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ON A WEIGHTED SUM OF DISTANCES FROM A WELL DISTRIBUTED SET OF POINTS

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Abstract. Suppose that ξ is a complex number, $t>0$ and $w_1,\dots,w_d \geqslant 0$. Let Δ be the modulus of the product of $d(d-1)/2$ distances between complex numbers $z_1,\dots,z_d$ labelled so that $|z_1-\xi| \geqslant \dots \geqslant |z_d-\xi|$. We prove that the sum $\frac{1}{d}\sum_{i=1}^d w_i|z_i-\xi|^t$ is at least

$$\frac{\sqrt{e}}{2}e^{-1/d}\Delta^{2t/d(d-1)}d^{-t/(d-1)}\prod_{i=1}^{d-1} w_i^{2(d-i)/d(d-1)}$$

and show that this inequality is sharp for certain choice of weights w_i. This inequality is then applied to sets of conjugate algebraic integers.

1. Introduction

Let $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Q}$ be the sets of complex numbers, real numbers and rational numbers, respectively. Given a positive number δ, we say that a set of $d \geqslant 2$ numbers $z_1,z_2,\dots,z_d \in \mathbb{C}$ is δ*-distributed* if

$$\prod_{1\leqslant i<j\leqslant d} |z_i-z_j| \geqslant \delta.$$

For example, a full set of d conjugates of an algebraic integer of degree $d \geqslant 2$ over $\mathbb{Q}$ is 1-distributed. For a given collection of non-negative weights $w_1,\dots,w_d \in \mathbb{R}$ and a positive number t, we are interested how small can the sum $\frac{1}{d}\sum_{i=1}^d w_i|z_i-\xi|^t$ be when ξ runs through $\mathbb{C}$ and the set $z_1,\dots,z_d$ is, say, 1-distributed? In particular, for $w_1=\dots=w_d=1$ and $t=2$, we ask the following question of elementary geometry:

QUESTION 1. *Let $d \geqslant 2$ be an integer, and let $A_1,\dots,A_d$ be a set of 1-distributed points on the plane, i.e., $\prod_{1\leqslant i<j\leqslant d}|A_iA_j| \geqslant 1$. Is it true that*

$$\frac{|PA_1|^2+\dots+|PA_d|^2}{d} \geqslant d^{-2/(d-1)} \tag{1}$$

for every point P?

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Evidently, the answer to Question 1 is 'yes' for $d=2$. Equality in (1) is attained for the midpoint P of the segment A_1A_2 of length 1. Note that inequality (1) if proved would be the best possible. Indeed, if $A_1,\dots,A_d$ are the vertices of a regular polygon inscribed into a circle of radius $d^{-1/(d-1)}$ and center P then $\prod_{1\leqslant i<j\leqslant d}|A_iA_j|=1$ (see equality (5) below) and $|PA_1|=\dots=|PA_d|=d^{-1/(d-1)}$, so the left hand side of (1) equals $d^{-2/(d-1)}$.

The best known lower bound in (1) is with the constant $\sqrt{e}/2-\varepsilon$ on the right hand side of (1) for each sufficiently large d (see [3]). It is quite far from the expected constant $1-\varepsilon$. Adding weights $w_i\geqslant 0$ and replacing squares by arbitrary powers $t>0$, we prove the following:

THEOREM 2. *Let $d\geqslant 2$ be an integer, $t>0$, $w_1,\dots,w_d\geqslant 0$, $\xi\in\mathbb{C}$ and $z_1,\dots,z_d\in\mathbb{C}$ are labelled so that $|z_1-\xi|\geqslant\dots\geqslant|z_d-\xi|$. Then*

$$\frac{1}{d}\sum_{i=1}^{d}w_i|z_i-\xi|^t\geqslant\frac{\sqrt{e}}{2}e^{-1/d}\prod_{i=1}^{d-1}w_i^{2(d-i)/d(d-1)}\Big(\prod_{1\leqslant i<j\leqslant d}|z_i-z_j|\Big)^{2t/d(d-1)}d^{-t/(d-1)}. \tag{2}$$

