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# GENERALIZATIONS OF FATOU'S AND MARCINKIEWICZ'S RESULTS

MENG-KUANG KUO

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*Abstract.* Two new criteria for almost everywhere, uniform, and $L^1$-convergence of Fourier series are established. The first one is a Tauberian result for $(C,1)$ summability, which generalizes a result of Fatou. The second one is a norm analogue of Dini's test, which generalizes a result of Marcinkiewicz. It is also pointed out that our second result is not comparable with the Dini-Lipschitz test.

## 1. Introduction

Let $T=[-\pi,\pi]$ and $(X,\|\cdot\|)=(C(T),\|\cdot\|_\infty)$ or $(L^1(T),\|\cdot\|_1)$. Denote by $s_n(f;x)$ the $n$th partial sum of the Fourier series of $f\in X$. The $n$th Cesàro mean $\sigma_n(f;x)$ of $\{s_n(f;x)\}_{n=0}^\infty$ is defined by

$$\sigma_n(f;x)=\frac{s_0(f;x)+\cdots+s_n(f;x)}{n+1}\qquad (n\geqslant 0).$$

We have

$$\sigma_n(f;x)=\sum_{|j|\leqslant n}\left(1-\frac{|j|}{n+1}\right)\hat f(j)e^{ijx}=f*K_n(x), \tag{1.1}$$

where $K_n(x)$ is the $n$th Fejér kernel defined by

$$K_n(x)=\sum_{|j|\leqslant n}\left(1-\frac{|j|}{n+1}\right)e^{ijx}=\frac{1}{n+1}\left(\frac{\sin((n+1)x/2)}{\sin(x/2)}\right)^2. \tag{1.2}$$

As indicated in [1, 7], $s_n(f;x)$ may not converge to $f(x)$ in $(X,\|\cdot\|)$, even in the sense of pointwise convergence. What's the condition for such convergence? Several famous conclusions had been established. In [3, Vol. I, p. 106] (see also [1, Vol. I, p. 178]), Fatou proved that $(1.3)$ is sufficient for the almost everywhere convergence of $s_n(f;x)$ with $f\in L^1(T)$ and for the uniform convergence with $f\in C(T)$:

$$\frac{1}{n}\sum_{k=-n}^{n}|k\hat f(k)|=o(1)\quad\text{as}\quad n\to\infty. \tag{1.3}$$

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But the $L^1$-convergence of $s_n(f;x)$ was not discussed in Fatou's result. However, for the $L^1$-convergence, an integral analogue of the Dini-Lipschitz test was established (see [6, p. 78, Corollary 3.2.2] or [7, Vol. I, p. 180, Ex.7]). For the $\|\cdot\|_\infty$-convergence, the famous Dini-Lipschitz test was found, which is stated as follows: For $f \in C(T)$, $\|s_n(f)-f\|_\infty \to 0$ as $n \to \infty$ if

$$\omega(\delta) = o\left(\frac{1}{\log(1/\delta)}\right) \qquad \text{as} \quad \delta \to 0^+. \tag{1.4}$$

Here $\omega(\delta)$ is the modulus of continuity of $f$. For pointwise convergence, the so-called Dini test was found (see [1, Vol. I, p. 113] or [7, Vol. I, p. 52]), which is described in the way: $s_n(f;x_0) \longrightarrow f(x_0)$ as $n \to \infty$ if there exists some $\delta > 0$ such that

$$\int_0^\delta \frac{|\phi_{x_0}(t)|}{t}\,dt < \infty. \tag{1.5}$$

Here $\phi_x(t) = f(x+t)+f(x-t)-2f(x)$. In [5, p. 7], Marcinkiewicz proved that a stronger form of $(1.6)$ with $\|\cdot\| = \|\cdot\|_1$ ensures the almost everywhere convergence of $s_n(f;x)$ for $f \in L^1(T)$:

$$\int_0^\delta \frac{\|\phi_x(t)\|}{t}\,dt < \infty. \tag{1.6}$$

Here $\|\phi_x(t)\|$ denotes the $\|\cdot\|$-norm of the function $\phi_x(t)$ with variable $x$.

In this paper, we shall establish the following result and prove that it generalizes Fatou's and Marcinkiewicz's results.

