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THE MODULAR INTERPOLATION INEQUALITY IN SOBOLEV SPACES WITH VARIABLE EXPONENT ATTAINING THE VALUE 1

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Abstract. We prove a modular type interpolation inequality for functions belonging to Sobolev spaces with variable exponent attaining the value 1. The approach combines the original proof of the interpolation inequality by Nirenberg [19] with an inequality for averages over balls, avoiding the use of the norm interpolation inequality for variable exponent Sobolev spaces, known for exponents whose infimum is greater than 1.

1. Introduction

One of the useful properties in the study of the theory of Sobolev spaces is the possibility to estimate the L^p norm of the intermediate derivatives of a function u in terms of the L^p norms of the derivative of maximum order and of u itself. For a function u belonging to a classical Sobolev space $W^{2,p}(\Omega)$, the norm is defined as

$$||u||_{W^{2,p}(\Omega)} = ||u||_{L^p(\Omega)} + ||\,|Du|\,||_{L^p(\Omega)} + ||\,|D^2u|\,||_{L^p(\Omega)}.$$

If the domain Ω is sufficiently smooth and has bounded boundary, the norm $||\,|Du|\,||_{L^p(\Omega)}$ can be controlled by $||u||_{L^p(\Omega)}$ and $||\,|D^2u|\,||_{L^p(\Omega)}$ by means of the interpolation inequality

$$||\,|Du|\,||_{L^p(\Omega)} \leqslant \varepsilon||\,|D^2u|\,||_{L^p(\Omega)} + c(\varepsilon)||u||_{L^p(\Omega)} \tag{1.1}$$

and, due to (1.1), the space $W^{2,p}(\Omega)$ admits the equivalent norm

$$|||u|||_{W^{2,p}(\Omega)} = ||u||_{L^p(\Omega)} + ||\,|D^2u|\,||_{L^p(\Omega)}.$$

Investigations on interpolation inequalities have been started long ago and interested many authors. The earliest papers on the argument are due to Ehrling [7], Gagliardo [9] and Nirenberg [19] but we also refer to the books of Adams [1] and Maz'ja [16] for a scrupulous treatment of it.

In the last years variable exponent spaces have attracted many authors and have been considered in a series of papers, see e.g. [12], [8], [10], [20] and references

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therein. For a function u belonging to a variable Sobolev space $W^{2,p(\cdot)}(\Omega)$, the norm is defined as

$$||u||_{W^{2,p(\cdot)}(\Omega)} = ||u||_{L^{p(\cdot)}(\Omega)} + ||\,|Du|\,||_{L^{p(\cdot)}(\Omega)} + ||\,|D^2u|\,||_{L^{p(\cdot)}(\Omega)}$$

and, as it happens for constant exponents, if the domain is sufficiently regular, the norm of $|Du|$ in $L^{p(\cdot)}(\Omega)$ can be controlled by $||u||_{L^{p(\cdot)}(\Omega)}$ and $||\,|D^2u|\,||_{L^{p(\cdot)}(\Omega)}$ by means of the interpolation inequality

$$||\,|Du|\,||_{L^{p(\cdot)}(\Omega)} \leqslant \varepsilon||\,|D^2u|\,||_{L^{p(\cdot)}(\Omega)} + c(\varepsilon)||u||_{L^{p(\cdot)}(\Omega)} \tag{1.2}$$

which can be obtained as a direct consequence of the following general result due to J.L. Lions (see [14], [18]).

THEOREM 1.1. *Let B_i, $i = 1,2,3$, three Banach spaces such that*

$$B_1 \subset B_2 \subset B_3$$

algebraically and topologically. Suppose that the imbedding of B_1 in B_2 is compact. For every $\varepsilon > 0$, there exists a constant $c(\varepsilon)$ such that if $u \in B_1$ then

$$||u||_{B_2} \leqslant \varepsilon||u||_{B_1} + c(\varepsilon)||u||_{B_3}. \tag{1.3}$$

Choosing $B_1 = W^{2,p(\cdot)}(\Omega)$, $B_2 = W^{1,p(\cdot)}(\Omega)$ and $B_3 = L^{p(\cdot)}(\Omega)$, we can recognize in (1.3) the interpolation inequality (1.2). Let us stress that the proof of inequality (1.2) as consequence of (1.3) reduces to the problem of compact embedding between Sobolev spaces with variable exponent. A detailed study in this direction appears in a recent paper by Zang and Fu [21] (see also [11] and [5]), where the restriction $\operatorname{ess\,inf} p > 1$ is necessary, since the boundedness of the maximal operator is required (see [4]).