Suppose that $y_1\geqslant\dots\geqslant y_{d-1}>0$. It is easy to see that if $w_1,\dots,w_{d-1}$ are precisely $d-1$ elements of the multiset $y_1,\dots,y_{d-1}$ then the product $\prod_{i=1}^{d-1}w_i^{2(d-i)/d(d-1)}$ is maximal when $w_i=y_i$ for each $i=1,\dots,d-1$ and minimal when $w_i=y_{d-i}$ for each $i=1,\dots,d-1$. The same holds for the sum $\sum_{i=1}^{d}w_i|z_i-\xi|^t$, because $|z_1-\xi|^t\geqslant\dots\geqslant|z_d-\xi|^t$.

Let f be a continuous function defined in the interval $[0,1]$ such that $f(x)>0$ for $x\in[0,1)$. Set

$$I(f):=\int_0^1(1-x)\log f(x)\,dx. \tag{3}$$

For $w_i=f(i/d)$, we have

$$\log\Big(\prod_{i=1}^{d-1}w_i^{(d-i)/d(d-1)}\Big)=\frac{1}{d-1}\sum_{i=1}^{d-1}(1-i/d)\log f(i/d)=\frac{d}{d-1}R(I(f)),$$

where $R(I(f))=\frac{1}{d}\sum_{i=0}^{d-1}(1-i/d)\log f(i/d)$ the Riemann sum of the integral $I(f)$. Hence

$$\prod_{i=1}^{d-1}w_i^{2(d-i)/d(d-1)}\sim e^{2I(f)}\quad\text{as}\quad d\to\infty,$$

so Theorem 2 implies the following:

COROLLARY 3. *For every $\varepsilon>0$, there is a constant $d_0=d_0(\varepsilon)$ such that, for each $\xi\in\mathbb{C}$, each collection of 1-distributed points $z_1,\dots,z_d\in\mathbb{C}$, where $d\geqslant d_0(\varepsilon)$, labelled so that $|z_1-\xi|\geqslant\dots\geqslant|z_d-\xi|$, each continuous function f defined on $[0,1]$, positive on $[0,1)$, and each $t\in\mathbb{R}$ satisfying $0<t<d(\log d)^{-1-\varepsilon}$, we have*

$$\frac{1}{d}\sum_{i=1}^{d}f(i/d)|z_i-\xi|^t>\frac{1}{2}e^{1/2+2I(f)}-\varepsilon.$$

In fact, as it is stated in the proof of Theorem 2, the factor $e^{1/2-1/d}$ appears in (2) from the estimate $\frac{2}{d(d-1)}\sum_{i=1}^{d-1} i\log i \leqslant \log(d-1)-1/2+1/d$ for $d \geqslant 2$ (see Lemma 5). Under conditions of the theorem, we will prove the following slightly stronger inequality:

$$\frac{1}{d}\sum_{i=1}^{d} w_i|z_i-\xi|^t \geqslant \frac{d-1}{2}\prod_{i=1}^{d-1}\Big(\frac{w_i}{d-i}\Big)^{2(d-i)/d(d-1)}\Big(\prod_{1\leqslant i<j\leqslant d}|z_i-z_j|\Big)^{2t/d(d-1)} d^{-t/(d-1)}. \tag{4}$$

This inequality is sharp. Indeed, select an arbitrary $\xi \in \mathbb{C}$, $w_i = 1-i/d$ and $z_i = \xi + u\omega^i$ for $i = 1,\dots,d$, where u is a non-zero complex number and ω is the primitive dth root of unity. Then

$$\prod_{1\leqslant i<j\leqslant d}|z_i-z_j| = |u|^{d(d-1)/2}d^{d/2}. \tag{5}$$

The left hand side of (4) is

$$\frac{1}{d}\sum_{i=1}^{d} w_i|z_i-\xi|^t = \frac{1}{d}\sum_{i=1}^{d}(1-i/d)|u|^t = \frac{(d-1)|u|^t}{2d}.$$

