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A SUPPORT THEOREM FOR t–WRIGHT–CONVEX FUNCTIONS

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Abstract. The support theorems play a very important role in the theory of convex functions and have many consequences. In the present paper we give a necessary and sufficient conditions under which every t-Wright convex function has at arbitrary point a t-Wright affine support function.

Introduction and terminology

Let $t \in (0,1)$ be a fixed number, and let $L(t)$ be the smallest field containing a singleton $\{t\}$. Throughout the whole paper X will always denote a linear space over the field K, where $L(t) \subset K \subset \mathbb{R}$, and, moreover, $D \subset X$ will always be a non-empty t-convex set i.e. such set that $tD + (1-t)D \subset D$, and these assumptions will not be repeated in the sequel.

A function $f : D \to \mathbb{R}$ is said to be t-convex if

$$\bigwedge_{x,y \in D} f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y);$$

t-Wright convex if

$$\bigwedge_{x,y \in D} f(tx + (1-t)y) + f((1-t)x + ty) \leqslant f(x) + f(y). \tag{1}$$

If inequality (1) is satisfied for $t = \frac{1}{2}$ then we say that function f is Jensen-convex and in this case (1) has the following form

$$\bigwedge_{x,y \in D} f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x) + f(y)}{2}.$$

If $f : D \to \mathbb{R}$ is a function such that $-f$ is t-convex (t-Wright convex, Jensen-convex) then f is called t-concave (t-Wright concave, Jensen concave), respectively. One can easily observe that every t-convex function is obviously t-Wright convex but in general not converse [10]. It has been proved by N. Kuhn [7] and independently, by Z. Daróczy and Zs. Páles [16] that every t-convex function is Jensen-convex. The connection

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between t-Wright convex and Jensen-convex functions has been investigated in [9]. The authors solved a problem posed by J. Matkowski. In [10] Matkowski asked whether t-Wright convex function with a $t \in (0,1)$ has to be Jensen-convex? In [9] Gy. Maksa, K. Nikodem and Zs. Páles gave the positive answer to the problem of Matkowski for all rational $t \in (0,1)$ and certain algebraic values of t. However, they have proved that if t is either transcedental or the distance of some of the algebraic (maybe complex) conjugate of t from $\frac{1}{2}$ is at least $\frac{1}{2}$, then there is a function $f : D \to \mathbb{R}$ which is t-Wright convex but not Jensen convex. The function constructed in [9] has many pathological properties, in particular it must be discontinuous.

A survey of other results concerning t-Wright convex functions may be found in the papers [2], [10], [11], [14], [15].

In the sequel we will use the following definition

DEFINITION 1. A point $x_0 \in X$ is termed to be algebraically internal for a set $A \subset X$ if

$$\bigwedge_{y \in X \setminus \{0\}} \bigvee_{\varepsilon_y > 0} \quad x_0 + \lambda y \in A, \qquad \lambda \in (-\varepsilon_y, \varepsilon_y) \cap L(t).$$

The set of all algebraically internal points of A will be denoted by $\operatorname{algint}_{L(t)} A$.

If $f : D \to \mathbb{R}$ is at the same time t-Wright convex and t-Wright concave then we say that f is a t-Wright affine function. Obviously every t-Wright affine function satisfies the following functional equation

$$\bigwedge_{x,y \in D} f(tx + (1-t)y) + f((1-t)x + ty) = f(x) + f(y).$$

There is known a characterization of t-Wright affine functions. Theorem 1, below has been proved by K. Lajkó [8] for functions defined on an open interval and was extended in the paper [13] for functions defined on more general structures.

THEOREM 1. *Let $D \subset X$ be a s-convex set for all $s \in L(t) \cap (0,1)$, such that* $\operatorname{algint}_{L(t)} D \neq \emptyset$. *A function $f : D \to \mathbb{R}$ is a t-Wright affine if and only if f has the form*

$$f(x) = a_0 + a_1(x) + a_2(x,x), \qquad x \in D,$$

where $a_0 \in \mathbb{R}$ is a constant, $a_1 : X \to \mathbb{R}$ is an additive function, and $a_2 : X \times X \to \mathbb{R}$ is a biadditive and symetric function and, moreover,

$$a_2(tx, (1-t)x) = 0, \qquad x \in X.$$

In the first section we prove a separation theorem for t-Wright convex functions. An analogical theorem for Jensen convex functions ($t = \frac{1}{2}$) is well-known (see [5], [12]) and it is a conscequence of an abstract version of Hahn-Banach theorem due to G. Rodé [18]. In section 2 we prove some lemmas which are useful in the next section. The main result of the paper are presented in the third section. Using the results from sections 1 and 2 we give a necessary and sufficient condition under which for an arbitrary point $y \in D$ and a t-Wright convex function $f : D \to \mathbb{R}$ there exists a t-Wright affine support function of f at y i.e. the function $a_y : D \to \mathbb{R}$ satysfying the following conditions

$$\begin{aligned}
&(i) \quad a_y(tx+(1-t)z)+a_y((1-t)x+tz)=a_y(x)+a_y(z), \quad x,z \in D,\\
&(ii) \quad a_y(x) \leqslant f(x), \quad x \in D,\\
&(iii) \quad a_y(y)=f(y).
\end{aligned}$$

