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CHEBYSHEV TYPE INEQUALITIES FOR SYNCHRONOUS VECTORS IN BANACH SPACES

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Abstract. In [3] Chebyshev type inequalities are proved for separable finite sequences. In this paper, the applicability of that notion is extended and replaced by a relation of synchronicity. In consequence, new sufficient conditions for Chebyshev type inequalities are derived.

1. Introduction and summary

Let V be a real Banach space with V^*, the space of all linear continuous functionals on V. In this paper we mainly consider inequalities of the form (cf. [3, 9, 10])

$$\langle x,z\rangle \langle y,v\rangle \leqslant \langle y,z\rangle \langle x,v\rangle, \; x,y \in V, \; z,v \in V^*, \tag{1}$$

called here Chebyshev type inequalities. For instance, if $V = V^* = R^n$ and $x = v = (1,\dots,1) \in R^n$, then (1) becomes *Chebyshev sum inequality*; if $V = V^* = L^2$, the space of all 2-nd power integrable functions with respect to the Lebesgue measure μ on the unit interval $[0,1]$, and $x = v \equiv 1$, then (1) takes the form *Chebyshev integral inequality*. It is well known that Chebyshev sum (resp. integral) inequality holds for monotone, or more generally, for similarly ordered sequences (resp. functions) (see e.g. [1, sec. 2.5], [7, sec. 7.1]).

In [3] a notion of separable finite sequences is introduced. Replacing monotone or similarly ordered sequences by separable ones yields generalizations of Chebyshev type inequalities (see [3]) and other inequalities for convex functions (see [4]). In this paper, we introduce a relation of synchronicity which is more coherent for study of (1) than the property of separability for vectors. Simultaneously, the definition of separable vectors is extended beyond finite-dimensional spaces. This allows us to obtain further extensions of Chebyshev type inequalities.

In [9, 10] it is shown that some inequalities related to (1) hold under conditions of similar ordering for sequences or functions. Note, the term “synchronicity” is used there instead of “similar ordering”. As we show (see Remark 5 and Example 1), the similar ordering is a stronger condition for (1) to hold than conditions of synchronicity defined here.

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The paper is organized as follows. Section 2 contains basic notions. In Section 3 a framework for earlier mentioned inequalities is given and some general results are established. In Section 4 we introduce a relation of synchronous vectors and extend the notion of separable vectors onto Banach spaces with Schauder bases. The relation yields sufficient conditions for the fundamental inequality (1) to hold. Section 5 is devoted to applications. We discuss our results in Banach sequence and function spaces with various bases. It amounts to sufficient conditions for Chebyshev inequalities. Between others, we generalize some of the recent results of Niezgoda and Toader (see [3, 9, 11]).

2. Preliminaries

Throughout this note V is a real norm Banach space.

A convex cone is a nonempty set $D \subset V$ such that $\alpha D + \beta D \subset D$ for all nonnegative scalars α and β. The closure of convex cone of all nonnegative finite combinations in $H \subset V$ is denoted by cone H. Similarly, span H denotes the closure of subspace of all finite combinations in H. If $A, B \subset V$, then

$$\text{cone } A + \text{ cone } B = \text{cone } (A \cup B) \tag{2}$$

if only the left hand side of above equality is a closed set.

The symbol V^* stands for the dual space of V, i.e. the space of all linear continuous functionals on V. For $x \in V$ and $v \in V^*$ by $\langle x, v \rangle$ we mean acting of a functional v on x.

For subsets $C \subset V$ and $F \subset V^*$ we define dual cones as follows

$$\text{dual}\, C = \{f \in V^* : \langle C, f \rangle \geqslant 0\}, \quad \text{dual}^* F = \{x \in V : \langle x, F \rangle \geqslant 0\}.$$

In the case of self-dual spaces (i.e. $V = V^*$), the notions of dual cones dual$(\cdot)$ and dual$^*(\cdot)$ are equivalent. Particularly, if D is a subset of L^p, $p > 1$, the space of all p-integrable functions with respect to Lebesgue measure μ on the unit interval $[0,1]$, then we use the following unique symbol for dual cones

$$\text{dual}_q D = \left\{ f \in L^q : \int fg d\mu \geqslant 0, g \in D \right\},$$

where $1/p + 1/q = 1$. It is known, that dual cones are closed convex cones in norm topologies and

$$\text{dual}\, C = \text{dual}\,(\text{cone } C), \quad \text{dual}^* F = \text{dual}^*(\text{cone } F).$$

For $A, B \subset V$, the inclusion $A \subset B$ implies dual$\,B \subset$ dual$\,A$. If C and D are convex cones in V, then

$$\text{dual}\,(C + D) = \text{dual}\, C \cap \text{dual}\, D.$$

The same holds for dual*.

If a set $C \subset V$ is a closed convex cone, then

$$\operatorname{dual}^* \operatorname{dual} C = C, \tag{3}$$

(cf. [5, Lemma 2.1]).

In the reminder of this section we recall the notion of separability for vectors. It is quoted from [3]. Let V be a finite-dimensional real vector space with inner product $\langle \cdot, \cdot \rangle$ provided with a basis $e = \{e_1, \dots, e_n\}$. Let J_1 and J_2 be complementary subsets of $\{1, \dots, n\}$. Fix $x \in V$ and let γ be a scalar.

A vector $y \in V$ is said to be γ, x-separable on J_1 and J_2 (with respect to the basis e), if

$$\langle e_i, y - \gamma x \rangle \geqslant 0 \text{ for } i \in J_1, \text{ and } \langle e_j, y - \gamma x \rangle \leqslant 0 \text{ for } j \in J_2. \tag{4}$$

Let us observe that $y \in V$ is γ, x-separable on J_1 and J_2 if and only if

$$y - \gamma x \in \operatorname{dual}(E_1 - E_2), \tag{5}$$

where $E_1 = \operatorname{cone}\ \{e_i : i \in J_1\}$ and $E_2 = \operatorname{cone}\ \{e_j : j \in J_2\}$.

3. A solution of Chebyshev type inequality

Let A be a bounded linear operation $V \mapsto V$ with the adjoint A^*. The null space of A will be denoted by $N(A)$.

For $y \in V$ and $z \in V^*$ we first consider the inequality

$$\langle Ay, z \rangle \geqslant 0, \tag{6}$$

or equivalently, $\langle y, A^* z \rangle \geqslant 0$.

A general method of solution of (6) is established below.

THEOREM 1. *For vectors $y \in V$, $z \in V^*$ and a convex cone $C \subset V$ the following statements are mutually equivalent.*

i) *(6) holds for all $y \in C + N(A)$*

ii) $z \in \operatorname{dual} AC$

iii) $A^* z \in \operatorname{dual} C$.

Proof. Since *i)*, $0 \leqslant \langle Ay, z \rangle = \langle y, A^* z \rangle$ for all $y \in C$. For this reason $0 \leqslant \langle AC, z \rangle = \langle C, A^* z \rangle$. Hence $z \in \operatorname{dual} AC$ and $A^* z \in \operatorname{dual} C$. It proves *i)* $\Rightarrow$ *ii)*, *iii)*.