Since, by (5), $\Big(\prod_{1\leqslant i<j\leqslant d}|z_i-z_j|\Big)^{2t/d(d-1)} d^{-t/(d-1)} = |u|^t$, the right hand side of (4) is the same:

$$\frac{d-1}{2}\prod_{i=1}^{d-1}\Big(\frac{w_i}{d-i}\Big)^{2(d-i)/d(d-1)}\Big(\prod_{1\leqslant i<j\leqslant d}|z_i-z_j|\Big)^{2t/d(d-1)} d^{-t/(d-1)}$$

$$= \frac{d-1}{2}\Big(\prod_{i=1}^{d-1} d^{-2(d-i)/d(d-1)}\Big)|u|^t = \frac{(d-1)|u|^t}{2d}.$$

In particular, inequality (4) is sharp for the set of conjugates $z_1 = \alpha_1,\dots,z_d = \alpha_d$ of an algebraic number $\alpha = e^{\pi\sqrt{-1}/2^m}$ of degree $d = 2^m$. The discriminant of the cyclotomic polynomial $\Phi(x) = x^{2^m}+1 = (x-\alpha_1)\dots(x-\alpha_d)$ (which is the minimal polynomial of α over $\mathbb{Q}$) is equal to $\prod_{1\leqslant i<j\leqslant d}|\alpha_i-\alpha_j|^2 = d^d$ (see, e.g., [1]). In Section 4 we shall state two corollaries of the theorem to sets of conjugate algebraic integers.

ion Auxiliary results

The following lemma was proved by Remak [9]. See also [6] for another proof, [2] for a generalization and [4] for some applications of Remak's inequality.

LEMMA 4. *Let $d \geqslant 2$ be an integer and $v_1,\dots,v_d \in \mathbb{C}$, where $|v_1| \geqslant |v_2| \geqslant \dots \geqslant |v_d|$. Then*

$$\prod_{1\leqslant i<j\leqslant d}|v_i-v_j| \leqslant d^{d/2}|v_1|^{d-1}|v_2|^{d-2}\dots|v_{d-1}|.$$

Our next lemma is the following standard estimate:

LEMMA 5. *For every $d \geqslant 2$, we have*

$$\sum_{i=1}^{d-1} i \log i \leqslant \frac{d(d-1)\log(d-1)}{2} - \frac{(d-1)(d-2)}{4}.$$

Lemma 5 follows from the next stronger inequality

$$\sum_{i=1}^{d-1} i \log i \leqslant \frac{d(d-1)\log(d-1)}{2} - \frac{d(d-2)}{4} + \frac{\log(d-1)}{12}$$

for $d \geqslant 2$ (see, e.g., [3]).

We shall also need the following identity:

LEMMA 6. *Let $d \geqslant 2$ be an integer, $\xi, z_1, \dots, z_d \in \mathbb{C}$, and $w_1, \dots, w_d \in \mathbb{R}$. Then*

$$\sum_{1 \leqslant i < j \leqslant d} w_i w_j |z_i - z_j|^2 = \sum_{i=1}^{d} w_i \sum_{i=1}^{d} w_i |z_i - \xi|^2 - \Big| \sum_{i=1}^{d} w_i z_i - \xi \sum_{i=1}^{d} w_i \Big|^2.$$

In particular, if $\sum_{i=1}^{d} w_i \neq 0$ then

$$\sum_{1 \leqslant i < j \leqslant d} w_i w_j |z_i - z_j|^2 = \sum_{i=1}^{d} w_i \sum_{i=1}^{d} w_i |z_i - \zeta|^2 \tag{6}$$

for $\zeta = \sum_{i=1}^{d} w_i z_i / \sum_{i=1}^{d} w_i$.