As a special case of our theorem (for $t=\frac{1}{2}$) we obtain a well-known support theorem for Jensen-convex functions (see [3], [4]) but our proof is quite diffrent and it seems to bee easier. Moreover, the support function obtained in this paper may be estimated by the values of a given function f.

1. The separation theorem for t-Wright convex functions

In the sequel we will use the following notations:

$$M(x,y) := tx+(1-t)y, \qquad N(x,y) := (1-t)x+ty,$$

and if $f,g : D \to \mathbb{R}$ are arbitrary functions, $z \in D$, then

$$f_z(x) := f(M(x,z))+f(N(x,z))-g(z), \quad x \in D.$$

LEMMA 1. *Let $f : D \to \mathbb{R}$ be a t-Wright convex function, and let $g : D \to \mathbb{R}$ be a t-Wright concave function. If*

$$\bigwedge_{x\in D} \Big(\inf_{z\in D}\{f_z(x)\} > -\infty \Big),$$

then the function $h : D \to \mathbb{R}$ given by the formula

$$h(x) := \inf_{z\in D}\{f_z(x)\},\ x \in D, \tag{2}$$

is t-Wright convex.

Proof. Fix arbitrary $u,v \in D$ and $\varepsilon > 0$. According to (2) we can find points $\alpha,\beta \in D$ such that

$$h(u)+\varepsilon > f_\alpha(u) \qquad \text{and} \qquad h(v)+\varepsilon > f_\beta(v).$$

By the t-Wright convexity of the functions f and $-g$ we get

$$\begin{aligned}
h(u)+h(v)+2\varepsilon &> f_\alpha(u)+f_\beta(v)\\
&= f(M(u,\alpha))+f(N(u,\alpha))-g(\alpha)+f(M(v,\beta))+f(N(v,\beta))-g(\beta)\\
&\geqslant f(M(M(u,\alpha),M(v,\beta)))+f(N(M(u,\alpha),M(v,\beta)))-g(M(\alpha,\beta))\\
&\quad + f(M(N(u,\alpha),N(v,\beta)))+f(N(N(u,\alpha),N(v,\beta)))-g(N(\alpha,\beta))\\
&= f(M(M(u,v),M(\alpha,\beta)))+f(N(M(u,v),M(\alpha,\beta)))-g(M(\alpha,\beta))\\
&\quad +f(M(N(u,v),N(\alpha,\beta)))+f(N(N(u,v),N(\alpha,\beta)))-g(N(\alpha,\beta))\\
&= f_{M(\alpha,\beta)}(M(u,v))+f_{N(\alpha,\beta)}(N(u,v)) \geqslant h(M(u,v))+h(N(u,v)).
\end{aligned}$$

Letting $\varepsilon \to 0_+$ we infer that h is a t-Wright convex function so the proof of our Lemma is finished. $\square$

We apply this lemma to the proof of the following

THEOREM 2. *Let $f : D \to \mathbb{R}$ be a t-Wright convex function and let $g : D \to \mathbb{R}$ be a t-Wright concave function. If*

$$\bigwedge_{x\in D} g(x) \leqslant f(x),$$

then there exists a t-Wright affine function $a : D \to \mathbb{R}$ such that

$$g(x) \leqslant a(x) \leqslant f(x), \qquad x \in D.$$

Proof. Without loss of generality we may assume that

$$\alpha := \inf_{z\in D} [f(z) - g(z)] = 0, \tag{3}$$

otherwise we will consider the function $f(x) - \alpha$ instead of $f(x)$. Let us define the following family of functions

$$\mathscr{H} := \{h : D \to \mathbb{R} : \ h \text{ is a } t\text{-Wright convex function and } \ g(x) \leqslant h(x) \leqslant f(x),\ x \in D\}.$$