It is worth pointing out that the norm of a function which belongs to a variable L^p space does not coincide, up to a power, with its modular, as it happens in the case of a constant exponent. Therefore, in the context of variable exponents it is interesting to consider, besides norm inequalities, also modular inequalities. The papers in this directions show that the corresponding modular inequalities are true for a smaller class of exponents and, sometimes, only for constant exponents (see for example [13]).

In this paper we obtain a modular type estimate in the setting of variable exponent Sobolev spaces. We shall consider exponents $p(\cdot)$ belonging to a class of regular functions on domains $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$, defined in Section 2 and denoted with $\mathscr{P}(\Omega)$.

We shall use $D^j u$ to denote the vector $\dfrac{\partial^j u}{\partial_{\alpha_1}\partial_{\alpha_2}\cdots\partial_{\alpha_n}}$, $\alpha_1 + \cdots + \alpha_n = j$, of all derivatives of u of order j and $|D^j u|$ for its norm, that is $|D^j u| = \Big(\sum_{|\alpha|=j} (D^\alpha u)^2\Big)^{\frac{1}{2}}$. In particular we shall write Du instead of $D^1 u$ and u instead of $D^0 u$. Moreover, we shall denote by $\rho_{L^{p(\cdot)}(\Omega)}(u)$ the modular $\int_\Omega |u(x)|^{p(x)}\,dx$.

Our main result is the following

THEOREM 1.2. *Let $\Omega \subset \mathbb{R}^n$ $(n \geqslant 1)$ be a domain satisfying the uniform C^2-regularity property and let u be a real valued mapping in $W^{2,p(\cdot)}(\Omega)$, with $p(\cdot) \in \mathscr{P}(\Omega)$. If $\rho_{L^{p(\cdot)}(\Omega)}(u) < 1$ and $\rho_{L^{p(\cdot)}(\Omega)}(|D^2u|) < 1$, then there exists $c = c(n, p(\cdot), \Omega)$ such that for ε sufficiently small*

$$\int_\Omega |Du(x)|^{p(x)}dx \leqslant c\left(\varepsilon^{p_-}\int_\Omega |D^2u(x)|^{p(x)}dx \right.$$
$$\left. +\varepsilon^{-p_+}\int_\Omega |u(x)|^{p(x)}dx + \varepsilon^{-p_+}\max\{(p_+ - p_-)^{p_+}, (p_+ - p_-)^{p_-}\}\right).$$

The inequality in Theorem 1.2 represents an extension of the classical interpolation inequality (1.1): in fact, if $p(\cdot) \equiv p$, it reduces to

$$|| \, |Du| \, ||_p^p \leqslant c(\varepsilon^p || \, |D^2u| \, ||_p^p + \varepsilon^{-p}||u||_p^p)$$

for all $u : ||u||_p < 1, || \, |D^2u| \, ||_p < 1$, which in turn is equivalent to (1.1) without any restriction on the norms of u and $|D^2u|$ because of its homogeneity.

Before spending few words on the proof of Theorem 1.2, we want to observe that a direct consequence of the inequality in its statement is that $\rho_{L^{p(\cdot)}(\Omega)}(|Du|)$ is bounded by a constant independent of u. Under the same hypotheses, this fact obviously follows also from the inequality (1.2), however, our previous digression on inequality (1.2) shows that such approach imposes on the exponent the condition to be essentially greater than 1.

The proof of Theorem 1.2, given in Section 3, combines the original argument of the interpolation inequality by Nirenberg [19] with an inequality for averages over balls (see next Proposition 3.1), and admits exponents attaining the value 1. The proof of the inequality in Theorem 1.2 is obtained considering first the case of functions of one variable with continuous derivatives up to the order 2. Later, a result of convergence of type Meyers-Serrin will be used for functions in the Sobolev class $W^{2,p(\cdot)}(\Omega)$.

2. Notation and preliminaries

Let Ω be an open domain in $\mathbb{R}^n$ and suppose that there exists a locally finite collection of subdomains, called patches, which together with a compact subdomain Ω_0 cover Ω. Suppose also that the closure $\overline{\Omega}_i$ of each patch Ω_i may be mapped in a one to one way onto a cube, with $\partial\Omega \cap \overline{\Omega}_i$ mapped onto a set lying in an $(n-1)$-dimensional plane. If we assume that all such mappings and their respective inverses are j times, $j \geqslant 1$, continuously differentiable and with uniformly bounded Jacobian determinants, we refer to Ω as a domain with *uniform C^j-regularity property.*

Let $p : \Omega \to [1, \infty)$ be a measurable bounded function such that

$$p_- = p_-(\Omega) := \operatorname{ess\,inf}\{p(x) : x \in \Omega\} \geqslant 1,$$

$$p_+ = p_+(\Omega) := \operatorname{ess\,sup}\{p(x) : x \in \Omega\} < \infty.$$

The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ consists of all measurable functions f such that the modular

$$\rho_{L^{p(\cdot)}(\Omega)}(f)=\int_\Omega |f(x)|^{p(x)}\,dx \tag{2.1}$$

is finite. Note that the space $L^{p(\cdot)}(\Omega)$ is a particular case of the so called Musielak-Orlicz spaces, see [17] for details on these spaces.