Conversely, if $z \in \operatorname{dual} AC$, then for $y = c + x$, where $c \in C$ and x with $Ax = 0$ are arbitrary we get $\langle Ay, z \rangle = \langle Ac, z \rangle + \langle Ax, z \rangle = \langle Ac, z \rangle \geqslant 0$. By a similar argument; for $y \in C + N(A)$ we have $Ay \in AC$, hence $Ay = Ac$ for certain $c \in C$. If $A^* z \in \operatorname{dual} C$, then $\langle Ay, z \rangle = \langle Ac, z \rangle = \langle c, A^* z \rangle \geqslant 0$. From this we conclude that *ii)*, *iii)* $\Rightarrow$ *i)*, which completes the proof. □

For given $x \in V$, $v \in V^*$ and arbitrary $y \in V$ let us define the operation A as follows:

$$Ay = y \langle x, v \rangle - x \langle y, v \rangle.$$

In this situation (6) becomes inequality (1) which is considered in [3, 9] in the case of Euclidean spaces. It is easily seen that $A^*z = \langle x,v\rangle z - \langle x,z\rangle v$ and span $\{x\} \subset N(A)$. Particularly, $N(A) =$ span $\{x\}$ whenever $\langle x,v\rangle \neq 0$. From the above proof, it may be concluded that in the statement of Theorem 1 the null space of A can be replaced by a subset of $N(A)$. It implies Niezgoda's result [3, Theorem 3.1].

COROLLARY 1. *Fix $x \in V$, $v \in V^*$. For $y \in V$, $z \in V^*$ and convex cone $C \subset V$, the following statements are mutually equivalent.*

i) *(1) holds for all* $y \in C + \text{span } \{x\}$.

ii) $z \in \text{dual}\{c\langle x,v\rangle - x\langle c,v\rangle : c \in C\}$.

iii) $\langle x,v\rangle z - \langle x,z\rangle v \in \text{dual}\, C$. □

Now, employing the idea of separable vectors (see (4) and (5)), we introduce cones and their dual cones depending on decomposition of spaces onto direct sums of closed subspaces. This concept will be developed in the next section. Let us start from two auxiliary remarks.

Assume that V_1 and V_2 are closed complementary subspaces of V, i.e. $V_1 \cap V_2 = \{0\}$ and $V = V_1 + V_2$. The fact will be denoted as $V = V_1 \oplus V_2$. It is known that the projections $V \mapsto V_i, i = 1,2$ are continuous. In consequence, we have the following

REMARK 1. If $V = V_1 \oplus V_2$, then $V^* = \text{dual}\, V_1 \oplus \text{dual}\, V_2$.

Proof. If $f \in \text{dual}\, V_1 \cap \text{dual}\, V_2$, then $\langle V_1, f\rangle = 0 = \langle V_2, f\rangle$. Hence $0 = \langle V_1 \oplus V_2, f\rangle = \langle V, f\rangle \Rightarrow f = 0$. Since projections $\pi_i : V \mapsto V_i$, $i = 1,2$ are continuous, for any $f \in V^*$ we have $f_i \overset{def}{=} f \circ \pi_i \in \text{dual}\, V_{2-(1+i) \bmod 2}$ and $f_1 + f_2 = f \circ (\pi_1 + \pi_2) = f$. Clearly, $\text{dual}\, V_i$, $i = 1,2$ are closed. □

In general, the algebraic sum of closed convex cones is not closed.

REMARK 2. If $V = V_1 \oplus V_2$ and $C_i \subset V_i$, $i = 1,2$ are closed convex cones, then $C_1 + C_2$ is a closed convex cone.

Proof. Assume $x_n \in C_1 + C_2$, $x_n \to x$, $x_n = x_n^{(1)} + x_n^{(2)}$, $x_n^{(i)} \in V_i$, $i = 1,2$. Since the projections $\pi_i : V \mapsto V_i$ are continuous, $x_n^{(i)} = \pi_i x_n \to \pi_i x \overset{def}{=} x^{(i)}$. Clearly, $x^{(1)} + x^{(2)} = (\pi_1 + \pi_2)x = x$. By uniqueness of decomposition $x_n^{(1)} + x_n^{(2)} = x_n \in C_1 + C_2$ we get $x_n^{(i)} \in C_i$. Since C_i are closed, $x^{(i)} \in C_i$. Finally, $x = x^{(1)} + x^{(2)} \in C_1 + C_2$. □

LEMMA 1. *Let $V = V_1 \oplus V_2$ and let ε_i, $i = 1,2$ be such scalars that $\varepsilon_i^2 = 1$. If $E_i \subset V_i$ are closed convex cones, then*

$$\text{dual}\,(\varepsilon_1 E_1 + \varepsilon_2 E_2) = \varepsilon_1 F_1 + \varepsilon_2 F_2,$$

where

$$F_i = \text{dual}\, E_i \cap \text{dual}\, V_{2-(1+i) \bmod 2},\ i = 1,2. \tag{7}$$

Proof. Let $f \in \varepsilon_1 F_1 + \varepsilon_2 F_2$. There exist $f_i \in V^*$, $i = 1,2$, such that

$$\langle E_i, f_i \rangle \geqslant 0, \ \left\langle V_{2-(1+i) \bmod 2}, f_i \right\rangle = 0$$

and $f = \varepsilon_1 f_1 + \varepsilon_2 f_2$. For any $x \in \varepsilon_1 E_1 + \varepsilon_2 E_2$ there exist $x_i \in E_i$ such that $x = \varepsilon_1 x_1 + \varepsilon_2 x_2$. We have

$$\begin{aligned} \langle x, f \rangle &= \langle \varepsilon_1 x_1 + \varepsilon_2 x_2, \varepsilon_1 f_1 + \varepsilon_2 f_2 \rangle \\ &= \varepsilon_1^2 \langle x_1, f_1 \rangle + \varepsilon_1 \varepsilon_2 \langle x_1, f_2 \rangle + \varepsilon_2 \varepsilon_1 \langle x_2, f_1 \rangle + \varepsilon_2^2 \langle x_2, f_2 \rangle \geqslant 0. \end{aligned}$$

Thus $f \in \operatorname{dual}(\varepsilon_1 E_1 + \varepsilon_2 E_2)$.

Conversely, let $f \in \operatorname{dual}(\varepsilon_1 E_1 + \varepsilon_2 E_2)$. By Remark 1 there exist $f_1, f_2 \in V^*$ with $\langle V_2, f_1 \rangle = 0 = \langle V_1, f_2 \rangle$ and $f = \varepsilon_1 f_1 + \varepsilon_2 f_2$. For any $x_i \in E_i \subset V_i$ we get:

$$\begin{aligned} 0 &\leqslant \langle \varepsilon_1 x_1 + \varepsilon_2 x_2, f \rangle = \langle \varepsilon_1 x_1 + \varepsilon_2 x_2, \varepsilon_1 f_1 + \varepsilon_2 f_2 \rangle \\ &= \varepsilon_1^2 \langle x_1, f_1 \rangle + \varepsilon_1 \varepsilon_2 \langle x_1, f_2 \rangle + \varepsilon_2 \varepsilon_1 \langle x_2, f_1 \rangle + \varepsilon_2^2 \langle x_2, f_2 \rangle = \langle x_1, f_1 \rangle + \langle x_2, f_2 \rangle. \end{aligned}$$

On substituting 0 into x_i, $i = 1,2$ we get $\langle E_i, f_i \rangle \geqslant 0$, i.e. $f_i \in \operatorname{dual} E_i$. By assumption, $f_i \in \operatorname{dual} V_{2-(1+i) \bmod 2}$. Therefore $f_i \in \operatorname{dual} E_i \cap \operatorname{dual} V_{2-(1+i) \bmod 2} = F_i$, $i = 1,2$. Finally, $f = \varepsilon_1 f_1 + \varepsilon_2 f_2 \in \varepsilon_1 F_1 + \varepsilon_2 F_2$. □

REMARK 3. Under assumptions and notations as in Lemma 1

$$F_i = \operatorname{dual}(E_1 + E_2) \cap \operatorname{dual} V_{2-(1+i) \bmod 2}, \ i = 1,2.$$

Moreover, for $E = \varepsilon_1 E_1 + \varepsilon_2 E_2$ we have $\operatorname{dual} E = \operatorname{dual} E \cap \operatorname{dual} V_1 + \operatorname{dual} E \cap \operatorname{dual} V_2$.

