Mathematical
Inequalities
& Applications
Volume 14, Number 2 (2011), 459–467

AN EMBEDDING THEOREM WITH MODIFIED AND RELAXED CONDITIONS

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(*Communicated by J. Pečarić*)

Abstract. In this note we prove an embedding theorem on the interrelation between the class $W^rH_\beta^\omega$ and the class $H(\boldsymbol{\lambda},p,r,\omega)$, defined by strong means of Fourier series, continuing the recent investigations due to S. Tikhonov [5] and R. J. Le and S. P. Zhou [1].

1. Introduction

The aim of this paper is to continue the recent investigations made by S. Tikhonov [5] and R. J. Le and S. P. Zhou [1] pertaining to the strong approximation of Fourier series and embedding theorems. For a short historical survey on this theme we refer to Introduction of Tikhonov's paper.

In order to recall the theorems due to the authors cited above and formulate our new theorem we have to enroll some notions and notations.

Let $f(x)$ be an odd continuous and 2π-periodic function, and let

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx \tag{1.1}$$

be its Fourier series. Let $s_n(x) = s_n(f,x)$ denote the nth partial sum of (1.1). We denote by $\|.\|$ the usual supremum norm.

Let $\omega(\delta)$ be a modulus of continuity function, in symbol $\omega \in \Omega$.

The modulus of smoothness of order β, $\beta > 0$, is defined by

$$\omega_\beta(f,t) := \sup_{|h|\leqq t} \left\| \sum_{\nu=0}^{\infty} (-1)^\nu \binom{\beta}{\nu} f(x+(\beta-\nu)h) \right\|.$$

We shall use the notion $L \ll R$ $(L \gg R)$ at inequalities if there exists a positive constant K such that $L \leqq KR$ $(KL \geqq R)$ holds, not necessarily the same at each occurrence.

A sequence $\mathbf{c} := \{c_n\}$ of positive terms will be called almost monotone increasing $(\mathbf{c} \in AMS)$ if there exists a constant $K(\mathbf{c}) \geqq 1$ such that $K(\mathbf{c})c_n \geqq c_m$ holds for any $n \geqq m$.

Mathematics subject classification (2010): 42A20, 42A32.
Keywords and phrases: Fourier series, strong approximation, generalized monotone sequences.

Paper MIA-14-38

A nonnegative *sequence* $\mathbf{c}$ is said to be *classical quasimonotone* (in symbol $\mathbf{c} \in CQMS$) if, for some $\alpha \geqq 0$, the sequence $\{c_n n^{-\alpha}\}$ is nonincreasing.

A null-sequence $\mathbf{c} := \{c_n\}$ $(c_n \to 0)$ satisfying the inequalities

$$\sum_{n=m}^{\infty} |\Delta c_n| \leqq K(\mathbf{c}) c_m \quad (\Delta c_n := c_n - c_{n+1})$$

with a positive constant $K(\mathbf{c})$ is said to be a *rest bounded variation sequence* (in symbol $\mathbf{c} \in RBVS$).

In [4] we showed that the classes $CQMS$ and $RBVS$ are not comparable.

Recently S. P. Zhou, P. Zhou and D.S. Yu [6] defined a new class of sequences, called *mean value bounded variation sequence* $(MVBVS)$ as follows:

A nonnegative sequence $\mathbf{c} := \{c_n\}$ is said to be a *mean value bounded variation sequence* $(\mathbf{c} \in MVBVS)$ if there is a $\lambda \geqq 2$ such that

$$\sum_{k=n}^{2n} |\Delta c_n| \leqq \frac{K(\mathbf{c})}{n} \sum_{k=[\lambda^{-1}n]}^{[\lambda n]} c_k$$

holds, where $[y]$ means the integral part of y.

In [6] it is shown that the class $MVBVS$ contains both the classes $CQMS$ and $RBVS$, among others.