THEOREM 1.1. *Let $f \in X$ and $x_0 \in T$. Then the following two assertions hold.*

*(i) $s_n(f) \to f$ in $X$ if and only if*

$$\lim_{N\to\infty}\left\{\sup_{n\geqslant N}\frac{n}{N}\left\|\sigma_{n+N-1}(f;x)-\sigma_{n-1}(f;x)\right\|\right\} = 0. \tag{1.7}$$

*(ii) $s_n(f;x_0) \to f(x_0)$ if and only if $\sigma_n(f;x_0) \to f(x_0)$ and*

$$\lim_{N\to\infty}\left\{\sup_{n\geqslant N}\frac{n}{N}\left|\sigma_{n+N-1}(f;x_0)-\sigma_{n-1}(f;x_0)\right|\right\} = 0. \tag{1.8}$$

The conclusion $(ii)$ indicates that Theorem 1.1 is a Tauberian result for the $(C,1)$-summability (cf. [2, 4] for definitions). With help of $(1.1)$, we have

$$\begin{aligned}\frac{n}{N}\left\|\sigma_{n+N-1}(f;x)-\sigma_{n-1}(f;x)\right\| &\leqslant \sup_{k\geqslant n}\left\{k\left\|\sigma_k(f;x)-\sigma_{k-1}(f;x)\right\|\right\}\left(\frac{n}{N}\sum_{k=n}^{n+N-1}\frac{1}{k}\right)\\ &\leqslant \sup_{k\geqslant n}\left\{\frac{1}{k+1}\sum_{\ell=-k}^{k}|\ell\hat{f}(\ell)|\right\}.\end{aligned}$$

This inequality still holds if we replace $\|\cdot\|$ by $|\cdot|$. According to Lebesgue's theorem (see [1, Vol. I, p. 139] or [7, Vol. I, p. 90]), we see that Theorem 1.1 generalizes Fatou's result. As proved in Theorem 3.3, $(1.6) \Longrightarrow (1.7)$. Thus, Theorem 1.1 also includes Marcinkiewicz's result as a special case. Theorem 3.3 not only generalizes the result of Marcinkiewicz, but also provides us a new criterion for the $\|\cdot\|$-convergence of the Fourier series of $f \in X$ which can be regarded as the norm analogue of Dini's test. We also point out that our result is not comparable with the Dini-Lipschitz test (see §3).

## 2. Proof of Theorem 1.1

The proof of Theorem 1.1 is based on the following lemma.

LEMMA 2.1. *Let $f \in X$. Then for any fixed $N \geqslant 1$, we have*

$$\lim_{n\to\infty}\left\|\frac{1}{N}\sum_{k=n}^{n+N-1} s_k(f;x) - s_n(f;x)\right\| = 0. \tag{2.1}$$

*Moreover, $(2.1)$ still holds if $x \in T$ and $\|\cdot\|$ is replaced by $|\cdot|$.*

*Proof.* By the Riemann-Lebesgue theorem, we infer that

$$\begin{aligned}\left\|\frac{1}{N}\sum_{k=n}^{n+N-1} s_k(f;x) - s_n(f;x)\right\| &\leqslant \frac{N-1}{2}\left\{\sup_{\ell>n}\left\|s_\ell(f;x) - s_{\ell-1}(f;x)\right\|\right\}\\ &\leqslant \frac{N-1}{2}\left\{\sup_{\ell>n}\left(\left|\hat{f}(\ell)\right| + \left|\hat{f}(-\ell)\right|\right)\right\}\\ &\longrightarrow 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

To replace $\|\cdot\|$ in the above inequalities by $|\cdot|$, we get the other case. □

*Proof of Theorem 1.1.* The left side of $(1.7)$ can be written in the following way:

$$\frac{n}{N}\Big(\sigma_{n+N-1}(f;x) - \sigma_{n-1}(f;x)\Big) = \frac{1}{N}\sum_{k=n}^{n+N-1} s_k(f;x) - \sigma_{n+N-1}(f;x). \tag{2.2}$$

This implies

$$\begin{aligned}\left\|s_n(f;x) - f(x)\right\| \leqslant &\left\|\frac{1}{N}\sum_{k=n}^{n+N-1} s_k(f;x) - s_n(f;x)\right\| + \left\|\sigma_{n+N-1}(f;x) - f(x)\right\|\\ &+ \frac{n}{N}\left\|\sigma_{n+N-1}(f;x) - \sigma_{n-1}(f;x)\right\|.\end{aligned} \tag{2.3}$$

With the help of Lemma 2.1 and Fejér theorem (see [7, Vol. I, p. 89]), we get the "if" part of $(i)$. Obviously, the "only if" part of $(i)$ follows from $(2.2)$. This completes the proof of $(i)$. To replace Fejér theorem by $|\sigma_n(f;x_0) - f(x_0)| \to 0$ and $\|\cdot\|$ in $(2.3)$ by $|\cdot|$, we find that the above argument also leads us to $(ii)$. □

## 3. Norm analogues of Dini's condition

In this section, we pass (1.7) to (1.6). To do so, we need the following two lemmas.