Proof. Note that

$$\begin{aligned} |z_i - z_j|^2 &= (z_i - \xi - z_j + \xi)(\overline{z_i} - \overline{\xi} - \overline{z_j} + \overline{\xi}) \\ &= |z_i - \xi|^2 + |z_j - \xi|^2 - (z_i - \xi)(\overline{z_j} - \overline{\xi}) - (\overline{z_i} - \overline{\xi})(z_j - \xi). \end{aligned}$$

Hence

$$\begin{aligned} 2 \sum_{1 \leqslant i < j \leqslant d} w_i w_j |z_i - z_j|^2 &= \sum_{i,j=1}^{d} w_i w_j |z_i - z_j|^2 \\ &= \sum_{i,j=1}^{d} w_i w_j \big(|z_i - \xi|^2 + |z_j - \xi|^2 - (z_i - \xi)(\overline{z_j} - \overline{\xi}) - (\overline{z_i} - \overline{\xi})(z_j - \xi)\big) \\ &= 2 \sum_{i=1}^{d} w_i \sum_{i=1}^{d} w_i |z_i - \xi|^2 - 2 \Big| \sum_{i=1}^{d} w_i (z_i - \xi) \Big|^2 \\ &= 2 \sum_{i=1}^{d} w_i \sum_{i=1}^{d} w_i |z_i - \xi|^2 - 2 \Big| \sum_{i=1}^{d} w_i z_i - \xi \sum_{i=1}^{d} w_i \Big|^2. \end{aligned}$$

This completes the proof of the lemma. □

2. Proof of Theorem 2

We will first prove (4). For $d=2$, the right hand side of (4) is equal to $w_1|z_1-z_2|^t 2^{-1-t}$. Since $|z_1-\xi| \geqslant |z_2-\xi|$, by the triangle inequality $|z_1-z_2| \leqslant |z_2-\xi| + |z_1-\xi|$, we have $|z_1-\xi| \geqslant |z_1-z_2|/2$. So the left hand side of (4) is

$$\frac{w_1|z_1-\xi|^t + w_2|z_2-\xi|^t}{2} \geqslant \frac{w_1|z_1-\xi|^t}{2} \geqslant \frac{w_1(|z_1-z_2|/2)^t}{2} = w_1|z_1-z_2|^t 2^{-1-t},$$

which is the right hand side of (4). The result is also trivial if at least one of the weights $w_1,\dots,w_{d-1}$ is equal to zero. This proves (4) for $d=2$ and for $w_1\dots w_{d-1}=0$. Therefore, we may assume that $d\geqslant 3$ and $w_1,\dots,w_{d-1}>0$.

Put

$$\Delta := \prod_{1\leqslant i<j\leqslant d} |z_i-z_j|$$

and $u_i := |z_i-\xi|$. We have $u_1 \geqslant u_2 \geqslant \dots \geqslant u_d$. Note that $\Delta = \prod_{1\leqslant i<j\leqslant d}|z_i-z_j| = \prod_{1\leqslant i<j\leqslant d}|(z_i-\xi)-(z_j-\xi)|$ is the modulus of the Vandermonde determinant consisting of the powers of $z_1-\xi,\dots,z_d-\xi$. Hence, by Lemma 4 with $v_i := z_i-\xi$ (so that $u_i=|v_i|$) for $i=1,\dots,d$, we obtain

$$\Delta \leqslant d^{d/2} u_1^{d-1} u_2^{d-2} \dots u_{d-1}.$$

We may assume that $u_{d-1}>0$, for otherwise $u_{d-1}=u_d=0$, so $z_{d-1}=z_d=\xi$. Then both (2) and (4) are trivial, because $\Delta=0$. Since $u_1,\dots,u_{d-1}$ are all positive, we have

$$(d-1)\log u_1 + (d-2)\log u_2 + \dots + \log u_{d-1} \geqslant \log\Delta - \frac{d\log d}{2}.$$

If this inequality is strict, we can replace each u_i by u_i/ν with some $\nu>1$, so that equality

$$(d-1)\log u_1 + (d-2)\log u_2 + \dots + \log u_{d-1} = \log\Delta - \frac{d\log d}{2} \tag{7}$$

holds. In order to minimize the sum $\sum_{i=1}^{d-1} w_i u_i^t$ with the restriction on the variables $u_1 \geqslant \dots \geqslant u_{d-1} > 0$ given by (7) we shall use the method of Lagrange multipliers.