The expression

$$||f||_{L^{p(\cdot)}(\Omega)}=\inf\{\lambda>0:\rho_{L^{p(\cdot)}(\Omega)}(f/\lambda)\leqslant 1\} \tag{2.2}$$

is a norm on $L^{p(\cdot)}(\Omega)$ which makes it a Banach space. It is possible to verify that $\rho_{L^{p(\cdot)}(\Omega)}(f)\leqslant 1$ if and only if $||f||_{L^{p(\cdot)}}\leqslant 1$, see [8], [12].

The exponent $p(\cdot)$ is said to be *locally log-Hölder continuous* if there exists a constant $C>0$ such that

$$|p(x)-p(y)|\leqslant \frac{C}{\log(1/|x-y|)} \tag{2.3}$$

for all points $x,y\in\Omega$ with $|x-y|<\frac{1}{2}$. It turns out that this condition plays a relevant role in the theory of variable exponent Sobolev spaces as Diening and Samko noticed in [6] and [20] respectively.

The exponent $p(\cdot)$ is said to be *log-Hölder continuous at infinity* if there exists a constant $C>0$ such that

$$|p(x)-p(y)|\leqslant \frac{C}{\log(e+|x|)} \tag{2.4}$$

for all points $x,y\in\Omega$ with $|y|\geqslant|x|$. This condition has been introduced by Cruz-Uribe, Fiorenza and Neugebauer [4] when studying the boundedness of the maximal operator in not necessarily bounded domains.

In the following we shall refer to $\mathscr{P}(\Omega)$ as the set of the exponents $p(\cdot)$ for which conditions (2.3) and (2.4) hold. Note that each $p(\cdot)\in\mathscr{P}(\Omega)$ is always continuous.

The *variable exponent Sobolev space* $W^{j,p(\cdot)}(\Omega)$ is the subspace of $L^{p(\cdot)}(\Omega)$ of functions f such that $D^\alpha f\in L^{p(\cdot)}(\Omega)$ for every multi-index α with $|\alpha|\leqslant j$. The norm

$$||f||_{W^{j,p(\cdot)}(\Omega)}=\sum_{|\alpha|\leqslant j}||\,|D^\alpha f|\,||_{L^{p(\cdot)}(\Omega)} \tag{2.5}$$

makes it a Banach space.

In the proof of our main result we shall need to approximate $u\in W^{2,p(\cdot)}(\Omega)$ by more regular functions. We shall use the following density lemma (see [3], Theorem 2.5) that in the case of classical Sobolev spaces is due to Meyers and Serrin.

LEMMA 2.1. *Let Ω be an open domain in $\mathbb{R}^n$ and let $p(\cdot)\in\mathscr{P}(\Omega)$. Then for all $j\geqslant 1$, $C^\infty(\Omega)\cap W^{j,p(\cdot)}(\Omega)$ is dense in $W^{j,p(\cdot)}(\Omega)$.*

In order to obtain a fundamental inequality for averages over balls, we shall need the two following lemmas. The first one is due to Diening [6] but a proof can be found also in [4], while the second is a variant of a result in [4] and it will be useful to replace an exponent function with another.

LEMMA 2.2. *Let Ω be an open set and $p(\cdot):\Omega\to[1,+\infty)$ a function which satisfies condition* (2.3)*. Then for any ball B such that $|B\cap\Omega|>0$,*

$$|B|^{p_-(B\cap\Omega)-p_+(B\cap\Omega)}\leqslant C.$$

LEMMA 2.3. *Given a set G and two non-negative functions $r(\cdot)$ and $s(\cdot)$, suppose that for each $y\in G$,*

$$0\leqslant s(y)-r(y)\leqslant\frac{c}{\log(e+|z(y)|)} \tag{2.6}$$

where $z:G\to\mathbb{R}^n$. Then there exists a constant C_t such that for every function u the following inequality holds

$$\int_G|u(y)|^{r(y)}dy\leqslant C_t\Big[\int_G|u(y)|^{s(y)}dy+\int_G[(s(y)-r(y))R_t(z(y))]^{r_-(G)}\,dy\Big]$$

where $R_t(x)=(e+|x|)^{-nt}$, $t>0$.