Let $\boldsymbol{\lambda} := \{\lambda_n\}$ be a sequence of positive terms and set

$$\Lambda_n := \sum_{k=1}^{n} \lambda_k,$$

furthermore denote

$$h(f, \boldsymbol{\lambda}, p) := \left\| \left(\Lambda_n^{-1} \sum_{k=1}^{n} \lambda_k |f(x) - s_k(f,x)|^p \right)^{1/p} \right\|, \quad p > 0.$$

Finally we define the classes of continuous functions to be considered later on:

$$H(\boldsymbol{\lambda}, p, r, \omega) := \{f \in C_{2\pi} : h_n(f, \boldsymbol{\lambda}, p) = O(n^{-r}\omega(n^{-1}))\},$$

$$W^r H_\beta^\omega := \{f \in C_{2\pi} : \omega_\beta(f^{(r)}, \delta) = O(\omega(\delta))\},$$

$$C_1 := \left\{ f \in C_{2\pi} : f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad \mathbf{b} \in CQMS \right\},$$

$$C_2 := \left\{ f \in C_{2\pi} : f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad \mathbf{b} \in RBVS \right\},$$

$$C_3 := \left\{ f \in C_{2\pi} : f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad \mathbf{b} \in MVBVS \right\},$$

where $C_{2\pi}$ denotes the class of continuous functions of period 2π.

Now we can recall the theorems stimulating our new version.

Tikhonov [5] proved his theorem for the classes C_1 and C_2.

THEOREM T. *Let* $\beta,\ p>0,\ r\geqq 0,\ \omega\in\Omega,\ \boldsymbol{\lambda}$ *be a positive sequence satisfying the conditions:*

$$\Lambda_{2n} \ll \Lambda_n, \tag{1.2}$$

and

$$\Lambda_n \ll n\lambda_n. \tag{1.3}$$

If

$$\lambda_n \omega^p(1/n) n^{1-rp} \in AMS, \tag{1.4}$$

then

$$W^r H_\beta^\omega \cap C_j \subset H(\boldsymbol{\lambda}, p, r, \omega), \quad j=1,2.$$

Le and Zhou [1] generalized Tikhonov's theorem to class C_3, namely it is clear that $C_1 \cup C_2 \subset C_3$ subsequent to the fact that the class $MVBVS$ is wider than either C_1 or C_2.

THEOREM LZ. *Under the assumptions of Theorem* T

$$W^r H_\beta^\omega \cap C_3 \subset H(\boldsymbol{\lambda}, p, r, \omega). \tag{1.5}$$

In this work we show that the embedding relation (1.5) holds under two modified assumptions, as well. One of our assumption is weaker than (1.3), and it will be required only if $p>1$, the other one will be an altered version of (1.4).

2. New theorem

Our theorem reads as follows:

THEOREM. *Let* $\beta,\ p>0,\ r\geqq 0,\ \omega\in\Omega,\ \boldsymbol{\lambda}$ *be a positive sequence satisfying* (1.2), *and if* $p>1$ *then additionally*

$$\sum_{n=1}^{m} \left(\frac{\Lambda_n}{\lambda_n}\right)^{p-1} \ll m^p. \tag{2.1}$$

If

$$\Lambda_n \omega^p(1/n) n^{-rp} \in AMS, \tag{2.2}$$

then (1.5) *maintains.*

REMARKS. **1.** We emphasize that condition (2.1) instead of (1.3) is required only if $p>1$; moreover (1.3) implies (2.1), but it is not true conversely.

2. Condition (2.1) gives greater freedom for the sequence $\boldsymbol{\lambda}$, namely certain terms of $\boldsymbol{\lambda}$ can be "small" if they appear rarely. E.g. if $0<\alpha<1/p$ and

$$\lambda_n := \begin{cases} n^k, & \text{if} \quad n \neq 2^k, \\ 1, & \text{if} \quad n = 2^k, \end{cases} \quad (k=1,2,\ldots) \tag{2.3}$$

then (2.1) holds, but (1.3) fails.

3. Condition (2.2) in a certain sense claims more then (1.4) does, but $\{\Lambda_n\}$ is always an increasing sequence on the contrary $\{n\,\lambda_n\}$, see e.g. the case given by (2.3).