Observe that $\mathscr{H} \neq \emptyset$, because $f \in \mathscr{H}$. The pair $(\mathscr{H}, \preceq)$ yields on partially ordered set, where the order is defined in the following manner

$$f_1 \preceq f_2 \iff \bigwedge_{x\in D} \Big(f_1(x) \leqslant f_2(x) \Big).$$

We will show that every chain has a lower bound in $\mathscr{H}$. Let $\mathscr{L} \subset \mathscr{H}$ be an arbitrary chain. Define the function $\widetilde{h} : D \to \mathbb{R}$ by the formula

$$\widetilde{h}(x) := \inf\{h(x) : h \in \mathscr{L}\}, \quad x \in D. \tag{4}$$

It follows from (4) that

$$g(x) \leqslant \widetilde{h}(x) \leqslant f(x), \quad x \in D.$$

It is suffices to show that $\widetilde{h}$ is a t-Wright convex function. To see it fix arbitrary points $x, y \in D$ and a number $\varepsilon > 0$. By the definition of $\widetilde{h}$ there exist $h_1, h_2 \in \mathscr{L}$ such that

$$\widetilde{h}(x) + \varepsilon > h_1(x) \qquad \text{and} \qquad \widetilde{h}(y) + \varepsilon > h_2(y).$$

Hence putting $h_3 := \min\{h_1, h_2\}$ we have

$$\begin{aligned} \widetilde{h}(x) + \widetilde{h}(y) + 2\varepsilon > h_1(x) + h_2(y) &\geqslant h_3(x) + h_3(y) \\ &\geqslant h_3(M(x,y)) + h_3(N(x,y)) \\ &\geqslant \widetilde{h}(M(x,y)) + \widetilde{h}(N(x,y)). \end{aligned}$$

Letting $\varepsilon \to 0_+$ we obtain hence the t-Wright convexity of the function $\widetilde{h}$ and on account of (4) this element is a lower bound of chain $\mathscr{L}$. By Kuratowski-Zorn Lemma there exists a minimal element H in $\mathscr{H}$. Observe that, because $H \in \mathscr{H}$ then the following inequalities

$$g(x)+g(z) \leqslant g(M(x,z))+g(N(x,z)) \leqslant H(M(x,z))+H(N(x,z)) \leqslant H(x)+H(z),$$

holds true for all $x,z \in D$. Therefore

$$\begin{aligned} g(x) &\leqslant g(M(x,z))+g(N(x,z))-g(z) \leqslant H(M(x,z))+H(N(x,z))-g(z) \\ &\leqslant H(x)+H(z)-g(z). \end{aligned}$$

It follows from (3) that

$$0 \leqslant \inf_{z\in D}[H(z)-g(z)] \leqslant \inf_{z\in D}[f(z)-g(z)] = 0,$$

whence

$$g(x) \leqslant \inf_{z\in D}[H(M(x,z))+H(N(x,z))-g(z)] \leqslant H(x), \quad x \in D. \tag{5}$$

On account of Lemma 1 the function $\widetilde{H}: D \to \mathbb{R}$ given by the formula

$$\widetilde{H}(x) := \inf_{z\in D}[H(M(x,z))+H(N(x,z))-g(z)], \quad x \in D,$$

is a t-Wright convex function and therefore by the inequality (5) it belongs to the family $\mathscr{H}$. By the minimality of H we get

$$\bigwedge_{x,z\in D} H(x) \leqslant H(M(x,z))+H(N(x,z))-g(z),$$

and then (replace x and z) by t-Wright convexity of H we get

$$\bigwedge_{x,z\in D} g(x) \leqslant H(M(x,z))+H(N(x,z))-H(z) \leqslant H(x).$$

However, because for an arbitrary fixed point $z \in D$ the function $H_z: D \to \mathbb{R}$ given by formula

$$H_z(x) := H(M(x,z))+H(N(x,z))-H(z),$$

is t-Wright convex then it belongs to the family $\mathscr{H}$ and according again to the minimality of H we get

$$\bigwedge_{x,z\in D} H(M(x,z))+H(N(x,z)) = H(x)+H(z).$$

This means that H is a t-Wright affine function and the proof of Theorem 2 is completed. $\square$

2. Some lemmas

In this section we prove some lemmas which will be useful in the next section. Let us define the sequences of means $M_n,\ N_n : D\times D \to D,\ n\in\mathbb{N}$ in the following manner:

$$M_1(x,y) := M(x,y) = tx+(1-t)y, \qquad N_1(x,y) := N(x,y) = (1-t)x+ty,$$

$$M_{n+1}(x,y) := M(M_n(x,y),N_n(x,y)),$$
$$N_{n+1}(x,y) := N(M_n(x,y),N_n(x,y)).$$

Obviously $N_n(x,y)=M_n(y,x)$ and $M_n(x,x)=N_n(x,x)=x$, for all $x,y\in D,\ n\in\mathbb{N}$, and an easy induction shows that

$$M_n(x,y) = s_nx+(1-s_n)y, \qquad N_n(x,y) = (1-s_n)x+s_ny, \tag{6}$$

where $s_n := \frac{(2t-1)^n+1}{2},\ n\in\mathbb{N}$.