LEMMA 3.1. *Let* $n,N \geqslant 1$ *and* $0<|t|\leqslant \pi$. *Then*

$$\left|K_{n+N-1}(t)-K_{n-1}(t)\right| \leqslant \frac{\pi^2}{(n+N)t^2}\left(\frac{N}{n}\left|\sin\left(\frac{nt}{2}\right)\right|+\left|\sin\left(\frac{Nt}{2}\right)\right|\right). \tag{3.1}$$

*Proof.* We have $|\sin t/2| \geqslant |t|/\pi$ for $0<|t|\leqslant\pi$. By (1.2), we obtain

$$\begin{aligned}\left|K_{n+N-1}(t)-K_{n-1}(t)\right| &= \left|\frac{-N\sin^2(\frac{n}{2}t)+n\sin(\frac{2n+N}{2}t)\sin(\frac{N}{2}t)}{n(n+N)\sin^2(\frac{t}{2})}\right| \\ &\leqslant \frac{\pi^2}{(n+N)t^2}\left(\frac{N}{n}\left|\sin\left(\frac{nt}{2}\right)\right|+\left|\sin\left(\frac{Nt}{2}\right)\right|\right).\end{aligned}$$

This is what we want. □

LEMMA 3.2. *Let* $g$ *be a nonnegative function defined on* $(0,\pi]$ *and integrable on every subinterval* $(a,\pi]$ *with* $0<a<\pi$. *Then for* $N\to\infty$,

$$\int_0^{1/N}\frac{g(t)}{t}\,dt = o(1) \qquad \Longrightarrow \qquad \int_{1/N}^{\pi}\frac{g(t)}{t^2}\,dt = o(N).$$

*Proof.* We can find a small $\delta>0$ such that $\int_0^\delta g(t)/t\,dt<\infty$. Then

$$\begin{aligned}\int_{1/N}^{\pi}\frac{g(t)}{t^2}\,dt &\leqslant \int_{1/N}^{\log N/N}\frac{g(t)}{t^2}\,dt+\int_{\log N/N}^{\delta}\frac{g(t)}{t^2}\,dt+O(1) \\ &\leqslant N\int_0^{\log N/N}\frac{g(t)}{t}\,dt+\frac{N}{\log N}\int_0^{\delta}\frac{g(t)}{t}\,dt+O(1).\end{aligned}$$

By Lebesgue's dominated convergence theorem, we have $N\int_0^{\log N/N} g(t)/t\,dt = o(N)$. Putting this with the above estimate, we are led to the conclusion. □

We have $\sigma_{n+N-1}(f)-\sigma_{n-1}(f) = f*(K_{n+N-1}-K_{n-1})$. Applying (3.1) to $|K_{n+N-1}(t)-K_{n-1}(t)|$, we get the following analogue of Dini's test.

THEOREM 3.3. *Let* $f\in X$. *If* (1.6) *holds, then* $s_n(f;x)\to f(x)$ *in* $X$ *and for almost all* $x\in T$.

*Proof.* First, we claim that $s_n(f)\to f$ in $X$. For $n\geqslant N\geqslant 1$, by Minkowski's integral inequality, we get

$$\begin{aligned}&\left\|\sigma_{n+N-1}(f;x)-\sigma_{n-1}(f;x)\right\| \\ &\leqslant \frac{1}{2\pi}\left\{\int_0^{1/n}+\int_{1/n}^{1/N}+\int_{1/N}^{\pi}\right\}\|\phi_x(t)\|\left|K_{n+N-1}(t)-K_{n-1}(t)\right|dt \\ &= I'_{n,N}+I''_{n,N}+I'''_{n,N}, \text{ say.}\end{aligned} \tag{3.2}$$