Proof. Let $G^{R_t}=\{y\in G:|u(y)|\geqslant(s(y)-r(y))R_t(z(y))\}$ and write

$$\int_G|u(y)|^{r(y)}dy=\int_{G^{R_t}}|u(y)|^{r(y)}dy+\int_{G\setminus G^{R_t}}|u(y)|^{r(y)}dy.$$

Let us estimate each integral separately. Since $\dfrac{s(y)-r(y)}{c}R_t(z(y))\leqslant 1$ by assumption (2.6), we have

$$\begin{aligned}\int_{G\setminus G^{R_t}}|u(y)|^{r(y)}dy&\leqslant\int_{G\setminus G^{R_t}}[(s(y)-r(y))R_t(z(y))]^{r(y)}\,dy\\&\leqslant C\int_{G\setminus G^{R_t}}[(s(y)-r(y))R_t(z(y))]^{r_-(G)}\,dy.\end{aligned} \tag{2.7}$$

On the other hand, if $y\in G^{R_t}$, then

$$\begin{aligned}\int_{G^{R_t}}|u(y)|^{r(y)}dy&=\int_{G^{R_t}}|u(y)|^{s(y)}|u(y)|^{r(y)-s(y)}dy\\&\leqslant C\int_{G^{R_t}}|u(y)|^{s(y)}\,[(s(y)-r(y))R_t(z(y))]^{-\frac{c}{\log(e+|z(y)|)}}\,dy\\&\leqslant C_t\int_{G^{R_t}}|u(y)|^{s(y)}dy.\end{aligned}$$

Combining the last estimate with (2.7) we get the conclusion. $\square$

3. The proof of the main result

In the proof of Theorem 1.2 we are inspired by the arguments used in the case of the classical Sobolev spaces (see Nirenberg, [19]). Our main difficulty is the lack of the homogeneity of the modular (2.1) which leads us to the use of the following variant of Theorem 4.1 in [2]:

PROPOSITION 3.1. *Let* $\Omega=(0,1)^n$ *and let* $p(\cdot):\Omega\to[1,\infty)$ *satisfy* (2.3). *Let* $u\in L^{p(\cdot)}(\Omega)$ *be such that* $\rho_{L^{p(\cdot)}(\Omega)}(u)\leqslant 1$. *Then for every ball* B *and every* $x\in B$, *denoting by* $B_\Omega=B\cap\Omega$ *and by* $\bar p(x)=p(x)/p_-$, *we have*

$$\Big(\frac{1}{|B|}\int_{B_\Omega}|u(y)|\,dy\Big)^{p(x)}$$

$$\leqslant C\Big[\Big(\frac{1}{|B|}\int_{B_\Omega}|u(y)|^{\bar p(y)}\,dy\Big)^{p_-}+\Big\{(\bar p_+(B_\Omega)-\bar p_-(B_\Omega))R_2(x)\Big\}^{p(x)}\Big]$$

where $C=C(n,p(\cdot))$ *and* $R_2(x)=(e+|x|)^{-2n}$.

Proof. Observe that $\bar p(x)\geqslant 1$ and that $\bar p$ verifies (2.3). It follows, by the boundedness of Ω, that there exists a constant c such that

$$|\bar p(x)-\bar p(y)|\leqslant\frac{c}{\log(e+|x|)}\ x,y\in\Omega,\ |y|\geqslant|x|. \tag{3.1}$$

Fix $x\in\Omega$ and a ball B of radius $r>0$ containing x. We shall carry on splitting the proof in two cases.

Case 1. $r<|x|/4$

By our assumption on r, if $y_1,y_2\in B_\Omega$, then $\log(e+|y_1|)\approx\log(e+|y_2|)$. Hence for all $y\in B_\Omega$ we have

$$0\leqslant\bar p(y)-\bar p_-(B_\Omega)\leqslant\frac{c}{\log(e+|y|)}.$$

Therefore, by Hölder's inequality and Lemma 2.3 with $s=\bar p$, $r=\bar p_-(B_\Omega)$, $z=y$, $t=2$, we deduce

$$\Big(\frac{1}{|B|}\int_{B_\Omega}|u(y)|\,dy\Big)^{p(x)}\leqslant\Big(\frac{1}{|B|}\int_{B_\Omega}|u(y)|^{\bar p_-(B_\Omega)}\,dy\Big)^{p(x)/\bar p_-(B_\Omega)}$$
$$\leqslant\ \Big(\frac{C}{|B|}\int_{B_\Omega}|u(y)|^{\bar p(y)}dy+\frac{C}{|B|}\int_{B_\Omega}[(\bar p(y)-\bar p_-(B_\Omega))R_2(y)]^{\bar p_-(B_\Omega)}dy\Big)^{\frac{p(x)}{\bar p_-(B_\Omega)}}.$$