Note that, if $f: D\to\mathbb{R}$ is a t-Wright convex function then for all $n\in\mathbb{N}$ we have

$$f(M_{n+1}(x,y))+f(N_{n+1}(x,y)) \leqslant f(M_n(x,y))+f(N_n(x,y)), \quad x,y\in D. \tag{7}$$

For arbitrary $y\in D$ we define D_y by the formula $D_y := D\cap(2y-D)$. Observe that t-convexity of D implies t-convexity of D_y. Moreover, D_y is symetric with respect to y.

LEMMA 2. *For an arbitrary $y\in D,\ x,z\in D_y$ the following formulas*

(i′) $M_n(M(x,z),2y-M(x,z)) = M(M_n(x,2y-x),M_n(z,2y-z))$,
(ii′) $N_n(M(x,z),2y-M(x,z)) = M(N_n(x,2y-x),N_n(z,2y-z))$,
(iii′) $M_n(N(x,z),2y-N(x,z)) = N(M_n(x,2y-x),M_n(z,2y-z))$,
(iv′) $N_n(N(x,z),2y-N(x,z)) = N(N_n(x,2y-x),N_n(z,2y-z))$.

(i″) $M(M(M_n(x,2y-x),M_n(z,2y-z)),N(N_n(x,2y-x),N_n(z,2y-z)))$
$= M_{n+1}(x,2y-x)$,
(ii″) $N(M(M_n(x,2y-x),M_n(z,2y-z)),N(N_n(x,2y-x),N_nz,2y-z)))$
$= N_{n+1}(z,2y-z)$,
(iii″) $M(M(N_n(x,2y-x),N_n(z,2y-z)),N(M_n(x,2y-x),M_n(z,2y-z)))$
$= N_{n+1}(x,2y-x)$,
(iv″) $N(M(N_n(x,2y-x),N_n(z,2y-z)),N(M_n(x,2y-x),M_n(z,2y-z)))$
$= M_{n+1}(z,2y-z)$.

holds true for all $n \in \mathbb{N}$.

Proof. Fix arbitrary $y \in D$, and take $x, z \in D_y$ and $n \in \mathbb{N}$. Then by (6) we get

$$M_n(x, 2y - x) = y + (2t-1)^n(x - y),$$
$$N_n(x, 2y - x) = y - (2t-1)^n(x - y).$$

Therefore

$$\begin{aligned}
&M_n(M(x,z), 2y - M(x,z)) \\
&\quad = y + (2t-1)^n\big(M(x,z) - y)y + (2t-1)^n[tx + (1-t)z - ty - (1-t)y] \\
&\quad = ty + t(2t-1)^n(x-y) + (1-t)y + (1-t)(2t-1)^n(z-y) \\
&\quad = t[y + (2t-1)^n(x-y)] + (1-t)[y + (2t-1)^n(z-y)] \\
&\quad = M(M_n(x, 2y-x), M_n(z, 2y-z)).
\end{aligned}$$

This proves (i').

We have also

$$\begin{aligned}
&M(M(M_n(x,2y-x), M_n(z,2y-z)), N(N_n(x,2y-x), N_n(z,2y-z))) \\
&\quad = tM(M_n(x,2y-x), M_n(z,2y-z)) + (1-t)N(N_n(x,2y-x), N_n(z,2y-z)) \\
&\quad = t[tM_n(x,2y-x) + (1-t)M_n(z,2y-z)] \\
&\qquad + (1-t)[(1-t)N_n(x,2y-x) + tN_n(z,2y-z)] \\
&\quad = t\{t[y + (2t-1)^n(x-y)] + (1-t)[y + (2t-1)^n(z-y)]\} \\
&\qquad + (1-t)\{(1-t)[y - (2t-1)^n(x-y)] + t[y - (2t-1)^n(z-y)]\} \\
&\quad = [t^2 + 2t(1-t) + (1-t)^2]y + [t^2 - (1-t)^2](2t-1)^n(x-y) \\
&\quad = y + (2t-1)^{n+1}(x-y) \\
&\quad = M_{n+1}(x, 2y-x).
\end{aligned}$$

This shows (i”). We omit the proofs of the remaining equalities because they runs similarly. □

For a function $f : D \to \mathbb{R}$ we put

$$W_f(x,z) := f(x) + f(z) - f(M(x,z)) - f(N(x,z)), \qquad x, z \in D.$$

Clearly, the function f is a t-Wright convex (t-Wright concave, t-Wright affine), if

$$W_f(x,z) \geqslant 0 \quad (W_f(x,z) \leqslant 0,\ W_f(x,z) = 0,\ \text{respectively}),\ \text{for all } x, z \in D.$$