Proof. Note that $\operatorname{dual} E = \operatorname{dual}(\varepsilon_1 E_1 + \varepsilon_2 E_2) = \varepsilon_1 \operatorname{dual} E_1 \cap \varepsilon_2 \operatorname{dual} E_2$. Since $E_i \subset V_i$, $\operatorname{dual} V_i \subset \operatorname{dual} E_i$. From this $\operatorname{dual} V_i = \varepsilon_i \operatorname{dual} V_i \subset \varepsilon_i \operatorname{dual} E_i$. For these reasons, by Lemma 1 we obtain

$$\begin{aligned} \operatorname{dual} E \cap \operatorname{dual} V_i &= \varepsilon_{2-(1+i) \bmod 2} \operatorname{dual} E_{2-(1+i) \bmod 2} \cap \operatorname{dual} V_i \\ &= \varepsilon_{2-(1+i) \bmod 2} F_{2-(1+i) \bmod 2}, \ i = 1,2. \end{aligned}$$

Takeing $\varepsilon_i = 1$, $i = 1,2$ gives the alternative form of F_i.

Moreover, after summing the above equations over i we obtain $\operatorname{dual} E = \operatorname{dual} E \cap \operatorname{dual} V_1 + \operatorname{dual} E \cap \operatorname{dual} V_2$, by Lemma 1. □

The case $\langle x, v \rangle = 0$ of (1) is not interesting. In the paper we assume $\langle x, v \rangle > 0$. The case $\langle x, v \rangle < 0$ can be easily obtained from the previous one by replacing v by $-v$ and reversing of inequality in (1). By Corollary 1 and Lemma 1 we get the following solution of (1) (cf. [3, Theorem 3.5]).

THEOREM 2. *Let $V = V_1 \oplus V_2$ and $E_i \subset V_i$. $i = 1,2$ be closed convex cones. For given $x \in V$ and $v \in V^*$ with $\langle x, v \rangle > 0$ the following statements are equivalent.*

i) Inequality (1) holds for all $y \in E_1 - E_2 + \operatorname{span}\{x\}$.

ii) $z - \lambda v \in F_1 - F_2$, where $\lambda = \langle x, z \rangle / \langle x, v \rangle$ and F_1, F_2 are defined by (7). □

4. Synchronous and separable vectors

Let $\{e_\xi\}$ be a basis (in the Schauder sense) of a Banach space V, i.e. for every $x \in V$ there exists a unique sequence of scalars $\{\alpha_\xi\}$ such that $x = \sum_\xi \alpha_\xi e_\xi$. The sequence of linear functionals $\{f_\eta\}$ defined by $\langle x, f_i\rangle = \alpha_i$ is called *the sequence of coefficient functionals associated to the basis* $\{e_\xi\}$, or shortly, *the associated sequence of coefficient functionals* (abbrev. a.s.c.f.). The fundamental fact due to S. Banach is that coefficient functionals associated to a basis of Banach space are continuous. If $\{e_\xi\}$ is a basis of V and $\{f_\eta\}$ is a.s.c.f., then every $x \in V$ has a unique expansion of the form $x = \sum_i \langle x, f_i\rangle e_i$. Moreover $\langle e_\xi, f_\eta\rangle = \delta_{\xi\eta}$, the Kronecker symbol.

For a Banach space V with a basis $\{e_\xi\}$ let V_+ denotes the convex cone associated to the basis (the nonnegative cone, in other terminology), i.e. $V_+ = \{x = \sum_\xi x_\xi e_\xi \in V : x_\xi \geqslant 0\}$. It is clear that V_+ coincides with cone $\{e_\xi\}$, the smallest closed convex cone containing the basis $\{e_\xi\}$.

It is known, if V^* is a separable space, then a.s.c.f. $\{f_\eta\}$ constitutes a basis of V^*. In this situation we may say about dual bases. Moreover,

$$V_+^* = \operatorname{dual} V_+ \text{ and } V_+ = \operatorname{dual}^* V_+^*. \tag{8}$$

From now on, we assume that V and V^* are separable and $\{e_\xi\} \subset V$, $\{f_\eta\} \subset V^*$ are dual Schauder bases. For $u \in V$ and $w \in V^*$ we have

$$\langle u, w\rangle = \sum_\xi \langle u, f_\xi\rangle \langle e_\xi, w\rangle. \tag{9}$$

Given $u \in V$ and $w \in V^*$, we introduce a relation of synchronicity with respect to dual bases $\{e_\xi\} \subset V$ and $\{f_\eta\} \subset V^*$ (abbrev. $u \sim w$) as follows

$$u \sim w \equiv \langle u, f_k\rangle \langle e_k, w\rangle \geqslant 0, \text{ for all } k. \tag{10}$$

By (9), it is easily seen that $u \sim w$ forces $\langle u, w\rangle \geqslant 0$. The sets $\{u \in V : u \sim w\}$ for fixed $w \in V^*$ and $\{w \in V^* : u \sim w\}$ for fixed $u \in V$ are closed convex cones. According to (8), $u \sim w$ whenever $u \in V_+$ and $w \in V_+^*$. In the sequel, we will describe more interesting situations for $u \in V$ and $w \in V^*$ to be synchronous. Simultaneously, a connection between the separability of vectors due to Niezgoda [3] and our synchronicity will be shown.

Let $\sigma = (\sigma_i)$ and $\rho = (\rho_i)$ be complementary subsequences of integers i.e. $\{\sigma_i\} \cap \{\rho_i\} = \emptyset$, $\{\sigma_i\} \cup \{\rho_i\} = N$ and $\sigma_1 < \sigma_2 < \ldots$, $\rho_1 < \rho_2 < \ldots$. Let us define closed subspaces of V and V^*:

$$\begin{aligned} V_\sigma &= \operatorname{span}\ \{e_i : i \in \sigma\}, \quad & V_\rho &= \operatorname{span}\ \{e_i : i \in \rho\}, \\ V_\sigma^* &= \operatorname{span}\ \{f_i : i \in \sigma\}, \quad & V_\rho^* &= \operatorname{span}\ \{f_i : i \in \rho\}. \end{aligned} \tag{11}$$

There is no difficulty to show that

$$\begin{aligned} \operatorname{dual} V_\sigma &= \{f \in V^* : f = \sum_i x_i f_i, x_{\sigma_i} = 0\}, \quad & \operatorname{dual}^* V_\sigma^* &= \{x \in V : x = \sum_i x_i e_i, x_{\sigma_i} = 0\}, \\ \operatorname{dual} V_\rho &= \{f \in V^* : f = \sum_i x_i f_i, x_{\rho_i} = 0\}, \quad & \operatorname{dual}^* V_\rho^* &= \{x \in V : x = \sum_i x_i e_i, x_{\rho_i} = 0\}. \end{aligned}$$

Moreover, $V_\sigma \cap V_\rho = \{0\}$, $V_\sigma^* \cap V_\rho^* = \{0\}$ and

$$\begin{aligned} V_\sigma &= \mathrm{dual}^* V_\rho^*, \qquad & V_\rho &= \mathrm{dual}^* V_\sigma^*, \\ V_\sigma^* &= \mathrm{dual}\, V_\rho, \qquad & V_\rho^* &= \mathrm{dual}\, V_\sigma. \end{aligned} \tag{12}$$

Now we additionally assume that $\{e_\xi\}$ is an unconditional basis of V, i.e. for every $x \in V$ the series $\sum_i \langle x, f_i \rangle e_i$ converges unconditionally. It is known, then a.s.c.f. $\{f_\eta\}$ is an unconditional basis of V^*. In this situation the spaces V and V^* decompose onto direct sums

$$V = V_\sigma \oplus V_\rho \quad \text{and} \quad V^* = V_\sigma^* \oplus V_\rho^*, \tag{13}$$

for any complementary σ and ρ, (see [8, Chap. II, Theorem 16.8]).