Since $r<\frac{|x|}{4}$, if $y\in B$ then $R_2(y)\leqslant cR_2(x)$ and therefore we can estimate the term in the right-hand side with

$$\Big(\frac{C}{|B|}\int_{B_\Omega}|u(y)|^{\bar p(y)}dy+\frac{C}{|B|}R_2(x)^{\bar p_-(B_\Omega)}\int_{B_\Omega}(\bar p(y)-\bar p_-(B_\Omega))^{\bar p_-(B_\Omega)}dy\Big)^{\frac{p(x)}{\bar p_-(B_\Omega)}}$$

$$\leqslant \Big(\frac{C}{|B|}\int_{B_\Omega}|u(y)|^{\bar{p}(y)}dy + C(\bar{p}_+(B_\Omega)-\bar{p}_-(B_\Omega))^{\bar{p}_-(B_\Omega)}R_2(x)^{\bar{p}_-(B_\Omega)}\Big)^{\frac{p(x)}{\bar{p}_-(B_\Omega)}}$$

$$\leqslant 2^{p_+}C\left(\frac{1}{|B|}\int_{B_\Omega}|u(y)|^{\bar{p}(y)}dy\right)^{\frac{p(x)}{\bar{p}_-(B_\Omega)}} + 2^{p_+}C[(\bar{p}_+(B_\Omega))-\bar{p}_-(B_\Omega))R_2(x)]^{p(x)}$$

where in the last inequality we have used that $p(x)/\bar{p}_-(B_\Omega)\leqslant p_+<+\infty$.

It remains to prove that the first term is dominated by a constant multiple of $\left(\frac{1}{|B|}\int_{B_\Omega}|u(y)|^{\bar{p}(y)}dy\right)^{p_-}$. To this aim observe that

$$\Big(\frac{1}{|B|}\int_{B_\Omega}|u(y)|^{\bar{p}(y)}dy\Big)^{p(x)/\bar{p}_-(B_\Omega)}$$

$$= \Big(\frac{1}{|B|}\int_{B_\Omega}|u(y)|^{\bar{p}(y)}dy\Big)^{p_-}\Big(\frac{1}{|B|}\int_{B_\Omega}|u(y)|^{\bar{p}(y)}dy\Big)^{(p(x)/\bar{p}_-(B_\Omega))-p_-}$$

$$\leqslant |B|^{-\left[\frac{p(x)}{\bar{p}_-(B_\Omega)}-p_-\right]/p_-}\Big(\int_{B_\Omega}|u(y)|^{p(y)}dy\Big)^{\left[\frac{p(x)}{\bar{p}_-(B_\Omega)}-p_-\right]/p_-}\Big(\frac{1}{|B|}\int_{B_\Omega}|u(y)|^{\bar{p}(y)}dy\Big)^{p_-}.$$

Note that

$$-\frac{1}{p_-}\Big[\frac{p(x)}{\bar{p}_-(B_\Omega)}-p_-\Big] = p(x)\Big[\frac{1}{p(x)}-\frac{1}{p_-(B_\Omega)}\Big]\leqslant 0.$$

Hence, if $|B|\geqslant 1$,

$$|B|^{-[(p(x)/\bar{p}_-(B_\Omega))-p_-]/p_-}\leqslant 1.$$

Otherwise, if $|B|\leqslant 1$, we observe that

$$p(x)\Big[\frac{1}{p(x)}-\frac{1}{p_-(B_\Omega)}\Big]\geqslant \frac{p_+}{p_-^2}(p_-(B_\Omega)-p_+(B_\Omega)) \tag{3.2}$$

and therefore, by Lemma 2.2,

$$|B|^{-[(p(x)/\bar{p}_-(B_\Omega))-p_-]/p_-}\leqslant |B|^{(p_+/p_-^2)(p_-(B_\Omega)-p_+(B_\Omega))}\leqslant C.$$

Similarly,

$$\frac{p(x)}{\bar{p}_-(B_\Omega)}-p_- = \frac{p(x)}{p_-(B_\Omega)}p_- - p_-\geqslant 0.$$

Hence, being $\rho_{L^{p(\cdot)}(\Omega)}(u)\leqslant 1$,

$$\Big(\int_{B_\Omega}|u(y)|^{p(y)}\Big)^{[(p(x)/\bar{p}_-(B_\Omega))-p_-]/p_-}\leqslant 1$$

and therefore the desired inequality.