LEMMA 3. *Let* $f : D \to \mathbb{R}$ *be a* t*-Wright convex function. If for some* $x, z \in D$ *we have*

$$\lim_{n\to\infty}[f(M_n(x,z)) + f(N_n(x,z))] > -\infty, \tag{8}$$

then

$$\lim_{n\to\infty} W_f(M_n(x,z), N_n(x,z)) = 0.$$

Proof. It follows from (7) that the limit (8) (finite or infinite) exists. Assume that

$$\lim_{n\to\infty}[f(M_n(x,z))+f(N_n(x,z))]=\alpha>-\infty.$$

For arbitrary positive integer n we have

$$\begin{aligned}&f(x)+f(z)-f(M_{n+1}(x,z))-f(N_{n+1}(x,z))\\&\quad=f(x)+f(z)-f(M_1(x,z))-f(N_1(x,z))\\&\qquad+\textstyle\sum_{i=1}^{n}[f(M_i(x,z))+f(N_i(x,z)-f(M_{i+1}(x,z))+f(N_{i+1}(x,z))].\end{aligned}$$

Letting $n\to\infty$ we get

$$0\leqslant f(x)+f(z)-\alpha=W_f(x,z)+\sum_{k=1}^{\infty}W_f(M_k(x,z),N_k(x,z))<\infty.$$

Therefore,

$$\lim_{n\to\infty}W_f(M_n(x,z),N_n(x,z))=0. \qquad \square$$

3. The support theorem

We start with the following

THEOREM 3. *Let $f:D\to\mathbb{R}$ be a t-Wright convex function, and let $y\in D$ be a fixed point. If*

$$\bigwedge_{x\in D_y}\left(\lim_{n\to\infty}[f(M_n(x,2y-x))+f(N_n(x,2y-x))]>-\infty\right), \tag{$\star$}$$

then the function $A_y:D_y\to\mathbb{R}$ given by the formula

$$A_y(x):=\lim_{n\to\infty}[f(M_n(x,2y-x))+f(N_n(x,2y-x))] \tag{9}$$

is a t-Wright affine function.

Proof. Define a sequence of functions $g_y^n:D_y\to\mathbb{R}$, $n\in\mathbb{N}$ by the formula

$$g_y^n(x):=f(M_n(x,2y-x))+f(N_n(x,2y-x)).$$

It follows from $(\star)$ and (7) that this sequence converges to a finite limit. Let us put

$$A_y(x):=\lim_{n\to\infty}g_y^n(x),\quad x\in D_y.$$

We will show that $W_{A_y}(x,z)=0$, for all $x,z\in D_y$. Using t-Wright convexity of the function f and conditions (i')-(iv') we get

$$\begin{aligned}
W_{g_y^n}(x,z) &= f(M_n(x,2y-x))+f(N_n(x,2y-x))+f(M_n(z,2y-z))+f(N_n(z,2y-z))\\
&\quad -f(M_n(M(x,z),2y-M(x,z)))-f(N_n(M(x,z),2y-M(x,z)))\\
&\quad -f(M_n(N(x,z),2y-N(x,z))-f(N_n(N(x,z),2y-N(x,z)))\\
&= f(M_n(x,2y-x))+f(M_n(z,2y-z))\\
&\quad -f(M(M_n(x,2y-x),M_n(z,2y-z)))-f(N(M_n(x,2y-x),M_n(z,2y-z)))\\
&\quad +f(N_n(x,2y-x))+f(N_n(z,2y-z))\\
&\quad -f(M(N_n(x,2y-x),N_nz,2y-z)))-f(N(N_n(x,2y-x),N_n(z,2y-z)))\\
&\geqslant 0.
\end{aligned}$$

On the other hand by (i")-(iv") we obtain

$$\begin{aligned}
W_{g_y^n}(x,z) &= f(M_n(x,2y-x))+f(N_n(x,2y-x))+f(M_n(z,2y-z))+f(N_n(z,2y-z))\\
&\quad -f(M(M_n(x,2y-x),M_n(z,2y-z)))-f(N(M_n(x,2y-x),M_n(z,2y-z)))\\
&\quad -f(M(N_n(x,2y-x),N_n(z,2y-z)))-f(N(N_n(x,2y-x),N_n(z,2y-z)))\\
&\leqslant f(M_n(x,2y-x))+f(N_n(x,2y-x))+f(M_n(z,2y-z))+f(N_n(z,2y-z))\\
&\quad -f(M(M(M_n(x,2y-x),M_n(z,2y-z)),N(N_n(x,2y-x),N_n(z,2y-z))))\\
&\quad -f(N(M(M_n(x,2y-x),M_n(z,2y-z)),N(N_n(x,2y-x),N_n(z,2y-z))))\\
&\quad -f(M(M(N_n(x,2y-x),N_n(z,2y-z)),N(M_n(x,2y-x),M_n(z,2y-z))))\\
&\quad -f(N(M(N_n(x,2y-x),N_n(z,2y-z)),N(M_n(x,2y-x),M_n(z,2y-z))))\\
&= f(M_n(x,2y-x))+f(N_n(x,2y-x))+f(M_n(z,2y-z))\\
&\quad +f(N_n(z,2y-z))-f(M_{n+1}(x,2y-x))-f(N_{n+1}(x,2y-x))\\
&\quad -f(M_{n+1}(z,2y-z))-f(N_{n+1}(z,2y-z))\\
&= W_f(M_n(x,2y-x),N_n(x,2y-x))+W_f(M_n(z,2y-z),N_n(z,2y-z)).
\end{aligned}$$