For σ and ρ as before let us define closed convex cones in V and V^*:

$$\begin{aligned} E_\sigma &= \mathrm{cone}\ \{e_i : i \in \sigma\}, \qquad & E_\rho &= \mathrm{cone}\ \{e_i : i \in \rho\}, \\ F_\sigma &= \mathrm{cone}\ \{f_i : i \in \sigma\}, \qquad & F_\rho &= \mathrm{cone}\ \{f_i : i \in \rho\}. \end{aligned} \tag{14}$$

By definitions (11) and (14) it is clear that

$$\begin{aligned} E_\sigma &\subset V_\sigma, \qquad & E_\rho &\subset V_\rho, \\ F_\sigma &\subset V_\sigma^*, \qquad & F_\rho &\subset V_\rho^*, \end{aligned} \tag{15}$$

moreover

$$\begin{aligned} E_\sigma &= \{x \in V : x = \sum_i x_i e_i, x_i \geqslant 0, x_{\rho_i} = 0\}, \; & F_\sigma &= \{f \in V^* : f = \sum_i x_i f_i, x_i \geqslant 0, x_{\rho_i} = 0\}, \\ E_\rho &= \{x \in V : x = \sum_i x_i e_i, x_i \geqslant 0, x_{\sigma_i} = 0\}, \; & F_\rho &= \{f \in V^* : f = \sum_i x_i f_i, x_i \geqslant 0, x_{\sigma_i} = 0\}. \end{aligned}$$

According to Remark 2, $\varepsilon_1 E_\sigma + \varepsilon_2 E_\rho$ and $\varepsilon_1 F_\sigma + \varepsilon_2 F_\rho$ are closed convex cones for any scalars $\varepsilon_i^2 = 1, i = 1,2$, by (13) and (15). By (2) and (14) we get

$$\begin{aligned} E_\sigma + E_\rho &= \mathrm{cone}\ \{e_\xi\} = V_+, \qquad & F_\sigma + F_\rho &= \mathrm{cone}\ \{f_\eta\} = V_+^*, \\ E_\sigma - E_\rho &= \mathrm{cone}\ (\{e_{\sigma_i}\} \cup \{-e_{\rho_i}\}), \qquad & F_\sigma - F_\rho &= \mathrm{cone}\ (\{f_{\sigma_i}\} \cup \{-f_{\rho_i}\}). \end{aligned} \tag{16}$$

Moreover

$$\begin{aligned} E_\sigma &= V_+ \cap V_\sigma, \qquad & E_\rho &= V_+ \cap V_\rho, \\ F_\sigma &= V_+^* \cap V_\sigma^*, \qquad & F_\rho &= V_+^* \cap V_\rho^*. \end{aligned} \tag{17}$$

We are now in a position to introduce a system of convex cones and their dual cones in V and V^*. Its idea is based on the notion of separable vectors (see (4) and (5)).

LEMMA 2. *Let V and V^* be separable spaces with dual unconditional bases $\{e_\xi\}$ and $\{f_\eta\}$, respectively. Assume that σ and ρ are complementary subsequences of integers. For $E_\sigma, E_\rho \subset V$ and $F_\sigma, F_\rho \subset V^*$ defined by (*14*) we have*

$$\mathrm{dual}\,(E_\sigma - E_\rho) = F_\sigma - F_\rho, \; \mathrm{dual}^*(F_\sigma - F_\rho) = E_\sigma - E_\rho.$$

Moreover,

$$V = \bigcup_{\sigma,\rho} (E_\sigma - E_\rho), \; V^* = \bigcup_{\sigma,\rho} (F_\sigma - F_\rho).$$

Proof. (13)–(15) ensure that the assumptions of Lemma 1 are fulfilled. Combining the lemma and Remark 3 with (16), (8), (12), (17) gives $\operatorname{dual}(E_\sigma - E_\rho) = \operatorname{dual}(E_\sigma + E_\rho) \cap \operatorname{dual} V_\rho - \operatorname{dual}(E_\sigma + E_\rho) \cap \operatorname{dual} V_\sigma = F_\sigma - F_\rho$. By Remark 2, convex cones (16) are closed. (3) forces $\operatorname{dual}^*(F_\sigma - F_\rho) = \operatorname{dual}^* \operatorname{dual}(E_\sigma - E_\rho) = E_\sigma - E_\rho$, by the proved part of this lemma.

Moreover, for any $u \in V$ set $\{\sigma_i\} = \{k : \langle u, f_k\rangle \geqslant 0\}$ and $\{\rho_i\} = \{k : \langle u, f_k\rangle < 0\}$. The sequences σ and ρ are complementary and $u \in \operatorname{dual}^*\{f_i : i \in \sigma\} \cap \operatorname{dual}^*\{-f_j : j \in \rho\} = \operatorname{dual}^* F_\sigma \cap \operatorname{dual}^*(-F_\rho) = \operatorname{dual}^*(F_\sigma - F_\rho) = E_\sigma - E_\rho$, by (14) and the first part of this lemma. The converse inclusion is obvious. The second identity is analogous. □

The system of cones introduced above ensures that for fixed $x \in V, v \in V^*$ and arbitrary $z \in V^*$ there exists $y \in V$ such that (1) holds. In fact, there exist complementary sequences σ and ρ with property $\langle x, v\rangle z - \langle x, z\rangle v \in F_\sigma - F_\rho = \operatorname{dual}(E_\sigma - E_\rho)$. According to the equivalence $i) \Leftrightarrow iii)$ of Corollary 1, (1) holds for $y \in E_\sigma - E_\rho + \operatorname{span}\{x\}$.

The following corollary is an application of Lemma 2 and a completion of Theorem 2.

COROLLARY 2. *Under assumptions as in Lemma 2, fix $x \in V$ and $v \in V^*$ with $\langle x, v\rangle > 0$. Suppose there exist complementary subsequences σ and ρ such that*

$$\langle e_i, z\rangle \geqslant \lambda \langle e_i, v\rangle \text{ and } \langle e_j, z\rangle \leqslant \lambda \langle e_j, v\rangle \text{ for all } i \in \sigma \text{ and } j \in \rho, \tag{18}$$

where $\lambda = \langle x, z\rangle / \langle x, v\rangle$ and there exists a scalar γ such that

$$\langle y, f_i\rangle \geqslant \gamma \langle x, f_i\rangle \text{ and } \langle y, f_j\rangle \leqslant \gamma \langle x, f_j\rangle \text{ for all } i \in \sigma \text{ and } j \in \rho, \tag{19}$$

then (1) holds for $x, y \in V$ and $z, v \in V^*$.

Proof. By (13)–(15) one can employ Theorem 2 to $E_1 = E_\sigma$ and $E_2 = E_\rho$. The same argument as in the proof of Lemma 2 leads to $F_1 = F_\sigma$ and $F_2 = F_\rho$. It is sufficient to show that $y \in E_\sigma - E_\rho + \operatorname{span}\{x\}$ is equivalent to (19) and $z - \lambda v \in F_\sigma - F_\rho$ is equivalent to (18). The proofs of both of equivalences are analogous. We focus on the first one. If $y \in E_\sigma - E_\rho + \operatorname{span}\{x\}$, then there exists a scalar γ with $y - \gamma x \in E_\sigma - E_\rho$. By Lemma 2 and (14) we have $y - \gamma x \in \operatorname{dual}^*(F_\sigma - F_\rho) = \operatorname{dual}^* F_\sigma \cap \operatorname{dual}^*(-F_\rho) = \operatorname{dual}^*\{f_i : i \in \sigma\} \cap \operatorname{dual}^*\{-f_j : j \in \rho\}$ or equivalently $\langle y - \gamma x, f_i\rangle \geqslant 0$ and $\langle y - \gamma x, f_j\rangle \leqslant 0$ for all $i \in \sigma$ and $j \in \rho$. It is nothing but (19). □

The system of cones established in Lemma 2 yields necessary and sufficient conditions for the synchronicity (10).

