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MONOTONICITY OF FUNCTIONS INVOLVING Q–GAMMA FUNCTIONS

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Abstract. We give monotonicity properties of some functions involving q-Gamma functions. The results are q-analogue of results concerning Gamma function and generalize some known results.

1. Introduction

Let $\Gamma(x)$ be the Gamma function defined by

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dt, \qquad x > 0, \tag{1.1}$$

and $\psi(x)$ be the logarithmic derivative of $\Gamma(x)$ the co-called psi function and defined by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \qquad x > 0. \tag{1.2}$$

The q-analogue of $\Gamma(x)$ is called q-Gamma function, was introduced in 1904 by Jackson and defined for $x > 0$ and $0 < q < 1$ by

$$\Gamma_q(x) = (1-q)^{1-x} \prod_{n=0}^{\infty} \frac{1-q^{n+1}}{1-q^{n+x}}. \tag{1.3}$$

Note that $\lim_{q\to 1^-} \Gamma_q(x) = \Gamma(x)$ and the q-Gamma function satisfies the functional equation

$$\Gamma_q(x+1) = \frac{1-q^x}{1-q}\Gamma_q(x), \qquad \Gamma_q(1) = 1. \tag{1.4}$$

The q-analogue of $\psi(x)$ is called q-psi function and is defined as the logarithmic derivative of $\Gamma_q(x)$,

$$\psi_q(x) = \frac{\Gamma_q'(x)}{\Gamma_q(x)} \tag{1.5}$$

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and from (1.5) in combination with (1.3) it follows

$$\psi_q(x) = -\log(1-q) + \log q \sum_{n=0}^{\infty} \frac{q^{n+x}}{1-q^{n+x}} = -\log(1-q) + \log q \sum_{n=1}^{\infty} \frac{q^{nx}}{1-q^{n}}. \quad (1.6)$$

In the recent years many authors [2-12 and references there in] have proved inequalities and monotonicity properties for functions involving $\Gamma(x)$, $\psi(x)$, $\Gamma_q(x)$ and $\psi_q(x)$. In this paper we present monotonicity properties and inequalities for some functions of $\Gamma_q(x)$, which generalize already known results [4,5,12].

2. Main results

PROPOSITION 2.1. *The function $\psi_q'(x)$ is strictly completely monotonic for $x > 0$ and $0 < q < 1$. This means that the inequality $(-1)^m(\psi_q'(x))^{(m)} > 0$ holds for $m = 0,1,2,\ldots$.*

Proof. Let $x > 0$ and $0 < q < 1$, then from (1.7) we can easily prove that

$$\psi_q'(x) = (\log q)^2 \sum_{n=1}^{\infty} \frac{nq^{nx}}{1-q^{n}} > 0, \quad (2.1)$$

that is the function $\psi_q(x)$ is an increasing function of x, so,

$$\psi_q(x) < \psi_q(y), \quad 0 < x < y. \quad (2.2)$$

From (2.1) we obtain

$$\psi_q''(x) = (\log q)^3 \sum_{n=1}^{\infty} \frac{n^2 q^{nx}}{1-q^{n}} < 0$$

$$\psi_q^{(3)}(x) = (\log q)^4 \sum_{n=1}^{\infty} \frac{n^3 q^{nx}}{1-q^{n}} > 0$$

and by induction we obtain that

$$\psi_q^{(m)}(x) = (\log q)^{m+1} \sum_{n=1}^{\infty} \frac{n^m q^{nx}}{1-q^{n}}, \quad m \geqslant 1,$$

which is positive for m odd and negative for m even, so we obtain the desired result.

REMARK 2.1. Proposition 2.1 is the q-analogue of the known result [1] that the function $\psi'(x)$ is completely monotonic for $x > 0$.

THEOREM 2.1. *Let $0 < q < 1$, $A \leqslant 0$, and $b \geqslant 0$. Then the function*

$$f_q(x) = x^A [\Gamma_q(1 + \frac{b}{x})]^x$$

decreases with respect to $x > 0$.

Proof. The function $f_q(x)$ is positive for $x > 0$. Taking the derivative of $f_q(x)$ with respect to x we obtain:

$$f_q'(x) = \left\{ \frac{A}{x} + \log\Gamma_q\left(1+\frac{b}{x}\right) - \frac{b}{x}\psi_q\left(1+\frac{b}{x}\right) \right\} f_q(x). \tag{2.3}$$

Using the same method which was used for the proof of theorem 2.2 of [5] it follows that the function

$$h_q(x) = \frac{A}{x} + \log\Gamma_q\left(1+\frac{b}{x}\right) - \frac{b}{x}\psi_q\left(1+\frac{b}{x}\right)$$

or by setting $y = \frac{1}{x}$

$$s_q(y) = Ay + \log\Gamma_q(1+by) - by\psi_q(1+by)$$

is negative for $y > 0$ if $A \leqslant 0$ because $s_q'(y) < 0$ and $s_q(y) < s_q(0) = 0$, for $y > 0$. This means that the function $h_q(x)$ is also negative for $x > 0$, so from (2.3) we obtain the desired result. □

REMARK 2.2. Theorem 2.1 extends the region of validity of A, to the negative values, since the theorems 2.2. and 2.3 in [5] deal with $A \geqslant 0$.