We have shown that

$$0\leqslant W_{g_y^n}(x,z)\leqslant W_f(M_n(x,2y-x),N_n(x,2y-x))+W_f(M_n(z,2y-z),N_n(z,2y-z)),\ n\in\mathbb{N}.$$

Letting $n\to\infty$ and using Lemma 3 we infer that

$$W_{A_y}(x,z)=0,\quad x,z\in D_y,$$

which was to be proved. □

REMARK 1. In the case $t=\frac{1}{2}$ we have

$$A_y(x)=2f(y),\quad x\in D_y.$$

Proof. By (6) we put $s_n=1-s_n=\frac{1}{2}$, $n\in\mathbb{N}$. Consequently

$$M_n(x,2y-x)=N_n(x,2y-x)=y,\quad n\in\mathbb{N}. \qquad \square$$

REMARK 2. If, moreover, a point $y \in \operatorname{algint}_{L(t)} D$ then the function A_y given by formula (9) is well defined and t-Wright affine on the whole space X.

Proof. Let us define the sequence of functions $g_y^n : X \to \mathbb{R}$ in the following manner

$$g_y^n(x) := \begin{cases} f(M_n(x,2y-x)) + f(N_n(x,2y-x)), & M_n(x,2y-x), N_n(x,2y-x) \in D_y, \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad, & \text{in the former case.} \end{cases}$$

By the above formula and on account of proof of Theorem 3 it is enought to show that

$$\bigwedge_{x\in X} \bigvee_{n(x)\in\mathbb{N}} \bigwedge_{n\geqslant n(x)} M_n(x,2y-x), N_n(x,2y-x) \in D_y.$$

To see it fix an arbitrary point $x \in X$. We know that

$$M_n(x,2y-x) = y + (2t-1)^n(x-y), \quad N_n(x,2y-x) = y - (2t-1)^n(x-y), \quad n \in \mathbb{N}.$$

Since $y \in \operatorname{algint}_{L(t)} D$ and $\lim_{n\to\infty}(2t-1)^n = 0$ then

$$\bigvee_{n(x)\in\mathbb{N}} \bigwedge_{n\geqslant n(x)} M_n(x,2y-x), N_n(x,2y-x) \in D_y. \qquad \square$$

THEOREM 4. *Let $f : D \to \mathbb{R}$ be a t-Wright convex function and let $y \in D$ be a fixed point. Then, the existens of a function $a_y : D_y \to \mathbb{R}$ satysfying conditions*

(i) $a_y(tx+(1-t)z) + a_y((1-t)x+tz) = a_y(x) + a_y(z), \quad x,z \in D_y,$
(ii) $a_y(x) \leqslant f(x), \quad x \in D_y,$
(iii) $a_y(y) = f(y).$

is equivalent to the following condition

$$\bigwedge_{x\in D_y} \Big(\lim_{n\to\infty} [f(M_n(x,2y-x)) + f(N_n(x,2y-x))] > -\infty \Big). \tag{$\star$}$$

If $(\star)$ holds true and, moreover, $y \in \operatorname{algint}_{L(t)} D$ then the function a_y can be extended to the function fulfilling conditions (i)-(iii) on the whole set D.

Proof. Assume that $a_y : D_y \to \mathbb{R}$ satisfies conditions (i), (ii) and (iii). Suppose that for certain $x \in D_y$ we have

$$\lim_{n\to\infty} [f(M_n(x,2y-x)) + f(N_n(x,2y-x))] = -\infty.$$

According to (i) and (ii) we have

$$\begin{aligned} a_y(x) + a_y(2y-x) &= a_y(M_n(x,2y-x)) + a_y(N_n(x,2y-x)) \\ &\leqslant f(M_n(x,2y-x)) + f(N_n(x,2y-x)), \end{aligned}$$

and hence

$$a_y(x)+a_y(2y-x)=-\infty.$$

Consequently, $a_y(x)=-\infty$ or $a_y(2y-x)=-\infty$, a contradiction.