PROPOSITION 1. *Under the hypothesis of Lemma 2, for $u \in V$ and $w \in V^*$ we have:*

i) $u \sim w$ whenever $u \in E_\sigma - E_\rho$ and $w \in F_\sigma - F_\rho$,

ii) if $u \sim w$, then there exist complementary sequences σ and ρ such that $u \in E_\sigma - E_\rho$ and $w \in F_\sigma - F_\rho$.

Proof. Sufficiency. By Lemma 2 and (14) we have $u \in E_\sigma - E_\rho = \text{dual}^*(F_\sigma - F_\rho) = \text{dual}^* F_\sigma \cap \text{dual}^*(-F_\rho) = \text{dual}^*\{f_i : i \in \sigma\} \cap \text{dual}^*\{-f_i :\in \rho\}$ or equivalently $\langle u, f_i\rangle \geqslant 0$ and $\langle u, f_j\rangle \leqslant 0$ for all $i \in \sigma$ and $j \in \rho$. Analogously, $w \in F_\sigma - F_\rho$ gives $\langle e_i, w\rangle \geqslant 0$ and $\langle e_j, w\rangle \leqslant 0$ for all $i \in \sigma$ and $j \in \rho$. Since σ and ρ are complementary, $\langle u, f_k\rangle \langle e_k, w\rangle \geqslant 0$ for all integer k, or equivalently, $u \sim w$.

Necessity. Set $\{\sigma_k\} = \{i : \langle u, f_i\rangle > 0\} \cup \{i : \langle u, f_i\rangle = 0,\ \langle e_i, w\rangle \geqslant 0\}$ and let ρ be complementary with σ, i.e. $\{\rho_k\} = \{j : \langle u, f_j\rangle < 0\} \cup \{j : \langle u, f_j\rangle \leqslant 0,\ \langle e_j, w\rangle < 0\}$. If $u \sim w$, then

$$\begin{cases} \langle u, f_i\rangle \geqslant 0\ ,\ i \in \sigma \\ \langle u, f_j\rangle \leqslant 0\ ,\ j \in \rho \end{cases} \text{and} \begin{cases} \langle e_i, w\rangle \geqslant 0\ ,\ i \in \sigma \\ \langle e_j, w\rangle \leqslant 0\ ,\ j \in \rho. \end{cases}, \tag{20}$$

because $\langle u, f_i\rangle > 0$ implies $\langle e_i, w\rangle \geqslant 0$ and $\langle u, f_i\rangle < 0$ implies $\langle e_i, w\rangle \leqslant 0$. The same argument as before guarantees that $u \in E_\sigma - E_\rho$ and $w \in F_\sigma - F_\rho$. $\square$

If $\{e_\xi\} \subset V$ and $\{f_\eta\} \subset V^*$ are conditional dual bases, then there exist complementary subsequences σ and ρ with $V \neq V_\sigma \oplus V_\rho$, i.e. there exists $u \in V$ such that $\sum_{i\in\sigma} \langle u, f_i\rangle e_i$ and $\sum_{j\in\rho} \langle u, f_j\rangle e_j$ are divergent. If $u \sim w$ with $\langle u, f_k\rangle \neq 0 \neq \langle e_k, w\rangle$ for all k and $\{\sigma_i\} = \{k : \langle u, f_k\rangle \geqslant 0\}$, then $u \notin E_\sigma - E_\rho$. Let us note, there is only one pair of complementary sequences σ and ρ of the form (20) in this situation. Thus the assumption of unconditionality is crucial for the necessity part of Proposition 1.

The synchronicity relation leads to sufficient conditions for (1) to hold.

PROPOSITION 2. *Let $\{e_\xi\}$, $\{f_\eta\}$ be dual Schauder bases of separable spaces V and V^*, respectively. Fix $x \in V$ and $v \in V^*$ with $\langle x, v\rangle > 0$.*

If for $y \in V$ and $z \in V^*$ there exists a scalar γ such that

$$y - \gamma x \sim z - \lambda v,$$

where $\lambda = \langle x, z\rangle / \langle x, v\rangle$, then (1) holds for $x, y \in V$ and $z, v \in V^*$.

Proof. If $y - \gamma x \sim z - \lambda v$, then by (9) we have $\langle y - \gamma x, z - \lambda v\rangle = \sum_k \langle y - \gamma x, f_k\rangle \langle e_k, z - \lambda v\rangle \geqslant 0$. On the other hand, $\langle y - \gamma x, z - \lambda v\rangle = \frac{1}{\langle x, v\rangle}[\langle y, z\rangle \langle x, v\rangle - \langle x, z\rangle \langle y, v\rangle]$. Since $\langle x, v\rangle > 0$, inequality (1) holds. $\square$

Analyzing of Corollary 2, Proposition 1 and their proofs gives

REMARK 4. Let V and V^* be separable spaces with dual bases $\{e_\xi\}$ and $\{f_\eta\}$, respectively. Assume that $x, y \in V$, $z, v \in V^*$, γ, λ are scalars and σ, ρ are complementary subsequences of integers.

The conditions (19) and (18) can be employed as definitions of separable vectors (cf. (4)) in V and V^*, respectively.

The synchronicity $y - \gamma x \sim z - \lambda v$ is equivalent to (18) and (19) for certain σ and ρ. In the case of unconditional bases, it is also equivalent to $y - \gamma x \in E_\sigma - E_\rho$ and $z - \lambda v \in F_\sigma - F_\rho$ (cf. (5)), where $E_\sigma, E_\rho, F_\sigma, F_\rho$ are defined by (14). $\square$

We say that a vector $x \in V$ is $\{f_\eta\}$-positive if $\langle x, f_\eta \rangle > 0$ for all η, analogously a functional $v \in V^*$ is $\{e_\xi\}$-positive if $\langle e_\xi, v \rangle > 0$. By assumptions of this positivity for x and v, Proposition 2 can be restated in a more interesting fractional form.

COROLLARY 3. *Under assumptions as in Proposition 2, let $x \in V$ be $\{f_\eta\}$-positive, $v \in V^*$ be $\{e_\xi\}$-positive with $\langle x, v \rangle > 0$ and $\lambda = \langle x, z \rangle / \langle x, v \rangle$.*

If there exists a scalar γ such that

$$\left[\frac{\langle y, f_i \rangle}{\langle x, f_i \rangle} - \gamma\right]\left[\frac{\langle e_i, z \rangle}{\langle e_i, v \rangle} - \lambda\right] \geqslant 0, \text{ for } i = 1, 2, \ldots, \tag{21}$$

then (1) holds for $x, y \in V$ and $z, v \in V^*$. □

According to Remark 4, the above corollary extends [3, Corollary 3.7].