Consider the Lagrange function

$$F(u_1,\dots,u_{d-1},\lambda) = \sum_{i=1}^{d-1} w_i u_i^t - \lambda\left(\sum_{i=1}^{d-1}(d-i)\log u_i - \log(\Delta d^{-d/2})\right).$$

Then $\frac{\partial F}{\partial u_i}=0$ leads to $w_i t u_i^{t-1} = \lambda(d-i)/u_i$, giving

$$u_i^t = \lambda(d-i)/w_i t \tag{8}$$

for $i=1,\dots,d-1$. Using $t\log u_i = \log((d-i)/w_i) + \log(\lambda/t)$, from (7) we deduce that

$$t\log(\Delta d^{-d/2}) = t\sum_{i=1}^{d-1}(d-i)\log u_i = \sum_{i=1}^{d-1}(d-i)\log((d-i)/w_i) + \frac{d(d-1)\log(\lambda/t)}{2}.$$

Multiplying this equality by $2/d(d-1)$, we obtain

$$\log(\lambda/t) = \frac{2t\log\Delta}{d(d-1)} - \frac{t\log d}{d-1} + \frac{2}{d(d-1)}\sum_{i=1}^{d-1}(d-i)\log(w_i/(d-i)).$$

Consequently,

$$\frac{\lambda}{t} = \Delta^{2t/d(d-1)}d^{-t/(d-1)}\prod_{i=1}^{d-1}\Big(\frac{w_i}{d-i}\Big)^{2(d-i)/d(d-1)}. \tag{9}$$

It is easily seen that the point $(u_1,\dots,u_{d-1})$ given by (8), with λ is given by (9), is the point of the global minimum of the function $\sum_{i=1}^{d-1} w_i u_i^t$. Thus, using (8) and (9), we find that

$$\frac{1}{d}\sum_{i=1}^{d} w_i|z_i-\xi|^t \geqslant \frac{1}{d}\sum_{i=1}^{d-1} w_i|z_i-\xi|^t \geqslant \frac{1}{d}\sum_{i=1}^{d-1} w_i u_i^t = \frac{\lambda}{dt}\sum_{i=1}^{d-1}(d-i)$$
$$= \frac{\lambda(d-1)}{2t} = \frac{d-1}{2}\Delta^{2t/d(d-1)}d^{-t/(d-1)}\prod_{i=1}^{d-1}\Big(\frac{w_i}{d-i}\Big)^{2(d-i)/d(d-1)}.$$

This proves (4).

By Lemma 5,

$$\prod_{i=1}^{d-1}(d-i)^{2(d-i)/d(d-1)} = \prod_{i=1}^{d-1} i^{2i/d(d-1)} \leqslant (d-1)e^{-(d-2)/2d} = (d-1)e^{-1/2+1/d}.$$

Thus

$$(d-1)\prod_{i=1}^{d-1}(d-i)^{-2(d-i)/d(d-1)} \geqslant e^{1/2-1/d}.$$

Hence (4) implies (2). $\square$

3. Examples and applications to algebraic integers

Set in Corollary 3 $f(x) = 1-x$ and assume that $z_i = \alpha_i$ $(i = 1,\dots,d)$ is the set of conjugates of an algebraic integer α over $\mathbb{Q}$. By (3),

$$I(f) = \int_0^1 (1-x)\log(1-x)\,dx = -1/4,$$

so $1/2 + 2I(f) = 0$ and we obtain the following:

COROLLARY 7. *For every $\varepsilon > 0$, there is a constant $d_1 = d_1(\varepsilon)$ such that, for each $\xi \in \mathbb{C}$, each algebraic integer α of degree d, where $d \geqslant d_1(\varepsilon)$, with conjugates $\alpha_1,\dots,\alpha_d$ labelled so that $|\alpha_1-\xi| \geqslant \dots \geqslant |\alpha_d-\xi|$, and each $t \in \mathbb{R}$ satisfying $0 < t < d(\log d)^{-1-\varepsilon}$, we have*

$$\frac{1}{d}\sum_{i=1}^{d}(1-i/d)|\alpha_i-\xi|^t > \frac{1}{2}-\varepsilon. \tag{10}$$