Now assume that

$$\lim_{n\to\infty}[f(M_n(x,2y-x))+f(N_n(x,2y-x))]>-\infty, \quad x\in D_y.$$

On account of Theorem 3 we infer that a function $A_y:D_y\to\mathbb{R}$ given by the formula

$$A_y(x):=\lim_{n\to\infty}[f(M_n(x,2y-x))+f(N_n(x,2y-x))],$$

is a t-Wright affine. Let us define a function $g_y:D_y\to\mathbb{R}$ by formula

$$g_y(x):=A_y(x)-f(2y-x).$$

Note that g_y is a t-Wright concave function (as a sum of t-Wright affine and t-Wright concave function). Moreover,

$$g_y(x)\leqslant f(x), \quad x\in D_y, \quad \text{and}$$
$$g_y(y)=f(y).$$

On account of Theorem 2 there exists a t-Wright affine function $a_y:D_y\to\mathbb{R}$ such that

$$g_y(x)\leqslant a_y(x)\leqslant f(x), \quad x\in D_y.$$

Clearly, a_y satisfies conditions (i)-(iii).

Now, assume addtionally that $y\in \operatorname{algint}_{L(t)}D$. We will show that a_y can be extended to the support function defined on the whole domain D. Let us define a sequence of sets in the following manner

$$H_t^0(D_y):= \quad D_y,$$
$$H_t^{n+1}(D_y):=\Big\{x\in D:\ M(x,y),N(x,y)\in H_t^n(D_y)\Big\}, \quad n\in\mathbb{N}.$$

Observe that $\{H_t^n(D_y)\}_{n\in\mathbb{N}}$ has the following properties

(l) $H_t^n(D_y)$ is a t-convex set , $n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$,

(ll) $H_t^n(D_y)\subset H_t^{n+1}(D_y)$, $n\in\mathbb{N}_0$,

(lll) $\bigcup_{n=0}^{\infty}H_t^n(D_y)=D$.

The simply proof of the properties (l) and (ll) runs by induction. We proof only the condition (lll). Without loss of generality we may assume that $y=0$. It is easy to see that the set $H_t^n(D_0)$ may be rewriten in the following form

$$H_t^n(D_0)=\Big\{x\in D:\ t^k(1-t)^{n-k}x\in D_0, \quad k\in\{0,\dots,n\}\Big\}, \quad n\in\mathbb{N}_0.$$

We will show that $\bigcup_{n=0}^{\infty} H_t^n(D_0) = D$. Take an arbitrary $x \in X$. Since $0 \in \operatorname{algint}_{L(t)} D_0$ then there exists $\varepsilon_x > 0$ such that

$$\lambda x \in D_0, \qquad \lambda \in L(t) \cap (-\varepsilon_x, \varepsilon_x). \tag{9}$$

Put $a := \max\{t, 1-t\}$. Note that $a \in (0,1)$. By definition of a we have

$$\bigwedge_{n\in\mathbb{N}} \bigwedge_{k\in\{0,\ldots,n\}} 0 < t^k(1-t)^{n-k} \leqslant a^n,$$

and, consequently

$$\bigvee_{n_0\in\mathbb{N}} \bigwedge_{n\geqslant n_0} \bigwedge_{k\in\{0,\ldots,n\}} t^k(1-t)^{n-k} \in L(t) \cap (-\varepsilon_x, \varepsilon_x).$$

Therefore by (9) we get

$$\bigwedge_{n\geqslant n_0} \bigwedge_{k\in\{0,\ldots,n\}} t^k(1-t)^{n-k} x \in D_0,$$

which implies that

$$x \in \bigcup_{n=0}^{\infty} H_t^n(D_0),$$

and finishes the proof of condition (lll).

Now, we define a sequence of functions $a_y^n : H_t^n(D_y) \to \mathbb{R}$, $n \in \mathbb{N}_0$ in the following way

$$a_y^0(x) := a_y(x), \quad x \in D_y,$$
$$a_y^{n+1}(x) := a_y^n(M(x,y)) + a_y^n(N(x,y)) - a_y^n(y), \quad x \in H_t^{n+1}(D_y).$$

It is not hard to check that for all $n \in \mathbb{N}_0$ we have

(a) a_y^n is t-Wright affine function,

(b) $a_y^n(y) = f(y)$,

(c) $a_y^{n+1}(x) = a_y^n(x), \quad x \in H_t^n(D_y)$,

(d) $a_y^n(x) \leqslant f(x), \quad x \in H_t^n(D_y)$.

Finally, let $\overline{a}_y : D \to \mathbb{R}$ be defined by formula

$$\overline{a}_y(x) := \lim_{n\to\infty} a_y^n(x), \qquad x \in D.$$

By conditions (a)-(d) we infer that the function $\overline{a}_y$ is a t-Wright affine support f at the point y on the whole domain D. $\square$

REMARK 3. A t-Wright convex function constructed in the paper [9] don't satisfy the condition $(\star)$, for all $y \in \mathbb{R}$, so the assumption $(\star)$ in our theorems is essential and can not be omitted.