The next considerations show a close connection between synchronicity and similar ordering of sequences. Under assumptions and notations as in the above corollary, let σ be defined as follows

$$\{\sigma_k\} = \{i : \lambda \leqslant \langle e_i, z \rangle / \langle e_i, v \rangle\}$$

and ρ be complementary with σ. In this situation we have

$$\frac{\langle e_j, z \rangle}{\langle e_j, v \rangle} < \lambda \leqslant \frac{\langle e_i, z \rangle}{\langle e_i, v \rangle}, \text{ for all } i \in \sigma \text{ and } j \in \rho. \tag{22}$$

If sequences of fractions $\left\{\frac{\langle e_\xi, z \rangle}{\langle e_\xi, v \rangle}\right\}_{\xi=1}^{\infty}$ and $\left\{\frac{\langle y, f_\eta \rangle}{\langle x, f_\eta \rangle}\right\}_{\eta=1}^{\infty}$ are similarly ordered i.e.

$$\left[\frac{\langle e_i, z \rangle}{\langle e_i, v \rangle} - \frac{\langle e_j, z \rangle}{\langle e_j, v \rangle}\right]\left[\frac{\langle y, f_i \rangle}{\langle x, f_i \rangle} - \frac{\langle y, f_j \rangle}{\langle x, f_j \rangle}\right] \geqslant 0 \text{ for all integers } i, j,$$

then

$$\frac{\langle y, f_j \rangle}{\langle x, f_j \rangle} \leqslant \gamma \leqslant \frac{\langle y, f_i \rangle}{\langle x, f_i \rangle} \text{ for all } i \in \sigma,\ j \in \rho, \tag{23}$$

where $\gamma = \inf_{i \in \sigma} \frac{\langle y, f_i \rangle}{\langle x, f_i \rangle}$. (22) and (23) imply (21). It is equivalent to $y - \gamma x \sim z - \lambda v$. In consequence, (1) holds. This proves the following remark.

REMARK 5. Under assumptions as in Corollary 3, if $\left\{\frac{\langle e_\xi, z \rangle}{\langle e_\xi, v \rangle}\right\}_{\xi=1}^{\infty}$ and $\left\{\frac{\langle y, f_\eta \rangle}{\langle x, f_\eta \rangle}\right\}_{\eta=1}^{\infty}$ are similarly ordered, then $y - \gamma x \sim z - \lambda v$ for certain γ and (1) holds for $x, y \in V$ and $z, v \in V^*$. □

5. Applications

In this section we shall discuss our results in concrete Banach spaces endowed with various bases.

For convenience, given any infinite real sequences $x=(x_1,x_2,\dots)$ and $y=(y_1,y_2,\dots)$, let $|x|=(|x_1|,|x_2|,\dots)$, $x\cdot y=(x_1y_1,x_2y_2,\dots)$ and $x/y=(x_1/y_1,x_2/y_2,\dots)$, provided that $y_i\neq 0$.

We first analyze the case of $V=l^p,\ p>1$, the classic space of all real absolutely p-summable sequences with the dual space $V^*=l^q,\ 1/p+1/q=1$.

The sequence of unit vectors $\{e_\xi=(\delta_{1\xi},\delta_{2\xi},\dots):\xi=1,2,\dots\}$ is an unconditional basis of every such space. Let us observe that the positivity of a vector $x=(x_1,x_2,\dots)\in l^p,\ p>1$ w.r.t. the unit vector basis means that $x_i>0$ for all i. In this situation, Corollary 3 yields

COROLLARY 4. *Let $x,y\in l^p$ and $z,v\in l^q$ with $x_i,v_i>0,\ i=1,2,\dots$ and $\sum_{k=1}^{\infty}x_iv_i>0$, where $p,q>1$ and $1/p+1/q=1$. If there exists a scalar γ such that*

$$\left[\frac{y_i}{x_i}-\gamma\right]\left[\frac{z_i}{v_i}-\frac{\sum_{k=1}^{\infty}x_iz_i}{\sum_{k=1}^{\infty}x_iv_i}\right]\geqslant 0,\ \text{for } i=1,2,\dots,$$

then the following inequality holds

$$\sum_{i=1}^{\infty}x_iz_i\sum_{i=1}^{\infty}y_iv_i\leqslant\sum_{i=1}^{\infty}y_iz_i\sum_{i=1}^{\infty}x_iv_i. \quad \square \tag{24}$$

According to Remark 4, the above corollary extends Niezgoda's result [3, Corollary 4.1] from finite separable sequences to infinite ones (see also Toader [9]).

Set $x=v=(1,\frac{1}{2},\frac{1}{3},\dots)$. Since $\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$ (Euler, 1735), (24) takes the form

$$\sum_{k=1}^{\infty}\frac{z_k}{k}\sum_{k=1}^{\infty}\frac{y_k}{k}\leqslant\frac{\pi^2}{6}\sum_{k=1}^{\infty}y_kz_k. \tag{25}$$

and is valid whenever there exists a scalar γ such that

$$[iy_i-\gamma]\left[iz_i-\sum_{k=1}^{\infty}\frac{6z_k}{k\pi^2}\right]\geqslant 0,\ \text{for } i=1,2,\dots.$$

Particularly, by Remark 5, inequality (25) holds if only $\{kz_k\}$ and $\{ky_k\}$ are similarly ordered.

EXAMPLE 1. Let $p=(p_1,p_2,\dots)$ with $p_i\neq 0$ be such sequence that $p\in l^r$ for all $r>1$. Evidently, $|p|$ is positive w.r.t the unit vector basis. By the Hölder inequality, if $y\in l^r,\ r>1$, then $p\cdot y\in l^1\subset l^r$.

Applying Corollary 4 to $x=v=|p|$ and $|p|\cdot y,\ |p|\cdot z$ in place of $y,\ z$, respectively, for any $y\in l^r,\ z\in l^s$, where $r,s>1$ and $1/r+1/s=1$ we get

$$\sum_{i=1}^{\infty}p_i^2y_i\sum_{i=1}^{\infty}p_i^2z_i\leqslant\sum_{i=1}^{\infty}p_i^2\sum_{i=1}^{\infty}p_i^2y_iz_i \tag{26}$$

whenever there exists a scalar γ such that

$$[y_i-\gamma]\left[z_i-\frac{\sum_{k=1}^{\infty}p_k^2z_k}{\sum_{k=1}^{\infty}p_k^2}\right]\geqslant 0,\ i=1,2,\dots. \tag{27}$$

This is the weighted version of Chebyshev inequality for synchronous (or separable, according to Remark 4) infinite sequences (cf. [3, Example 4.2]).

By Remark 5 it is known that similarly ordered $y\in l^r$ and $z\in l^s$ satisfy (27). In consequence, inequality (26) is met, where $r,s>1$ and $1/r+1/s=1$.

Now, we will show there exist sequences fulfilling (27) which are not similarly ordered. For this purpose, let $a\in l^r$ and $z\in l^s$ be strictly decreasing, where $r,s>1$ and $1/r+1/s=1$. Set

$$\sigma=\left\{i:z_i\geqslant\frac{\sum_{k=1}^{\infty}p_k^2z_k}{\sum_{k=1}^{\infty}p_k^2}\right\},\ \gamma=\inf_{i\in\sigma}a_i$$

and let ρ be complementary with σ. In this situation, we have

$$\begin{cases} z_j\leqslant\frac{\sum_{k=1}^{\infty}p_k^2z_k}{\sum_{k=1}^{\infty}p_k^2}\leqslant z_i, \\ a_j\leqslant\gamma\leqslant a_i, \end{cases}\text{for all } i\in\sigma,\ j\in\rho. \tag{28}$$

Let π_σ and π_ρ be permutations of the set of all positive integers with

$$\pi_\sigma(j)=j,\ \pi_\rho(i)=i,\ \text{for all } i\in\sigma,\ j\in\rho.$$

The following properties of π_σ and π_ρ are easy to checking:

i) $\pi_\sigma^{-1}(j)=j,\ \pi_\rho^{-1}(i)=i$, for all $i\in\sigma,\ j\in\rho$;

ii) $\pi_\sigma(\sigma)=\sigma,\ \pi_\sigma^{-1}(\sigma)=\sigma,\ \pi_\rho(\rho)=\rho,\ \pi_\rho^{-1}(\rho)=\rho$;

iii) $\pi_\sigma\circ\pi_\rho=\pi_\rho\circ\pi_\sigma$.

