Mathematical
Inequalities
& Applications
Volume 14, Number 2 (2011), 455–458

CONVEXITY OF $f(A) = (\det A)^m$

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(*Communicated by S. Puntanen*)

Abstract. Sufficient conditions for the convexity of the function $f(A) = (\det A)^m$ $(m \geqslant 1)$ has been obtained.

1. Introduction

We use conventional notions and notations, as in [1], where M_n denotes the set of $n \times n$ matrices. A matrix $X \in M_n$ is said to be positive definite if $\operatorname{Re}(x^* X x) > 0$ for all nonzero $x \in \mathbb{C}^n$. The convex set of positive definite matrices is denoted by M_n^+.

DEFINITION 1. A real valued function f defined on M_n^+ is said to be convex if $f(\alpha A + \beta B) \leqslant \alpha f(A) + \beta f(B)$, and concave if $f(\alpha A + \beta B) \geqslant \alpha f(A) + \beta f(B)$ for all $0 < \alpha < 1, \alpha + \beta = 1$ and all $A, B \in M_n^+, A \neq B$.

It has been proved by Horn, Johnson [1, p. 466] that the function $f(A) = \log(\det A)$ is strictly concave function on the convex set of positive definite Hermitian matrices in M_n. Therefore

$$\log\det(\alpha A + \beta B) \geqslant \alpha \log\det A + \beta \log\det B, \tag{1}$$

for positive definite matrices A, $B \in M_n$ and α, $\beta > 0$ with $\alpha + \beta = 1$. It is interesting to notice that the function $f(A) = (\det A)^{1/n}$ is also concave on the set of positive definite Hermitian matrices i.e.

$$[\det(\alpha A + (1-\alpha)B)]^{1/n} \geqslant \alpha(\det A)^{1/n} + (1-\alpha)(\det B)^{1/n}. \tag{2}$$

This follows directly from the famous Minkowski inequality which states that:

If $A, B \in M_n(R)$ are real positive definite matrices, then

$$[\det(A+B)]^{1/n} \geqslant (\det A)^{1/n} + (\det B)^{1/n}. \tag{3}$$

Although we get from (3) that

$$\det(A+B) \geqslant \det A + \det B, \tag{4}$$

Mathematics subject classification (2010): 15A15.
Keywords and phrases: convexity; power of determinant.

Paper MIA-14-37

it is clear that inequality (4) does not imply that $f(A) = \det A$ is concave.

In fact the function $f(A) = (\det A)^m$ is generally not concave for $m \neq \frac{1}{n}$. Recently Zhan [2] obtained some sufficient conditions so that

$$[\det(A+B)]^t \geqslant (\det A)^t + (\det B)^t, \tag{5}$$

where $t \in R$ and $t \geqslant \frac{2}{n}$

The question of convexity of the function $f(A) = \det A$ has not been treated up to the knowledge of the author. The purpose of this article is to discuss this question.

2. Main results

For the proof of the main results we need the following:

LEMMA 2. *If $\alpha, \beta > 0$ such that $\alpha + \beta = 1$, and $\lambda_i > 1$ for $i = 1, 2, ..., n$ or $0 < \lambda_i < 1$ for $i = 1, 2, ..., n$, then*

$$\prod_{i=1}^{n} (\alpha + \beta \lambda_i) \leqslant \alpha + \beta \prod_{i=1}^{n} \lambda_i. \tag{6}$$

Proof. For $\alpha, \beta > 0$, and $\alpha + \beta = 1$, the identity

$$(\alpha + \beta u)(\alpha + \beta v) = (\alpha + \beta uv) + \alpha\beta(u-1)(1-v) \tag{7}$$

implies that inequality (6) is true for $n = 2$.

Assume that (6) is true for $n = m$, we prove that it is true for $n = m+1$. Since (6) holds for $n = m$ we have

$$\begin{aligned} \prod_{i=1}^{m+1} (\alpha + \beta \lambda_i) &= (\alpha + \beta \lambda_{m+1}) \prod_{i=1}^{m} (\alpha + \beta \lambda_i) \\ &\leqslant (\alpha + \beta \lambda_{m+1}) (\alpha + \beta \prod_{i=1}^{m} \lambda_i). \end{aligned} \tag{8}$$

Applying the identity (7) we obtain

$$(\alpha + \beta \lambda_{m+1}) (\alpha + \beta \prod_{i=1}^{m} \lambda_i) = (\alpha + \beta \lambda_{m+1} \prod_{i=1}^{m} \lambda_i) + + \alpha\beta(\lambda_{m+1} - 1)(1 - \prod_{i=1}^{m} \lambda_i). \tag{9}$$

Since the second term in (9) is negative for $0 < \lambda_1, ..., \lambda_{m+1} < 1$ or for $\lambda_1, ..., \lambda_{m+1} > 1$. It follows from (8) and (9) that

$$\prod_{i=1}^{m+1} (\alpha + \beta \lambda_i) \leqslant (\alpha + \beta \lambda_{m+1} \prod_{i=1}^{m} \lambda_i)$$

and consequently that inequality (6) holds for $n = m+1$. □

THEOREM 3. *Let $A, B \in M_n$ be positive definite matrices, λ_j $(j = 1, 2, ..., n)$ be the eigenvalues of $A^{-1/2}\, B\, A^{-1/2}$. If all $\lambda_1, ..., \lambda_n > 1$ or $\lambda_1, ..., \lambda_n < 1$, then*

$$\det(\alpha A + \beta B) \leqslant \alpha \det A + \beta \det B, \tag{10}$$

where $\alpha, \beta > 0$ with $\alpha + \beta = 1$.