Case 2. $r \geqslant |x|/4$

First of all observe that for every $y \in B_\Omega$, since $|x| \leqslant 1$,

$$0 \leqslant \bar{p}(y) - \bar{p}_-(B_\Omega) \leqslant \bar{p}_+ - \bar{p}_- \leqslant \frac{C}{\log(e+|x|)}.$$

As in the previous case, we apply Hölder's inequality and Lemma 2.3 with $r = \bar{p}_-(B_\Omega)$, $s = \bar{p}$, $z = x$, $t = 2$ having

$$\Big(\frac{1}{|B|}\int_{B_\Omega} |u(y)|\,dy\Big)^{p(x)}$$

$$\begin{aligned}
&\leqslant \Big(\frac{C}{|B|}\int_{B_\Omega} |u(y)|^{\bar{p}(y)}dy + \frac{C}{|B|}\int_{B_\Omega} [(\bar{p}(y) - \bar{p}_-(B_\Omega))R_2(x)]^{\bar{p}_-(B_\Omega)}dy\Big)^{p(x)/\bar{p}_-(B_\Omega)} \\
&\leqslant \Big(\frac{C}{|B|}\int_{B_\Omega} |u(y)|^{\bar{p}(y)}dy + \frac{C}{|B|}\int_{B_\Omega} [(\bar{p}_+(B_\Omega) - \bar{p}_-(B_\Omega))R_2(x)]^{\bar{p}_-(B_\Omega)}dy\Big)^{p(x)/\bar{p}_-(B_\Omega)} \\
&\leqslant 2^{p_+}\Big(\frac{C}{|B|}\int_{B_\Omega} |u(y)|^{\bar{p}(y)}dy\Big)^{p(x)/\bar{p}_-(B_\Omega)} + 2^{p_+}C[(\bar{p}_+(B_\Omega) - \bar{p}_-(B_\Omega))R_2(x)]^{p(x)}
\end{aligned}$$

where we have used that $p(x)/\bar{p}_-(B_\Omega) \leqslant p_+ < +\infty$. In order to prove that the first term is dominated by a constant multiple of $\Big(\frac{1}{|B|}\int_{B_\Omega} |u(y)|^{\bar{p}(y)}dy\Big)^{p_-}$ we argue as in case 1. □

REMARK 3.2. Observe that the second term of the right hand side of the inequality in Proposition 3.1 is a function belonging to $L^1(\mathbb{R}^n)$ since $p_- \geqslant 1$.

We are now in a position to prove our main result.

Proof of Theorem 1.2.

Step 1. Let us start assuming u with continuous derivatives up to order 2 in an interval Ω, say $\Omega = (0,1)$.

Fix $\varepsilon, 0 < \varepsilon < 1$, and divide Ω into a number of subintervals such that the length of each is bounded by

$$\frac{\varepsilon}{4} \leqslant b_i - a_i \leqslant \frac{\varepsilon}{2}.$$

Divide now such a subinterval, say (a_1, b_1), into three successive intervals of lenghts α, 2α and α, so that $b_1 - a_1 = 4\alpha$.

If t_1 and t_2 are points in the first and third intervals we have, by Lagrange's theorem, that at some point t_0,

$$Du(t_0) = \frac{u(t_2) - u(t_1)}{t_2 - t_1}$$

so that for a fixed $x \in (a_1, b_1)$,

$$\begin{aligned}
|Du(x)| &\leqslant |Du(t_0)| + \Big|\int_{t_0}^{x} D^2u(t)\,dt\Big| \\
&\leqslant \frac{|u(t_2)| + |u(t_1)|}{2\alpha} + \int_{a_1}^{b_1} |D^2u(t)|\,dt. \qquad (3.3)
\end{aligned}$$

Since we have

$$\int_{b_1-\alpha}^{b_1} dt_2 \int_{a_1}^{a_1+\alpha} \frac{|u(t_2)|+|u(t_1)|}{\alpha}\, dt_1$$

$$= \frac{1}{\alpha}\int_{b_1-\alpha}^{b_1}\left(\alpha|u(t_2)|+\int_{a_1}^{a_1+\alpha}|u(t_1)|\,dt_1\right)dt_2$$
$$= \int_{b_1-\alpha}^{b_1}|u(t_2)|\,dt_2+\int_{a_1}^{a_1+\alpha}|u(t_1)|\,dt_1 \leqslant \int_{a_1}^{b_1}|u(t)|\,dt$$

integrating (3.3) separately on $(a_1,a_1+\alpha)$ with respect to t_1 and on $(b_1-\alpha,b_1)$ with respect to t_2 we find

$$\alpha^2\,|Du(x)| \leqslant \frac{1}{2}\int_{a_1}^{b_1}|u(t)|dt+\alpha^2\int_{a_1}^{b_1}|D^2u(t)|dt.$$