Set $\pi=\pi_\sigma\circ\pi_\rho$. It is easily seen that,

$$\pi(\sigma)=\sigma,\ \pi(\rho)=\rho. \tag{29}$$

Let us define

$$y_i=a_{\pi(i)}.$$

By (28) and (29) we get

$$y_j\leqslant\gamma\leqslant y_i,\ \text{for all } i\in\sigma,\ j\in\rho. \tag{30}$$

Assume π is not the identity permutation. Then y and z are not similarly ordered.

To prove that, let $k=\min\{i:\pi(i)\neq i\}$. Set $\pi(k)=l$, evidently $k<l$. There exists m, such that $\pi(m)=k$. Clearly, $k<m$. In conclusion, there exist $k<m$ with $\pi(k)=l>k=\pi(m)$. Since a and z are strictly decreasing, $z_k-z_m>0$ and $y_k-y_m=a_{\pi(k)}-a_{\pi(m)}<0$. Therefore y and z are not similarly ordered.

However, by (28) and (30), the synchronicity condition (27) is satisfied and inequality (26) holds for y and z. $\square$

Now, given a real sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ with nonzero entries, we define

$$d_n = \alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n, \; n = 1, 2, \ldots, \tag{31}$$

where $\{e_n\}$ is the unit vector basis. It is known that $\{d_n\}$ is a basis of l^p, $p > 1$ if and only if there exists a constant M such that

$$\sum_{i=1}^{n} |\alpha_i|^p / |\alpha_{n+1}|^p \leqslant M, \text{ for all } n \tag{32}$$

(see e.g. [8, Proposition 4.3]). The sequence of coefficient functionals associated to the basis $\{d_n\}$ has the form

$$g_n = \frac{1}{\alpha_n} e_n - \frac{1}{\alpha_{n+1}} e_{n+1}, \; n = 1, 2, \ldots, \tag{33}$$

and constitutes a basis of l^q, $1/p + 1/q = 1$. It is easy to see that $\{g_n\}$-positivity of $x \in l^p$ means that the sequence x/α is strictly decreasing i.e. $x_1/\alpha_1 > x_2/\alpha_2 > \ldots$, on the other side, $\{d_n\}$-positivity of $v \in l^q$ is equivalent to $\alpha_1 v_1 + \ldots + \alpha_n v_n > 0$ for any integer n. By Corollary 3 we get

COROLLARY 5. *Fix $(\alpha_1, \alpha_2, \ldots)$, $\alpha_i \neq 0$, $i = 1, 2, \ldots$ with property (32). Let $\{d_n\}$ and $\{g_n\}$ be dual bases of the form (31) and (33), respectively. Assume $x, y \in l^p$ and $z, v \in l^q$, $1/p + 1/q = 1$ provided that $\sum_i x_i v_i > 0$, $\sum_{i=1}^{n} \alpha_i v_i > 0$, $n = 1, 2, \ldots$ and x/α is strictly decreasing.*

If there exists a scalar γ such that

$$\left[\frac{y_i/\alpha_i - y_{i+1}/\alpha_{i+1}}{x_i/\alpha_i - x_{i+1}/\alpha_{i+1}} - \gamma \right] \left[\frac{\sum_{k=1}^{i} \alpha_k z_k}{\sum_{k=1}^{i} \alpha_k v_k} - \frac{\sum_{k=1}^{\infty} x_k z_k}{\sum_{k=1}^{\infty} x_k v_k} \right] \geqslant 0 \text{ for } i = 1, 2, \ldots,$$

then (24) holds for $x, y \in l^p$ and $z, v \in l^q$. □

The above corollary is a counterpart of [3, Corollary 4.4] for infinite sequences.

EXAMPLE 2. Let $a = (a_1, a_2, \ldots)$ and $b = (b_1, b_2, \ldots)$ be infinite sequences, where a is nonconstant. We say that b is convex with respect to a (abbrev. $a \triangleleft b$) if

$$\begin{vmatrix} 1 & a_i & b_i \\ 1 & a_j & b_j \\ 1 & a_k & b_k \end{vmatrix} \geqslant 0 \text{ whenever } a_i \leqslant a_j \leqslant a_k,$$

(for the general definition see e.g. [2, Definition 1.]). If $a \triangleleft b$, then

$$\frac{b_l - b_n}{a_l - a_n} \leqslant \frac{b_k - b_l}{a_k - a_l}, \text{ whenever } a_n \leqslant a_k, \; a_k \neq a_l \neq a_n, \tag{34}$$

(see [2, Lemma 2]).

Under assumptions as in Corollary 5 let $x/\alpha \lhd y/\alpha$. Since x/α is strictly decreasing, for any integer i we get

$$\frac{y_{i+1}/\alpha_{i+1} - y_{i+2}/\alpha_{i+2}}{x_{i+1}/\alpha_{i+1} - x_{i+2}/\alpha_{i+2}} \leqslant \frac{y_i/\alpha_i - y_{i+1}/\alpha_{i+1}}{x_i/\alpha_i - x_{i+1}/\alpha_{i+1}},$$

by (34), i.e. the sequence $\{\frac{y_i/\alpha_i - y_{i+1}/\alpha_{i+1}}{x_i/\alpha_i - x_{i+1}/\alpha_{i+1}}\}_{i=1}^{\infty}$ is decreasing. If additionally $\{\sum_{k=1}^{j} \alpha_k z_k / \sum_{k=1}^{j} \alpha_k v_k\}_{j=1}^{\infty}$ is decreasing, then (24) holds, by Remark 5.

If we assume $\alpha_i > 0$, then the condition $x/\alpha \lhd y/\alpha$ can equivalently be expressed as

$$\begin{vmatrix} \alpha_i & x_i & y_i \\ \alpha_j & x_j & y_j \\ \alpha_k & x_k & y_k \end{vmatrix} \geqslant 0 \text{ whenever } x_i/\alpha_i \leqslant x_j/\alpha_j \leqslant x_k/\alpha_k. \quad \square$$

For given $(p_1, p_2, \ldots) \in l^r$, $r > 1$ with $p_i \neq 0$, let us define

$$\begin{aligned} c_k &= p_k e_k + p_{k+1} e_{k+1} + \ldots, \qquad k = 1,2,\ldots, \\ h_1 &= \tfrac{1}{p_1} e_1, \;\; h_n = \tfrac{1}{p_n} e_n - \tfrac{1}{p_{n-1}} e_{n-1},\; n > 1. \end{aligned} \tag{35}$$

Let us observe that $\langle c_k, h_n \rangle = \delta_{kn}$ and for any $f \in l^s$, $\langle c_k, f \rangle = 0$, $k = 1,2,\ldots$ implies $f = 0$, where $s > 1$ and $1/r + 1/s = 1$. Hence $\{c_k\} \subset l^r$ and $\{h_n\} \subset l^s$ are dual bases.

A sequence $x \in l^r$ is $\{h_n\}$-positive if only x/p is strictly increasing with positive entries, on the other side, $\{c_n\}$-positivity of $v \in l^s$ means that $\sum_{i=k}^{\infty} p_i v_i > 0$ for any integer k. In this situation, we are able to obtain a counterpart of Corollary 5 from Corollary 3. Moreover, if x/p is strictly increasing with positive entries, $\sum_{i=k}^{\infty} p_i v_i > 0$ for any integer k, $x/p \lhd y/p$ and $\{\sum_{i=k}^{\infty} p_i z_i / \sum_{i=k}^{\infty} p_i v_i\}_{k=1}^{\infty}$ is increasing, then (24) is valid for $x, y \in l^r$, $z, v \in l^s$, $1/r + 1/s = 1$ (cf. Example 2). The details are left to the reader.