Proof. If we multiply both sides of the asserted inequality (10) from the right and left by $\det A^{-1/2}$, the desired inequality is then equivalent to

$$\prod_{i=1}^{n} (\alpha + \beta \lambda_i) \leqslant \alpha + \beta \prod_{i=1}^{n} \lambda_i$$

which is true by Lemma 2. □

COROLLARY 4. *If all the eigenvalues of $A^{-1/2}\, B\, A^{-1/2}$ are either > 1 or < 1 for any two positive definite matrices A, B, then $f(A) = (\det A)^m$ is a convex function over the set M_n^+ for $m > 1$.*

Proof. Since t^m is convex over any positive interval for $m > 1$, it follows $(\alpha t + \beta s)^m \leqslant \alpha t^m + \beta s^m$ for $\alpha > 0$, $\alpha + \beta = 1$ and any positive real numbers t and s. Putting $t = \det A$ and $s = \det B$, it follows that

$$(\alpha \det A + \beta \det B)^m \leqslant \alpha (\det A)^m + \beta (\det B)^m. \tag{11}$$

From (3) and (11) it follows that

$$(\det(\alpha A + \beta B))^m \leqslant \alpha (\det A)^m + \beta (\det B)^m,$$

and the proof is complete. □

In fact the conditions on $\lambda_1, ..., \lambda_n$ in Theorem (3) are essential as the following example shows.

EXAMPLE 5. Assume that

$$A = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}.$$

Then

$$A^{-1/2}\, B\, A^{-1/2} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} B \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 14 \\ 14 & 22 \end{pmatrix}$$

has the eigenvalues $\lambda_1 = 0.06 < 1$ and $\lambda_2 = 30.94 > 1$. From the other side we can easily verify for $\alpha = \beta = \frac{1}{2}$ that

$$\det \frac{1}{2}(A + B) \not\leqslant \frac{1}{2} \det A + \frac{1}{2} \det B.$$

THEOREM 6. *Let* $A,\ B \in M_n^+$, $\lambda_j(A)$, $\lambda_j(B)$, $j = 1,2,\ldots,n$ *be the eigenvalues of* A *and* B, $\alpha,\beta > 0$ *with* $\alpha+\beta = 1$. *Then*

$$\det(\alpha A + \beta B) \leqslant \alpha \det A + \beta \det B \tag{12}$$

holds true if one of the following conditions is satisfied

$$(i)\ \lambda_j(B) < \lambda_{\min}(A) \quad for \quad j = 1,2,\ldots,n,$$
$$(i)\ \lambda_j(B) > \lambda_{\max}(A) \quad for \quad j = 1,2,\ldots,n,$$

where $\lambda_{\min}(A) = \min\limits_{1\leqslant j\leqslant n} \lambda_j(A)$ *and* $\lambda_{\max}(A) = \max\limits_{1\leqslant j\leqslant n} \lambda_j(A)$.

Proof. It is known from [1, p. 466] that for two positive definite matrices A and B there exists a nonsingular matrix C such that $A = CC^*$ and $B = C\Lambda C^*$, where $\Lambda = diag\ (\lambda_1,\ldots,\lambda_n)$. The asserted inequality (12) is then equivalent to

$$\det(\alpha + \beta\Lambda) \leqslant \alpha + \beta \det\Lambda$$

or

$$\prod_{i=1}^{n}(\alpha+\beta\lambda_i) \leqslant \alpha + \beta\prod_{i=1}^{n}\lambda_i$$

which is true by Lemma2 if all $\lambda_j > 1$ or all $\lambda_j < 1$ for $j = 1,2,\ldots,n$. To complete the proof, it is sufficient to show that either $\lambda_1,\ldots,\lambda_n > 1$ or $\lambda_1,\ldots,\lambda_n < 1$. Observing that A and CC^* has the same eigenvalues, it follows from Ostrowski Theorem [1, p. 224] that for each $j = 1,2,\ldots,n$, there exists $\theta_j > 0$ such that

$$\lambda_{\min}(A) \leqslant \theta_j \leqslant \lambda_{\max}(A) \tag{13}$$

and

$$\lambda_j(B) = \theta_j\, \lambda_j. \tag{14}$$

From (13) and (14) we conclude that

$$\frac{\lambda_j(B)}{\lambda_{\max}(A)} \leqslant \lambda_j \leqslant \frac{\lambda_j(B)}{\lambda_{\min}(A)}. \tag{15}$$

Therefore $\lambda_j < 1$ for $j = 1,2,\ldots,n$ if the condition (i) is satisfied and $\lambda_j > 1$ for $j = 1,2,\ldots,n$ if (ii) is satisfied. The proof is complete. □

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(Received July 19, 2009)

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