As a concequence of Theorem 4 we obtain following

THEOREM 5. *Let D be an algebraically open set (i.e. $D = \mathrm{algint}_{L(t)} D$) and let $f : D \to \mathbb{R}$ be a t-Wright convex function. The following conditions are equivalent*

$$(k_1) \qquad \bigvee_{y\in D} \bigwedge_{x\in D_y} \left(\lim_{n\to\infty} [f(M_n(x,2y-x)) + f(N_n(x,2y-x))] > -\infty \right),$$

(k_2) *there exists a t-Wright concave function $g : D \to \mathbb{R}$ such that*

$$\bigwedge_{x\in D} g(x) \leqslant f(x), \qquad x \in D,$$

$$(k_3) \qquad \bigwedge_{y\in D} \bigwedge_{x\in D_y} \left(\lim_{n\to\infty} [f(M_n(x,2y-x)) + f(N_n(x,2y-x))] > -\infty \right),$$

(k_4) *for every $y \in D$ there exists a t-Wright affine support function $a_y : D \to \mathbb{R}$.*

Proof. On account of Theorem 4 the implications: $(k_1)\Rightarrow(k_2)$, $(k_3)\Rightarrow(k_4)$, $(k_4)\Rightarrow(k_1)$ are obvious. We prove only the implication $(k_2)\Rightarrow(k_3)$. Fix an arbitrary $y \in D$ and $x \in D_y$. By (k_2), for all $n \in \mathbb{N}$, we obtain

$$\begin{aligned} g(x) + g(2y-x) &\leqslant g(M_n(x,2y-x)) + g(N_n(x,2y-x)) \\ &\leqslant f(M_n(x,2y-x)) + f(N_n(x,2y-x)), \end{aligned}$$

hence

$$-\infty < g(x) + g(2y-x) \leqslant \lim_{n\to\infty} [f(M_n(x,2y-x)) + f(N_n(x,2y-x))].$$

This completes the proof. □

REFERENCES

[1] R. GER, *On extension of polynomial functions*, Results in Math., **26** (1994), 281–289.
[2] Z. KOMINEK, *A continuity result on t-Wright convex functions*, Publ. Math., **63** (2003), 213–219.
[3] Z. KOMINEK, *Convex Functions in Linear Spaces*, Prace Naukowe Uniwersytetu Śląskiego w Katowicach nr 1087, Katowice, 1989.
[4] Z. KOMINEK, *On additive and convex functionals*, Radovi Mat., **3** (1987), 267–279.
[5] H. KÖNIG, *On the abstract Hahn-Banach theorem due to Rodé*, Aequationes Math., **34** (1987), 89–95.
[6] M. KUCZMA, *An introduction to the theory of functional equations and inequalities*, Polish Scientific Publishers and Silesian University Press, Warszawa-Kraków-Katowice, 1985.
[7] N. KUHN, *A note on t-convex functions*, General Inequalites, **4** (1984), 269–276.
[8] K. LAJKÓ, *On a functional equation of Alsina and Garcia-Roig*, Publ. Math, **52**, 3-4 (1998), 507–515.
[9] GY. MAKSA, K. NIKODEM, ZS. PÁLES, *Result on t-Wright convexity*, C. R. Math. Rep. Acad. Sci. Canada, **13** (1991), 274–278.
[10] J. MATKOWSKI, *On a-Wright convexity and the converse of Minkowki's inequalty*, Aequationes Math., **43** (1992), 106–112.
[11] J. MATKOWSKI, M. WRÓBEL, *A generalized α-Wright convexity and related functional equation*, Ann. Math. Sil. No., **10** (1996), 7–12.
[12] K. NIKODEM, *On the support of midconvex operators*, Aequationes Math., **42** (1991), 182–189.

[13] A. OLBRYŚ, *A characterization of* $(t_1,\ldots,t_n)$ *-Wright affine functions*, Commentationes Math., **XLVII**, 1 (2007), 47–56.
[14] A. OLBRYŚ, *On the measurability and the Baire property of* t *-Wright convex functions*, Aequationes Math., **68** (2004), 28–37.
[15] A. OLBRYŚ, *Some conditions implying the continuity of* t *-Wright convex functions*, Publ. Math. Debrecen, **68**, 3-4 (2006), 401–418.
[16] ZS. PÁLES, Z. DARÓCZY, *Convexity with given infinite weight sequences*, Stochastica, **11**, 1 (1987), 5–12.
[17] A. W. ROBERTS, D. E. VARBERG, *Convex Functions*, Academic Press, New York and London, 1973.
[18] G. RODÉ, *Eine abstrakte Version des Satzes von Hahn-Banach*, Arch. Math., **31** (1978).
[19] E. M. WRIGHT, *An inequality for convex functions*, Amer. Math. Monthly, **61** (1954), 620–622.

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