Hence

$$|Du(x)| \leqslant \frac{1}{2\alpha^2}\int_{a_1}^{b_1}|u(t)|dt+\int_{a_1}^{b_1}|D^2u(t)|dt$$

and

$$|Du(x)|^{p(x)} \leqslant 2^{p(x)-1}\left[\left(\frac{1}{2\alpha^2}\int_{a_1}^{b_1}|u(t)|\,dt\right)^{p(x)}+\left(\int_{a_1}^{b_1}|D^2u(t)|\,dt\right)^{p(x)}\right]. \tag{3.4}$$

Since $\rho_{L^{p(\cdot)}(\Omega)}(u)<1$ and $\rho_{L^{p(\cdot)}(\Omega)}(|D^2u|)<1$ by assumptions, we can estimate both terms in the right hand side of (3.4) by using Proposition 3.1 as follows

$$\begin{aligned}\left(\frac{1}{2\alpha^2}\int_{a_1}^{b_1}|u(t)|\,dt\right)^{p(x)} &= \left(\frac{2}{\alpha}\right)^{p(x)}\left(\frac{1}{4\alpha}\int_{a_1}^{b_1}|u(t)|\,dt\right)^{p(x)}\\ &\leqslant \left(\frac{2}{\alpha}\right)^{p(x)}\left[c(n,p(\cdot))\left(\frac{1}{4\alpha}\int_{a_1}^{b_1}|u(t)|^{p(t)/p_-}\,dt\right)^{p_-}+S(x)\right]\end{aligned}$$

and analogously

$$\begin{aligned}\left(\int_{a_1}^{b_1}|D^2u(t)|\,dt\right)^{p(x)} &= (4\alpha)^{p(x)}\left(\frac{1}{4\alpha}\int_{a_1}^{b_1}|D^2u(t)|\,dt\right)^{p(x)}\\ &\leqslant (4\alpha)^{p(x)}\left[c(n,p(\cdot))\left(\frac{1}{4\alpha}\int_{a_1}^{b_1}|D^2u(t)|^{p(t)/p_-}\,dt\right)^{p_-}+S(x)\right]\end{aligned}$$

where we have denoted by $S(x)=S_p(x)\in L^1(\mathbb{R})$ (see Remark 3.2) the expression

$$\left\{(\bar{p}_+-\bar{p}_-)R_2(x)\right\}^{p(x)} \tag{3.5}$$

with $\bar{p}(\cdot)=p(\cdot)/p_-$.

Therefore, by Hölder's inequality,

$$|Du(x)|^{p(x)} \leqslant 2^{p(x)-1}\Big\{ \left(\frac{2}{\alpha}\right)^{p(x)} \left[c(n,p(\cdot))\frac{1}{4\alpha}\int_{a_1}^{b_1} |u(t)|^{p(t)}\,dt + S(x)\right] + (4\alpha)^{p(x)}\left[c(n,p(\cdot))\frac{1}{4\alpha}\int_{a_1}^{b_1} |D^2u(t)|^{p(t)}\,dt + S(x)\right]\Big\}$$

and integrating in (a_1,b_1),

$$\int_{a_1}^{b_1} |Du(t)|^{p(t)}\,dt$$

$$\leqslant c(n,p(\cdot))\Big\{\frac{1}{4\alpha}\int_{a_1}^{b_1}\left(\frac{2}{\alpha}\right)^{p(t)} dt\int_{a_1}^{b_1} |u(t)|^{p(t)}dt + \left(\frac{2}{\alpha}\right)^{p_+}\int_{a_1}^{b_1} S(t)\,dt + \frac{1}{4\alpha}\int_{a_1}^{b_1}(4\alpha)^{p(t)}dt\int_{a_1}^{b_1} |D^2u(t)|^{p(t)}dt + (4\alpha)^{p_-}\int_{a_1}^{b_1} S(t)\,dt\Big\}.$$