COROLLARY 2.1. *Let* $0 < q < 1$. *Then the following inequalities hold:*

$$(i) \qquad \sqrt{x} \leqslant \left[\Gamma_q\left(1+\frac{1}{x}\right)\right]^x, \quad 0 < x \leqslant 1, \tag{2.4}$$

$$(ii) \qquad \sqrt{2x(1+q)} \leqslant \left[\Gamma_q\left(1+\frac{1}{x}\right)\right]^x, \quad 0 < x \leqslant 1/2, \tag{2.5}$$

$$(iii) \qquad [\Gamma_q(x+1)]^{1/x} < \frac{1-q^{1+x}}{1-q} < \frac{1}{1-q}, \quad x > 0 \tag{2.6}$$

and

$$(iv) \qquad \Gamma_q(x) < \frac{1-q^{1+x}}{1-q^x}\left[\frac{1-q^{1+x}}{1-q}\right]^{x-1}, \quad \Gamma_q(x) < \frac{(1-q)^{1-x}}{1-q^x}, \quad x > 0. \tag{2.7}$$

Proof. From theorem 2.1 for $A = -1/2$, $b = 1$ it follows that the function

$$\frac{[\Gamma_q(1+\frac{1}{x})]^x}{\sqrt{x}} \tag{2.8}$$

decreases with respect to $x > 0$. So, using (1.4) and the monotonicity of the function (2.8):

(i) for $0 < x \leqslant 1$ we have the inequality

$$\frac{[\Gamma_q(1+\frac{1}{x})]^x}{\sqrt{x}} \geqslant \Gamma_q(2) = \Gamma_q(1) = 1$$

which proves (2.4) and

(ii) for $0 < x \leqslant 1/2$ we have

$$\frac{[\Gamma_q(1+\frac{1}{x})]^x}{\sqrt{x}} \geqslant \sqrt{2\Gamma_q(3)} = \sqrt{2(1+q)}$$

from which (2.5) follows.

Also, from theorem 2.1, for $A = 0$, we can easily prove that the function $[\Gamma_q(1+bx)]^{1/x}$ increases with respect to x for $x > 0$. So, the following inequality holds:

$$[\Gamma_q(1+bx)]^{1/x} < [\Gamma_q(1+b+bx)]^{1/(1+x)}$$

or

$$[\Gamma_q(1+bx)]^{1+1/x} < [\Gamma_q(1+b+bx)] \tag{2.9}$$

(iii) From (2.9) taking $b = 1$ and using (1.4) we have

$$[\Gamma_q(x+1)]^{1+1/x} < \frac{1-q^{1+x}}{1-q}\Gamma_q(x+1). \tag{2.10}$$

The inequalities (2.6) follow from (2.10), taking in account that $1-q^{x+1} < 1$, for $x > 0$ and $0 < q < 1$.

(iv) From (2.6) using once more the relation (1.4), we obtain

$$\Gamma_q(x+1) < \left[\frac{1-q^{1+x}}{1-q}\right]^x \quad \text{and} \quad \Gamma_q(x+1) < \frac{1}{(1-q)^x}$$

or

$$\frac{1-q^x}{1-q}\Gamma_q(x) < \left[\frac{1-q^{1+x}}{1-q}\right]^x \quad \text{and} \quad \frac{1-q^x}{1-q}\Gamma_q(x) < \frac{1}{(1-q)^x} \tag{2.11}$$

from which the inequalities (2.7) follow. □

THEOREM 2.2. *(i) Let* $0 < q < 1$, $x > 0$, $Mx+N > 0$, $0 < a+bx \leqslant d+ex$ *and* A, c, f *real numbers with* $AM \leqslant 0$ *and* $0 < cb \leqslant fe$. *Then the function*

$$G_q(x) = (Mx+N)^A \frac{\left[\Gamma_q\left(a+bx\right)\right]^c}{[\Gamma_q(d+ex)]^f} \tag{2.12}$$

decreases with x.

(ii) Let $0 < q < 1$, $A \leqslant 0$, $M \geqslant 0$, $0 < a < d$ *and* $c > 0$. *Then the function*

$$g_q(x) = (Mx+N)^A \left[\frac{\Gamma_q(a+x)}{\Gamma_q(d+x)}\right]^c \tag{2.13}$$

is logarithmically completely monotonic for $x > \max\{0, -N/M\}$.