It follows from (1.6) and (3.1) that

$$I'_{n,N} \leqslant \frac{\pi}{2(n+N)}\left(\frac{N}{2}\int_0^{1/n}\frac{\|\phi_x(t)\|}{t}\,dt+\frac{N}{2}\int_0^{1/N}\frac{\|\phi_x(t)\|}{t}\,dt\right) \tag{3.3}$$
$$= o\left(\frac{N}{n+N}\right) \quad \text{as} \quad n \geqslant N \to \infty.$$

Similarly, (1.6), (3.1) and Lemma 3.2 together give

$$I''_{n,N} \leqslant \frac{\pi}{2(n+N)}\int_{1/n}^{1/N}\frac{\|\phi_x(t)\|}{t^2}\left(\frac{N}{n}+\sin(\frac{Nt}{2})\right)dt \tag{3.4}$$
$$\leqslant \frac{N\pi}{2n(n+N)}\int_{1/n}^{\pi}\frac{\|\phi_x(t)\|}{t^2}\,dt+\frac{N\pi}{4(n+N)}\int_0^{1/N}\frac{\|\phi_x(t)\|}{t}\,dt$$
$$= o\left(\frac{N}{n+N}\right) \quad \text{as} \quad n \geqslant N \to \infty.$$

On the other hand, (3.1) and Lemma 3.2 lead us to

$$I'''_{n,N} \leqslant \frac{\pi}{2(n+N)}\left(\frac{N}{n}+1\right)\int_{1/N}^{\pi}\frac{\|\phi_x(t)\|}{t^2}\,dt \tag{3.5}$$
$$= o\left(\frac{N}{n+N}\right) \quad \text{as} \quad n \geqslant N \to \infty.$$

Putting (3.2) – (3.5) together yields (1.7). By Theorem 1.1(i), we infer that $s_n(f) \to f$ in $X$. As for the almost everywhere convergence, applying Minkowski's integral inequality, we get

$$\left\|\int_0^{\delta}\frac{|\phi_x(t)|}{t}\,dt\right\| \leqslant \int_0^{\delta}\frac{\|\phi_x(t)\|}{t}\,dt < \infty.$$

Hence, (1.5) holds for almost all $x_0 \in T$. By Dini's test, we conclude that $s_n(f;x) \to f(x)$ almost everywhere. This completes the proof. □

Obviously, $\|\phi_x(t)\|_1 \leqslant 2w_1(f;t)$, where $w_1(f;x)$ is defined by [5, p. 7, Eq. (1.28)]. Hence, $\int_{[0,\pi]} w_1(f;t)dt/t < \infty \Longrightarrow$ (1.6). This indicates that Theorem 3.3 generalizes the result of Marcinkiewicz. It is clear that (1.6) is satisfied by those $f \in X$ with $\|\phi_x(t)\| \leqslant c_\alpha t^\alpha$ for all $0 \leqslant t \leqslant \pi$, where $0 < \alpha \leqslant 1$. Thus, $s_n(f) \to f$ in $X$ for such $f$. We know that the condition $\|\phi_x(t)\| = o\left(\frac{1}{\log(1/t)}\right)$ may not imply (1.6), so the Dini-Lipschitz test is not a corollary of Theorem 3.3. On the other hand, the function $(\log(1/\delta))^{-1}$ appeared in the right side of (1.4) goes smoothly to infinite as $\delta \to 0^+$. Hence, in general, the Dini-Lipschitz test does not deal with the case with $\|\phi_x(t)\| = o(h(t))$, where $h(t)$ has a big oscillation near the origin. For example, the Dini-Lipschitz test does not apply to the case

$$h(t) = \begin{cases} t^\alpha + t\,e^{1/t} & \text{on } \bigcup_{n=1}^{\infty}[1/n-\delta_n,\, 1/n], \\ t^\alpha & \text{otherwise,} \end{cases} \tag{3.6}$$

where $0 < \alpha \leqslant 1$, $0 < \delta_n < 1/n - 1/(n+1)$, and

$$\sum_{n=1}^{\infty} \int_{1/n-\delta_n}^{1/n} e^{1/t}\, dt < \infty.$$

However, $(3.6)$ implies $(1.6)$, so Theorem 3.3 works well for such $h(t)$. From the above argument, we see that Theorem 3.3 and the Dini-Lipschitz test are not comparable.

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*Meng-Kuang Kuo*
*Center for General Education*
*Jen-Teh Junior College of Medicine, Nursing and Management*
*NO, 79-9 Sha-Luen Hu Xi-Zhou Li Hou-Loung Town*
*Miaoli County, Republic of China*
*e-mail:* kuo53@seed.net.tw