The constant $1/2$ on the right hand side of (10) is best possible, because if α is a root of unity of degree $\deg\alpha = d$ over $\mathbb{Q}$ and $\xi = 0$ then $|\alpha_i - \xi| = 1$ for every $i = 1, \dots, d$, so

$$\frac{1}{d}\sum_{i=1}^{d}(1-i/d)|\alpha_i - \xi|^t = \frac{1}{d}\sum_{i=1}^{d}(1-i/d) = \frac{1}{2} - \frac{1}{2d}.$$

By Corollary 7, for every $\varepsilon > 0$, every $\xi \in \mathbb{C}$, every algebraic integer α of a sufficiently large degree d, and every $t \in (0, d(\log d)^{-1-\varepsilon})$,

$$\sum_{i=1}^{d}\mu_i|\alpha_i - \xi|^t > 1 - \varepsilon$$

for the choice of weights $\mu_i = 2(d-i)/d(d-1)$ satisfying $\sum_{i=1}^{d}\mu_i = 1$.

Applying Lemma 6 to the full set of conjugates of an algebraic integer α and to $t = 2$ we will derive the following statement:

COROLLARY 8. *For every $\varepsilon > 0$, there is a constant $d_2 = d_2(\varepsilon)$ such that, for each algebraic integer α of degree d, where $d \geqslant d_2(\varepsilon)$, with conjugates $\alpha_1, \dots, \alpha_d$ and each continuous non-decreasing positive function $f(x)$ on $[0,1]$, we have*

$$\sum_{1 \leqslant i < j \leqslant d}\mu_{ij}|\alpha_i - \alpha_j|^2 > \frac{e^{1/2+2I(f)}}{J(f)} - \varepsilon,$$

where $\mu_{ij} := f(i/d)f(j/d)/S$, $S := \sum_{1\leqslant i<j\leqslant d} f(i/d)f(j/d)$ and $J(f) := \int_0^1 f(x)\,dx$.

Proof. Take $w_i = f(i/d)$, $z_i = \alpha_i$ and $\zeta = \big(\sum_{i=1}^{d} f(i/d)\alpha_i\big)/\big(\sum_{i=1}^{d} f(i/d)\big)$ in Lemma 6. Then, by (6),

$$\sum_{1\leqslant i<j\leqslant d} f(i/d)f(j/d)|\alpha_i - \alpha_j|^2 = \sum_{i=1}^{d} f(i/d)\sum_{i=1}^{d} f(i/d)|\alpha_i - \zeta|^2. \tag{11}$$

Let σ be a permutation of the set $\{1, \dots, d\}$ such that $|\alpha_{\sigma(1)} - \zeta| \geqslant \dots \geqslant |\alpha_{\sigma(d)} - \zeta|$. By Theorem 2 with $t = 2$, for each $\varepsilon > 0$ and each sufficiently large d, the sum

$$\sum_{i=1}^{d} f(i/d)|\alpha_i - \zeta|^2 = \sum_{i=1}^{d} f(\sigma(i)/d)|\alpha_{\sigma(i)} - \zeta|^2$$

is at least $(1-\varepsilon)d\frac{\sqrt{e}}{2}\prod_{i=1}^{d-1} f(\sigma(i)/d)^{2(d-i)/d(d-1)}$. Since $f(x)$ is non-decreasing and $f(0) > 0$, we have

$$\begin{aligned}\prod_{i=1}^{d-1} f(\sigma(i)/d)^{2(d-i)/d(d-1)} &= \prod_{i=1}^{d} f(\sigma(i)/d)^{2(d-i)/d(d-1)} \\ &\geqslant \prod_{i=1}^{d} f(i/d)^{2(d-i)/d(d-1)} = \prod_{i=1}^{d-1} f(i/d)^{2(d-i)/d(d-1)} \sim e^{2I(f)}\end{aligned}$$

as $d \to \infty$, by (3). Note that $S/d^2 \sim \frac{1}{2}J(f)^2$ and $\frac{1}{d}\sum_{i=1}^{d} f(i/d) \sim J(f)$ as $d \to \infty$. Hence, by (11),

$$\sum_{1\leqslant i<j\leqslant d} \mu_{ij}|\alpha_i - \alpha_j|^2 > (1-2\varepsilon)\frac{e^{1/2+2I(f)}}{J(f)}.$$