Proof. (i) The derivative of the function $G_q(x)$ with respect of x is

$$G_q'(x) = \left\{ \frac{AM}{Mx+N} + cb\psi_q(a+bx) - fe\psi_q(d+ex) \right\} G_q(x) \tag{2.14}$$

Using proposition (2.1) for $n = 0$ and if $AM \leqslant 0$ and $0 < cd \leqslant fe$ from (2.14) we obtain the desired result.

(ii) The function $g_q(x)$ becomes from $G_q(x)$ for $b = e = 1$ and $f = c > 0$ so (2.14) gives

$$g_q'(x) = \left\{ \frac{AM}{Mx+N} + c[\psi_q(a+x) - \psi_q(d+x)] \right\} g_q(x) < 0 \tag{2.15}$$

Let $\alpha(x) = \frac{AM}{Mx+N}$ and $\beta(x) = \psi_q(x+a) - \psi_q(x+d)$, then (2.15) can be written as

$$\frac{g_q'(x)}{g_q(x)} = \alpha(x) + c\beta(x)$$

or

$$(\ln g_q(x))' = \alpha(x) + c\beta(x) < 0 \tag{2.16}$$

We can verify by induction that $\alpha^{(k)}(x) = \frac{(-1)^k k! AM^{k+1}}{(Mx+N)^{k+1}}$, for $k = 0, 1, 2, \ldots$, so the function $\alpha'(x)$ is completely monotonic, for $A \leqslant 0$ and $M \geqslant 0$. Also using proposition 2.1 the function $\beta'(x)$ is completely monotonic for $x > 0$, if $0 < a < d$, so from (2.16) we obtain the desired result. □

REMARK 2.3. From theorem 2.2 (i), for $A = 0$, we obtain the results given in [12] which generalize the results of [10].

COROLLARY 2.2. *Let* $0 < q < 1$, A, B, a, d *real numbers and* $x > \max\{0, -A, -a, -d\}$. *The positive function*

$$F_q(x) = (x+A)^B \frac{\Gamma_q(a+x)}{\Gamma_q(d+x)} \tag{2.17}$$

(i) *is an increasing function of x for* $x > \max\{0, -A, -d\}$, *if* $a > d$ *and* $B \geqslant 0$, **(ii)** *is a logarithmically completely monotonic function of x for* $x > \max\{0, -A, -a\}$, *if* $a < d$ *and* $B \leqslant 0$ *and* **(iii)** *the function*

$$H_q(x) = \frac{\Gamma_q(d+1+x)}{\Gamma_q(d+x)} \tag{2.18}$$

is completely monotonic function for $x > d > 0$.

Proof. We note that the function $F_q(x)$ becomes from (2.12) by setting $c = f = 1$, $b = e = 1$ and $M = 1$, so

$$(i) \quad F_q'(x) = \left\{ \frac{B}{x+A} + \psi_q(x+a) - \psi_q(x+d) \right\} F_q(x) \tag{2.19}$$

and if $a > d > 0$, $B \geqslant 0$, $x > \max\{0, -A, -d\}$ and using proposition (2.1) for $n = 0$, from (2.19) we obtain the desired result.

(ii) It follows immediately from theorem 2.2 (ii) for $c = 1$ and $M = 1$.

(iii) The function $H_q(x) = \frac{\Gamma_q(d+1+x)}{\Gamma_q(d+x)}$ becomes from $F_q(x)$ for $B = 0$ and $a = d+1$. We note that

$$H_q(x) = \frac{1-q^{x+d}}{1-q} \tag{2.20}$$

taking in account the functional equation (1.4). From (2.20), if $d > 0$ follows easily the desired result, since $0 < q < 1$. □

REMARK 2.4. Corollary 2.2 is the q-analogue of the result proved in [4], concerning the function $(x+A)^B \frac{\Gamma(x+a)}{\Gamma(x+b)}$.

COROLLARY 2.3. *The inequality*

$$\Gamma_q(x+1) > \frac{1}{x+1}\left[\frac{1-q}{1-q^{x+1}}\right]^2 \tag{2.21}$$

holds for $x > 0$.

Proof. From theorem 2.2 (i) for $A = -1/2$, $a = 1$, $d = 2$ $c = 1/2$ and $f = 1$ it follows that the function $\sqrt{\frac{\Gamma_q(1+x)}{1+x}}\frac{1}{\Gamma_q(x+2)}$ decreases with $x > 0$, so

$$\sqrt{\frac{\Gamma_q(1+x)}{1+x}}\frac{1}{\Gamma_q(x+2)} < \frac{\sqrt{\Gamma_q(1)}}{\Gamma_q(2)}$$

or

$$\frac{\Gamma_q(1+x)}{1+x} < [\Gamma_q(x+2)]^2 \tag{2.22}$$

and using (1.4), from (2.22) we obtain the inequality (2.21).

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