\|_{L_w^p} \leqslant c(p,r)2^{\nu r}\left\|S_{2^{\nu+1}}(x)-S_{2^{\nu}}(x)\right\|_{L_w^p} \leqslant c(p,r)2^{\nu r+1}E_{2^{\nu}}(f)_{L_w^p}$$

and

$$\left\|S_1^{(r)}\right\|_{L_w^p} = \left\|S_1^{(r)}-S_0^{(r)}\right\|_{L_w^p} \leqslant c(p,r)E_0(f)_{L_w^p}.$$

Then from (6) we have

$$\Omega_r(S_{2^m},\pi/(n+1))_{L_w^p} \leqslant c(p,r)\left(\frac{\pi}{n+1}\right)^{r}\left\{E_0(f)_{L_w^p}+\left(\sum_{\nu=0}^{m-1}2^{(\nu+1)r\gamma}E_{2^{\nu}}^{\gamma}(f)_{L_w^p}\right)^{\frac{1}{\gamma}}\right\}.$$

It is easily seen that

$$2^{(\nu+1)r\gamma}E_{2^{\nu}}^{\gamma}(f)_{L_w^p} \leqslant c(r)\sum_{\mu=2^{\nu-1}+1}^{2^{\nu}}\mu^{\gamma r-1}E_{\mu}^{\gamma}(f)_{L_w^p}, \quad \nu=1,2,3,.... \tag{7}$$

Therefore,

$$\begin{aligned}
&\Omega_r(S_{2^m},\pi/(n+1))_{L^p_w}\\
&\leqslant c(p,r)\left(\frac{\pi}{n+1}\right)^r\left\{E_0(f)_{L^p_w}+2^rE_1(f)_{L^p_w}+c(r)\left(\sum_{\nu=0}^{m-1}\sum_{\mu=2^{\nu-1}+1}^{2^\nu}\mu^{\gamma r-1}E_\mu^\gamma(f)_{L^p_w}\right)^{\frac{1}{\gamma}}\right\}\\
&\leqslant c(p,r)\left(\frac{\pi}{n+1}\right)^r\left\{E_0(f)_{L^p_w}+\left(\sum_{\mu=1}^{2^m}\mu^{\gamma r-1}E_\mu^\gamma(f)_{L^p_w}\right)^{\frac{1}{\gamma}}\right\}\\
&\leqslant c(p,r)\left(\frac{\pi}{n+1}\right)^r\left(\sum_{\nu=0}^{2^m-1}(\nu+1)^{\gamma r-1}E_\nu^\gamma(f)_{L^p_w}\right)^{\frac{1}{\gamma}}.
\end{aligned}$$

If we choose $2^m\leqslant n+1\leqslant 2^{m+1}$, then

$$\Omega_r(S_{2^m},\pi/(n+1))_{L^p_w}\leqslant\frac{c(p,r)}{(n+1)^r}\left(\sum_{\nu=0}^{n}(\nu+1)^{\gamma r-1}E_\nu^\gamma(f)_{L^p_w}\right)^{\frac{1}{\gamma}}.$$

Taking also the relation

$$E_{2^m}(f)_{L^p_w}\leqslant E_{2^{m-1}}(f)_{L^p_w}\leqslant\frac{c(p,r)}{(n+1)^r}\left(\sum_{\nu=0}^{n}(\nu+1)^{\gamma r-1}E_\nu^\gamma(f)_{L^p_w}\right)^{\frac{1}{\gamma}}$$

into account we obtain the required inequality of Theorem 3. □

Proof of Theorem 4. If T_n is the best approximating polynomial of f, then by [10, Theorem 1]

$$\left\|T^{(\alpha)}_{2^{m+1}}-T^{(\alpha)}_{2^m}\right\|_{L^p_w}\leqslant c(p,\alpha)2^{(m+1)\alpha}E_{2^m}(f)_{L^p_w}$$