3. Lemmas

LEMMA 3.1. ([2]) *Let* $a_n \geqq 0$, $\lambda_n > 0$. *If* $p \geqq 1$, *then*

$$\sum_{n=1}^{\infty} \lambda_n \left(\sum_{\nu=n}^{\infty} a_\nu\right)^p \ll \sum_{n=1}^{\infty} \lambda_n^{1-p} a_n^p \left(\sum_{\nu=1}^{n} \lambda_\nu\right)^p. \tag{3.1}$$

If $0 < p < 1$, $a_n \downarrow$, *then*

$$\sum_{n=1}^{\infty} \lambda_n \left(\sum_{\nu=n}^{\infty} a_\nu\right)^p \ll \sum_{n=1}^{\infty} n^{p-1} a_n^p \left(n\lambda_n + \sum_{\nu=1}^{n-1} \lambda_\nu\right). \tag{3.2}$$

LEMMA 3.2. *If* $f \in W^r H_\beta^\omega$, *then*

$$\omega(1/n) \gg n^{-(\beta+1)} \sum_{\nu=1}^{n} \nu^{r+\beta+1} b_\nu.$$

This lemma is proved in [5] implicitly.

LEMMA 3.3. *Let* $p > 0$, $r \geqq 0$, $\boldsymbol{\lambda} := \{\lambda_n\}$ *be a positive sequence satisfying* (1.2), *and if* $p > 1$, *then the sequence satisfies both* (1.2) *and* (2.1). *Let* $\omega \in \Omega$, $\mathbf{b} := \{b_n\} \in MVBVS$ *and* $b_n \ll n^{-r-1}\omega(1/n)$, *then the Fourier series* (1.1) *converges to* $f(x)$ *uniformly, i.e.*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

Furthermore, if $\Lambda_n \omega^p(1/n) n^{-rp} \in AMS$, *then*

$$f(x) \in H(\boldsymbol{\lambda}, p, r, \omega). \tag{3.3}$$

Proof. We shall follow the common method of the authors of [1] and [5] combining with a procedure used e.g. in [3] by us.

The first part of Lemma 3.3 follows already from the conditions $\mathbf{b} \in MVBVS$ and $nb_n \to 0$, see Theorem 5 in [6].

Thus we have to prove only (3.3).

Since $s_k(f,0) = s_k(f,\pi) = 0$, we may restrict $x \in (0,\pi)$, say, $\frac{\pi}{m+1} < x \leqq \frac{\pi}{m}$. Applying Abel's transformation, we get for $k < m$

$$\begin{aligned} |f(x) - s_k(x)| &\leqq \left|\sum_{\nu=k+1}^{m} b_\nu \sin \nu x\right| + \left|\sum_{\nu=m+1}^{\infty} b_\nu \sin \nu x\right| \\ &\ll \left|x \sum_{\nu=k+1}^{m} \nu b_\nu\right| + \sum_{\nu=m+1}^{\infty} |b_\nu - b_{\nu+1}||\widetilde{D}_\nu(x)| \qquad (3.4) \\ &\ll \frac{1}{m} \sum_{\nu=k+1}^{m} \nu b_\nu + m \sum_{\nu=m+1}^{\infty} |\Delta b_\nu|, \end{aligned}$$

where $\widetilde{D}_\nu(x) := \sum_{k=1}^{\nu} \sin kx$, and $|\widetilde{D}_\nu(x)| = O\left(\frac{1}{x}\right)$.
If $k \geqq m$, then

$$|f(x) - s_k(x)| \ll m \sum_{\nu=k+1}^{\infty} |\Delta b_\nu|.$$

If $n > m$, then we have

$$\sum_{k=1}^{n} \lambda_k |f(x) - s_k(x)|^p = \left(\sum_{k=1}^{m} + \sum_{k=m+1}^{n} \right) \lambda_k |f(x) - s_k(x)|^p =: I_1 + I_2.$$

From (3.4) we obtain that

$$I_1 \ll \sum_{k=1}^{m} \lambda_k \left(\frac{1}{m} \sum_{\nu=k+1}^{m} \nu b_\nu \right)^p + \sum_{k=1}^{m} \lambda_k m^p \left(\sum_{\nu=m+1}^{\infty} |\Delta b_\nu| \right)^p =: I_{11} + I_{12}.$$

First, we estimate I_{11}. If $p \geqq 1$, then (3.1) (setting $a_\nu = \nu b_\nu$ for $\nu \leqq m$ and $a_\nu = 0$ for $\nu > m$) gives that

$$I_{11} \ll \frac{1}{m^p} \sum_{k=1}^{m} \lambda_k^{1-p} \left(\sum_{\nu=1}^{k} \lambda_\nu \right)^p (kb_k)^p. \tag{3.5}$$