EXAMPLE 3. Fix $p = (p_1, p_2, \ldots) \in l^1$ with $p_i \neq 0$. Clearly, $p \in l^r$ for $r > 1$. By Hölder inequality, if $y \in l^r$, $r > 1$, then $p \cdot y \in l^1 \subset l^r$. Note that $p = c_1$, so p is not $\{h_n\}$-positive. In fact, $\langle p, h_n \rangle = \delta_{1n}$, $n = 1,2,\ldots$.

Applying Proposition 2 to $x = v = p$ and $p \cdot y, p \cdot z$ in place of y, z, respectively, gives the following statement.

Inequality (26) is valid for any $y \in l^r$, $z \in l^s$, $1/r + 1/s = 1$ whenever

$$[y_i - y_{i-1}] \left[\frac{\sum_{k=i}^{\infty} p_k^2 z_k}{\sum_{k=i}^{\infty} p_k^2} - \frac{\sum_{k=1}^{\infty} p_k^2 z_k}{\sum_{k=1}^{\infty} p_k^2} \right] \geqslant 0 \text{ for } i = 1,2,\ldots \tag{36}$$

with the convention that $y_0 = y_1 = \gamma$ (note: γ is a scalar occurring in the statement of Proposition 2).

Particularly, if $\{\sum_{k=j}^{\infty} p_k^2 z_k / \sum_{k=j}^{\infty} p_k^2\}_{j=1}^{\infty}$ decreases, then $\sum_{k=j}^{\infty} p_k^2 z_k \leqslant \lambda \sum_{k=j}^{\infty} p_k^2$ for all integer j, where $\lambda = \sum_{k=1}^{\infty} p_k^2 z_k / \sum_{k=1}^{\infty} p_k^2$. It is equivalent to $\langle c_j, p\cdot z - \lambda v\rangle \leqslant 0$ for all integer j, where $v = p$. If we assume $y_1 \geqslant y_2 \geqslant \ldots$, then $\langle p\cdot y - \gamma x, h_j\rangle \leqslant 0$ for all j, where $\gamma = y_1, x = p$. Hence $\langle p\cdot y - y_1 p, h_j\rangle \langle c_j, p\cdot z - \lambda p\rangle \geqslant 0$ for $j = 1,2\ldots$. It is equivalent to (36). In consequence, (26) is valid. Analogously, (26) holds whenever $\{\sum_{k=j}^{\infty} p_k^2 z_k / \sum_{k=j}^{\infty} p_k^2\}_{j=1}^{\infty}$ is increasing and $y_1 \leqslant y_2 \leqslant \ldots$ (cf. [11, Theorem 3.]. □

In the reminder of this section we assume that $V = L^p$, $p > 1$, the space of all p-power integrable functions with respect to the Lebesgue measure μ on the unit interval $[0,1]$. The dual space $V^* = L^q$, where $1/p + 1/q = 1$. The characteristic function of measurable set $A \subset [0,1]$ is denoted by I_A.

The Haar system:

$$\chi_0^0 = 1,\ t \in [0,1]$$

$$\chi_n^k(t) = \begin{cases} 2^{n/2}, & \frac{2k-2}{2^{n+1}} \leqslant t < \frac{2k-1}{2^{n+1}} \\ -2^{n/2}, & \frac{2k-1}{2^{n+1}} \leqslant t < \frac{2k}{2^{n+1}} \\ 0, & \text{otherwise} \end{cases},\ n = 0,1,\ldots,\ k = 1,2,\ldots,2^n \tag{37}$$

is an unconditional basis of L^p, $p > 1$, (see e.g. [8, Theorem 14.1]. Let us transform the system onto the usual sequence $\{\chi_k\}_{k=1}^{\infty}$ by the componentwise order, i.e. $\chi_1 = \chi_0^0, \chi_2 = \chi_0^1, \chi_3 = \chi_1^1, \ldots$. Set $\mathscr{H} = \{\chi_2, \chi_3, \ldots\}$. It is known that $\langle \chi_k, \chi_l\rangle = \delta_{kl}$. Hence Haar systems in L^p and L^q with $p,q > 1$ and $1/p + 1/q = 1$ are dual unconditional bases.

Let $D^p \subset L^p$, $p > 1$ be the closed convex cone of all nonincreasing μ *a.e.* functions. It is known (see [5, Theorem 3.1 and 3.3]) that:

$$\begin{aligned} D^q &= \text{cone } \left(\{I_{[0,s]} : 0 \leqslant s \leqslant 1\} \cup \{-I_{[0,1]}\}\right), \\ \text{dual}_p D^q &= \text{cone } \{I_\Pi - I_{\Pi+\varepsilon} : \varepsilon > 0, \Pi, \Pi + \varepsilon \subset [0,1]\}, \end{aligned} \tag{38}$$

where $p,q > 1$, $1/p + 1/q = 1$ and Π stands for an interval.

By Proposition 2 we obtain

COROLLARY 6. *Let $x,y \in L^p$ and $z,v \in L^q$ with $\int xv d\mu > 0$, where $p,q > 1$ and $1/p + 1/q = 1$.*

The following inequality holds

$$\int xz d\mu \int yv d\mu \leqslant \int yz d\mu \int xv d\mu \tag{39}$$

whenever there exists a scalar γ such that

$$\left[\int \chi_k y d\mu - \gamma \int \chi_k x d\mu\right]\left[\int \chi_k z d\mu - \lambda \int \chi_k v d\mu\right] \geqslant 0, \text{ for } k = 1,2,\ldots, \tag{40}$$

where $\lambda = \int xz d\mu / \int xv d\mu$. □

EXAMPLE 4. Under assumptions as in Corollary 6, set $x = v = \chi_1$. Then (39) becomes *Chebyshev integral inequality*

$$\int z d\mu \int y d\mu \leqslant \int yz d\mu. \tag{41}$$

On the other hand, condition (40) means

$$\begin{cases} [\int y d\mu - \gamma][\int z d\mu - \int z d\mu] \geqslant 0, \; k = 1 \\ \int y\chi_k d\mu \int \chi_k z d\mu \geqslant 0, \; k > 1 \end{cases}$$

and one can set $\gamma = \int y d\mu$. In this way, (41) holds whenever

$$\int y\chi_k d\mu \int \chi_k z d\mu \geqslant 0 \text{ for } k > 1,$$

where $y \in L^p$ and $z \in L^q$ provided that $p, q > 1$ and $1/p + 1/q = 1$. This result for the case $p = 2$ is obtained in [6] in another way. Particularly, (41) is valid if $\int y\chi d\mu \geqslant 0$ and $\int \chi z d\mu \geqslant 0$ for all $\chi \in \mathscr{H}$, i.e. $y \in \text{dual}_p \mathscr{H}$ and $z \in \text{dual}_q \mathscr{H}$. By (38), $\mathscr{H} \subset \text{dual}_p D^q$. Hence by (3), $D^q = \text{dual}_q \text{dual}_p D^q \subset \text{dual}_q \mathscr{H}$, $q > 1$. Summarizing, *Chebyshev integral inequality* holds whenever $y \in D^p$ and $z \in D^q$. This is a classic result. □

EXAMPLE 5. Let $x = \chi_1$ and v is measurable and bounded with $\int v d\mu > 0$. Now, applying Corollary 6 to $y \in L^p$ and zv in place of $z \in L^q$ gives

$$\int zv d\mu \int yv d\mu \leqslant \int yzv d\mu \int v d\mu \tag{42}$$

whenever there exists a scalar γ such that

$$y - \gamma\chi_1 \sim zv - \frac{\int zv d\mu}{\int v d\mu} v. \tag{43}$$

Inequality (42) is the *weighted version of Chebyshev integral inequality*.

Observe that $\gamma\chi_1 \sim zv - \frac{\int zv d\mu}{\int v d\mu} v$ for any γ. Hence (43) can be equivalently replaced by

$$y \sim zv - \frac{\int zv d\mu}{\int v d\mu} v. \quad \square$$

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