and hence by this inequality, (7) and hypothesis of Theorem 4 we have

$$\begin{aligned}
\sum_{m=1}^{\infty}\|T_{2^{m+1}}-T_{2^m}\|_{W^\alpha_{p,w}(\mathbf{T})}&=\sum_{m=1}^{\infty}\|T_{2^{m+1}}-T_{2^m}\|_{L^p_w}+\sum_{m=1}^{\infty}\left\|T^{(\alpha)}_{2^{m+1}}-T^{(\alpha)}_{2^m}\right\|_{L^p_w}\\
&\leqslant c(p,\alpha)\sum_{m=1}^{\infty}2^{(m+1)\alpha}E_{2^m}(f)_{L^p_w}\\
&\leqslant c(p,\alpha)\sum_{m=1}^{\infty}\sum_{j=2^{m-1}+1}^{2^m}j^{\alpha-1}E_j(f)_{L^p_w}\\
&\leqslant c(p,\alpha)\sum_{j=2}^{\infty}j^{\alpha-1}E_j(f)_{L^p_w}<\infty.
\end{aligned}$$

Therefore

$$\sum_{m=1}^{\infty} \|T_{2^{m+1}} - T_{2^m}\|_{W^{\alpha}_{p,w}(\mathbf{T})} < \infty,$$

which implies that $\{T_{2^m}\}$ is a Cauchy sequence in $W^{\alpha}_{p,w}(\mathbf{T})$. Since $T_{2^m} \to f$ in the Banach space $L^p_w(T)$, we have $f \in W^{\alpha}_{p,w}(\mathbf{T})$.

It is clear that

$$\begin{aligned} E_n(f^{(\alpha)})_{L^p_w} &\leqslant \left\| f^{(\alpha)} - S_n f^{(\alpha)} \right\|_{L^p_w} \\ &\leqslant \left\| S_{2^{m+2}} f^{(\alpha)} - S_n f^{(\alpha)} \right\|_{L^p_w} + \left\| \sum_{k=m+2}^{\infty} \left[S_{2^{k+1}} f^{(\alpha)} - S_{2^k} f^{(\alpha)} \right] \right\|_{L^p_w} . \end{aligned}$$

By [10, Theorem 1]

$$\begin{aligned} \left\| S_{2^{m+2}} f^{(\alpha)} - S_n f^{(\alpha)} \right\|_{L^p_w} &\leqslant c(p,\alpha) 2^{(m+2)\alpha} E_n(f)_{L^p_w} \\ &\leqslant c(p,\alpha)(n+1)^{\alpha} E_n(f)_{L^p_w} \end{aligned}$$

for $2^m < n < 2^{m+1}$.

On the other hand, following the method given in the proof of Theorem 3, we get

$$\left\| \sum_{k=m+2}^{\infty} \left[S_{2^{k+1}} f^{(\alpha)} - S_{2^k} f^{(\alpha)} \right] \right\|_{L^p_w} \leqslant c \left(\sum_{k=m+2}^{\infty} \left\| S^{(\alpha)}_{2^{k+1}}(x) - S^{(\alpha)}_{2^k}(x) \right\|^{\gamma}_{L^p_w} \right)^{\frac{1}{\gamma}}, \quad \gamma = \min\{p,2\}$$

Since by [10, Theorem 1]

$$\left\| S^{(\alpha)}_{2^{k+1}}(x) - S^{(\alpha)}_{2^k}(x) \right\|_{L^p_w} \leqslant c(p,\alpha) 2^{k\alpha} \left\| S_{2^{k+1}}(x) - S_{2^k}(x) \right\|_{L^p_w} \leqslant c(p,\alpha) 2^{k\alpha+1} E_{2^k}(f)_{L^p_w},$$

we get

$$\left\| \sum_{k=m+2}^{\infty} \left[S_{2^{k+1}} f^{(\alpha)} - S_{2^k} f^{(\alpha)} \right] \right\|_{L^p_w} \leqslant c \left(\sum_{k=m+2}^{\infty} 2^{\gamma k\alpha+1} E^{\gamma}_{2^k}(f)_{L^p_w} \right)^{\frac{1}{\gamma}} .$$

Therefore, we have

$$\left\| \sum_{k=m+2}^{\infty} \left[S_{2^{k+1}} f^{(\alpha)} - S_{2^k} f^{(\alpha)} \right] \right\|_{L^p_w} \leqslant c \left(\sum_{k=n+1}^{\infty} k^{\gamma\alpha-1} E^{\gamma}_{k}(f)_{L^p_w} \right)^{\frac{1}{\gamma}}$$

for $2^m < n < 2^{m+1}$. This completes the proof. $\square$

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