Hence, in the special case $p = 1$, we get that

$$I_{11} \ll \frac{1}{m} \sum_{k=1}^{m} \Lambda_k k b_k. \tag{3.6}$$

By using (2.2), (3.6) and $b_k \ll k^{-r-1} \omega(1/k)$, we obtain that

$$I_{11} \ll \Lambda_m m^{-r} \omega(1/m). \tag{3.7}$$

If $p > 1$, (3.5), (2.1), (2.2) and $b_k \ll k^{-r-1} \omega(1/k)$ imply that

$$\begin{aligned} I_{11} &\ll \frac{1}{m^p} \sum_{k=1}^{m} \lambda_k^{1-p} k^p k^{-(r+1)p} \omega^p(1/k) \Lambda_k^p \\ &= \frac{1}{m^p} \sum_{k=1}^{m} \Lambda_k k^{-rp} \omega^p(1/k) \left(\frac{\Lambda_k}{\lambda_k} \right)^{p-1} \\ &\ll \Lambda_m m^{-rp} \omega^p(1/m). \end{aligned} \tag{3.8}$$

If $0<p<1$, by using (3.2), (2.2) and $b_k \ll k^{-r-1}\omega(1/k)$, we have

$$\begin{aligned} I_{11} &\ll \frac{1}{m^p}\sum_{k=1}^{m}\lambda_k\left(\sum_{\nu=k}^{m}\nu^{-r}\omega(1/\nu)\right)^p \\ &\ll \frac{1}{m^p}\sum_{k=1}^{m}k^{p-1}k^{-rp}\omega^p(1/k)\left(k\lambda_k+\sum_{\nu=1}^{k-1}\lambda_\nu\right) \\ &\ll \frac{1}{m^p}\sum_{k=1}^{m}\Lambda_k k^{-rp}\omega^p(1/k)\left(k^p\frac{\lambda_k}{\Lambda_k}+k^{p-1}\right) \\ &\ll \Lambda_m m^{-rp}\omega^p(1/m)\left(1+\frac{1}{m^p}\sum_{k=1}^{m}k^p\frac{\lambda_k}{\Lambda_k}\right). \end{aligned} \tag{3.9}$$

To estimate the sum

$$\sum_{k=1}^{m}k^p\frac{\lambda_k}{\Lambda_k}$$

we utilize (1.2). If $2^\mu < m \leqq 2^{\mu+1}$, then, by (1.2),

$$\begin{aligned} \sum_{k=1}^{m}k^p\frac{\lambda_k}{\Lambda_k} &\leqq \sum_{i=0}^{\mu}\sum_{k=2^i}^{2^{i+1}}k^p\frac{\lambda_k}{\Lambda_k} \ll \sum_{i=0}^{\mu}2^{ip}\frac{\Lambda_{2^{i+1}}}{\Lambda_{2^i}} \\ &\ll \sum_{i=0}^{\mu}2^{ip} \ll m^p. \end{aligned} \tag{3.10}$$

This and (3.9) imply that

$$I_{11} \ll \Lambda_m m^{-rp}\omega^p(1/m) \tag{3.11}$$

holds for $0<p<1$, too.
Collecting the results (3.7), (3.8) and (3.11) we obtain that

$$I_{11} \ll \Lambda_m m^{-rp}\omega^p(1/m) \tag{3.12}$$

holds for any $p>0$.

Before estimating I_{12} we recall the notable inequality

$$\sum_{\nu=k}^{\infty}|\Delta b_\nu| \ll \omega(1/k)k^{-r-1}, \tag{3.13}$$

verified in the paper [1] for any $k \geqq m$ under the assumptions $\mathbf{b}\in MVBVS$ and $b_n \ll n^{-r-1}\omega(1/n)$.

By (3.13) it is trivial that

$$I_{12} \ll \Lambda_m m^{-rp}\omega^p(1/m).$$

Summing up we have proved that

$$I_1 \ll \Lambda_m m^{-rp}\omega^p(1/m). \tag{3.14}$$

The estimate of I_2 is easier. Namely, by (2.2), (3.4) and (3.13), we have

$$
\begin{aligned}
I_2 &\ll \sum_{k=m+1}^{n} \lambda_k m^p \omega^p(1/k) k^{-(r+1)p} \\
&\ll m^p \sum_{k=m+1}^{n} \frac{\lambda_k}{\Lambda_k} \Lambda_k \omega^p(1/k) k^{-rp} k^{-p} \\
&\ll m^p \Lambda_n \omega^p(1/n) n^{-rp} \sum_{k=m+1}^{n} \frac{\lambda_k}{\Lambda_k} k^{-p}.
\end{aligned}
$$