This completes the proof. $\square$

The sum of weights μ_{ij} in Corollary 8 is equal to 1, i.e., $\sum_{1\leqslant i<j\leqslant d} \mu_{ij} = 1$. Taking $f(x) = 1$, by (3), we obtain $I(f) = 0$ and $J(f) = \int_0^1 f(x)\,dx = 1$. This recovers our earlier result [3] stating that, for each $\varepsilon > 0$ and each algebraic integer α of degree $d \geqslant d_3(\varepsilon)$ with conjugates $\alpha_1, \dots, \alpha_d$,

$$\frac{2}{d(d-1)} \sum_{1\leqslant i<j\leqslant d}^{d} |\alpha_i - \alpha_j|^2 > \sqrt{e} - \varepsilon. \tag{12}$$

A corresponding problem for the conjugates of a totally real algebraic integer have been considered by Schur [10]. See also [8] for a solution of another Schur's problem from [10].

Unlike the constant $1/2$ in (10), it seems likely that the constant $e^{1/2+2I(f)}/J(f)$ of Corollary 8 is not optimal. Moreover, with this method, we cannot get anything better than $\sqrt{e}$. Indeed, for example, by considering two Riemann sums

$$R(I(f)) = \frac{1}{N}\sum_{i=0}^{N-1}(1-i/N)\log f(i/N), \quad R(J(f)) = \frac{1}{N}\sum_{i=0}^{N-1} f(i/N) \geqslant \prod_{i=0}^{N-1} f(i/N)^{1/N}$$

of the integrals $I(f) = \int_0^1 (1-x)\log f(x)\,dx$ and $J(f) = \int_0^1 f(x)\,dx$, respectively, and letting $N \to \infty$, one can show that $e^{2I(f)} \leqslant J(f)$ for each continuous non-decreasing positive function $f(x)$ on $[0,1]$, so that $e^{1/2+2I(f)}/J(f) \leqslant \sqrt{e}$. For any positive function $f(x)$ on $[0,1)$, one can show that $e^{1/2+2I(f)}/J(f) \leqslant 2$.

The expected constant in (12) instead of $\sqrt{e}$ is 2. This constant would be the best possible, because if $\mu_{ij} = 2/d(d-1)$ for every pair i, j satisfying $1 \leqslant i < j \leqslant d$ and if α is a root of unity of trace zero, i.e., $\alpha_1 + \ldots + \alpha_d = 0$, then, by (6) with $\zeta = 0$ and $w_i = 1$, we have $\frac{2}{d(d-1)}\sum_{1\leqslant i<j\leqslant d} |\alpha_i - \alpha_j|^2 = 2$.

Langevin [5] proved earlier that, for every ε and every algebraic integer α of degree $d \geqslant d_4(\varepsilon)$, the inequality $\max_{1\leqslant i<j\leqslant d} |\alpha_i - \alpha_j| > 2 - \varepsilon$ holds. This solved an old 1928 problem of Favard; see also [7]. In this context, the natural expected constant in the lower bound

$$\Big(\frac{2}{d(d-1)} \sum_{1\leqslant i<j\leqslant} |\alpha_i - \alpha_j|^t\Big)^{1/t} > C(t) - \varepsilon$$

for the 'average' mean (at least for $t \geqslant 2$) taken over the conjugates of an algebraic integer α of degree $d \geqslant d_5(\varepsilon, t)$ is

$$C(t) = 2\pi^{1/2t}\Gamma(t/2+1/2)^{1/t}\Gamma(t/2+1)^{-1/t},$$

where $\Gamma(z)$ stands for the Euler gamma function. One can check easily on roots of unity α that then the constant $C(t)$ would be the best possible.

Since Corollary 7 appears to be the only result of this kind with the optimal constant in (10), it would be of interest to obtain a sharp version of (10) for some other choice of weights $w_1, \ldots, w_d$.

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