To estimate the sum we use (1.2) and the method applied in (3.10) as follows:

$$
\begin{aligned}
\sum_{k=m+1}^{n} \frac{\lambda_k}{\Lambda_k} k^{-p} &\leqq \sum_{\nu=0}^{\infty} \sum_{k=2^\nu m}^{2^{\nu+1} m} \frac{\lambda_k}{\Lambda_k} k^{-p} \\
&\ll \sum_{\nu=0}^{\infty} (2^\nu m)^{-p} \frac{\Lambda_{2^{\nu+1} m}}{\Lambda_{2^\nu m}} \ll m^{-p}.
\end{aligned}
$$

Consequently

$$
I_2 \ll \Lambda_n \omega^p(1/n) n^{-rp}. \tag{3.15}
$$

The estimates (3.14) and (3.15) clearly imply that

$$
\sum_{k=1}^{n} \lambda_k |f(x) - s_k(x)|^p \ll \Lambda_n \omega^p(1/n) n^{-rp},
$$

whence

$$
f(x) \in H(\boldsymbol{\lambda}, p, r, \omega)
$$

follows.

The proof is complete. □

4. Proof of the Theorem

In the proof we shall follow the method of proof due to Le and Zhou [1].

By Lemma 3.2 we know that $f(x) \in W^r H_\beta^\omega$ implies

$$
\omega(1/n) \gg n^{-(\beta+1)} \sum_{\nu=1}^{n} \nu^{r+\beta+1} b_\nu,
$$

and $\mathbf{b} \in MVBVS$ follows from $f(x) \in C_3$.

Now put $m=[\lambda n]+1$. Using these facts we obtain that

$$\begin{aligned}\omega(1/n) &\geqq \omega(1/m) \gg m^{-(\beta+1)}\sum_{\nu=1}^{m}\nu^{r+\beta+1}b_\nu \\ &\gg m^{-(\beta+1)}\sum_{\nu=1}^{m}b_\nu\sum_{k=1}^{\nu}k^{r+\beta} = m^{-(\beta+1)}\sum_{k=1}^{m}k^{r+\beta}\sum_{\nu=k}^{m}b_\nu \\ &\gg (\lambda n)^{(\beta+1)}\sum_{k=1}^{\left[\frac{n}{\lambda}\right]}k^{r+\beta}\sum_{\nu=\left[\frac{n}{\lambda}\right]}^{[\lambda n]}b_\nu.\end{aligned} \tag{4.1}$$

Let $n\leqq k\leqq 2n$, then

$$b_n \leqq \sum_{\nu=n}^{k-1}|\Delta b_\nu| + b_k \leqq \sum_{\nu=n}^{2n}|\Delta b_\nu| + b_k \leqq K(\mathbf{b})n^{-1}\sum_{\nu=\left[\frac{n}{\lambda}\right]}^{[n\lambda]}b_\nu + b_k,$$

whence

$$nb_n \leqq \sum_{k=n}^{2n}b_n \leqq K(\mathbf{b})n^{-1}\sum_{k=n}^{2n}\sum_{\nu=\left[\frac{n}{\lambda}\right]}^{[n\lambda]}b_\nu + \sum_{k=n}^{2n}b_k \ll K(\mathbf{b})\sum_{\nu=\left[\frac{n}{\lambda}\right]}^{[n\lambda]}b_\nu.$$

This and (4.1) imply that

$$\omega(1/n) \gg n^{-(\beta+1)}\sum_{k=1}^{\left[\frac{n}{\lambda}\right]}k^{r+\beta}nb_n \gg n^{r+1}b_n,$$

thus

$$b_n \ll n^{-r-1}\omega(1/n) \tag{4.2}$$

holds. Herewith we verified that if $f(x)\in W^rH^\omega_\beta\cap C_3$ then (4.2) maintains, this and the assumptions of Theorem show that every condition of Lemma 3.3 is satisfied, consequently by Lemma 3.3 we know that

$$f(x)\in H(\boldsymbol{\lambda},p,r,\omega),$$

i.e. the embedding relation (1.5) is proved.

The proof is complete. □

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(Received May 27, 2009)

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