Since $\frac{\varepsilon}{4} < 4\alpha < \varepsilon$, we have

$$\frac{1}{4\alpha}\int_{a_1}^{b_1}\left(\frac{2}{\alpha}\right)^{p(t)} dt \leqslant \left(\frac{2}{\alpha}\right)^{p_+} \leqslant \left(\frac{32}{\varepsilon}\right)^{p_+}$$

$$\frac{1}{4\alpha}\int_{a_1}^{b_1}(4\alpha)^{p(t)}dt \leqslant (4\alpha)^{p_-} \leqslant \varepsilon^{p_-}$$

and finally that

$$\int_{a_1}^{b_1} |Du(t)|^{p(t)}dt$$

$$\leqslant c(n,p(\cdot))\left[\varepsilon^{-p_+}\int_{a_1}^{b_1} |u(t)|^{p(t)}dt + \varepsilon^{p_-}\int_{a_1}^{b_1} |D^2u(t)|^{p(t)}dt + \varepsilon^{-p_+}\int_{a_1}^{b_1} S(t)\,dt\right].$$

Summation over all the intervals yields

$$\int_{\Omega} |Du(t)|^{p(t)}dt$$

$$\leqslant c(n,p(\cdot))\left[\varepsilon^{-p_+}\int_{\Omega} |u(t)|^{p(t)}dt + \varepsilon^{p_-}\int_{\Omega} |D^2u(t)|^{p(t)}dt + \varepsilon^{-p_+}||S||_1\right].$$

Step 2. Consider now functions $u = u(x_1,\dots,x_n)$ in a cube Ω, let's say $\Omega = (0,1)^n$. Applying the inequality obtained in the previous step on line segments parallel to the edges, we have for all $i = 1,\dots,n$,

$$\int_0^1 \left|\frac{\partial u}{\partial x_i}\right|^{p(x)} dx_i \leqslant c(n,p(\cdot))\left[\varepsilon^{p_-}\int_0^1 \left|\frac{\partial^2 u}{\partial x_i^2}\right|^{p(x)} dx_i + \varepsilon^{-p_+}\int_0^1 |u|^{p(x)}\,dx_i + \varepsilon^{-p_+}||S||_1\right]. \tag{3.6}$$

Integrating each of the n inequalities in (3.6) with respect to all the other orthogonal directions, we get,

$$\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p(x)} dx \leqslant c(n,p(\cdot),\Omega)\Big[\varepsilon^{p_-}\int_\Omega \left|\frac{\partial^2 u}{\partial x_i^2}\right|^{p(x)} dx+\varepsilon^{-p_+}\int_\Omega |u(x)|^{p(x)}\,dx+\varepsilon^{-p_+}||S||_1\Big].$$

Now summing over $i=1,\ldots,n$ and observing that

$$\sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}\right|^p \geqslant n^{-p}|Du|^p$$

and

$$\sum_{i=1}^n \left|\frac{\partial^2 u}{\partial x_i^2}\right|^p \leqslant n\,|D^2u|^p,$$

we have an analogous estimate with $\frac{\partial}{\partial x_i}$ replaced with D and $\frac{\partial^2}{\partial x_i^2}$ with D^2.

Step 3. Let Ω be any domain with the uniform C^2-regularity property.

Let's say H_i the mappings such that

$$H_i \in C^2(\bar{\Omega}_i),$$

$$H_i(\bar{\Omega}_i) = \bar{Q}_+,$$

$$H_i^{-1}(\bar{Q}_+) = \bar{\Omega}_i,$$

$$H_i^{-1} \in C^2(\bar{Q}_+),$$

$$H_i(\partial\Omega\cap\bar{\Omega}_i) = Q_0,$$

where

$$Q = \{x=(x_1,\ldots,x_n)\in\mathbb{R}^n : |x_k|<1, k=1,\cdots,n\},$$

$$Q_+ = \{x\in Q, |x_i|<1 : x_n>0\},$$

$$Q_0 = \{x\in Q, |x_i|<1 : x_n=0\}.$$

Since the Jacobian determinants are uniformly bounded, we assume

$$m \leqslant |\mathrm{Jac}\, H_i| \leqslant M.$$

We shall need the following Lemma, whose proof is straightforward:

LEMMA 3.3. *Let $u\in W^{2,p(\cdot)}(\Omega_i)$ and $v_i(y)=u(H_i^{-1}(y))$ $\forall y\in Q_+$. Then, setting $\tilde{p}=p\circ H_i^{-1}$, it is $v_i\in W^{2,\tilde{p}(\cdot)}(Q_+)$.*

By using the assumptions $\rho_{L^{p(\cdot)}(\Omega)}(u) < 1$ and $\rho_{L^{p(\cdot)}(\Omega)}(|D^2u|) < 1$, it can be easily seen that $\rho_{L^{\tilde{p}(\cdot)}(Q_+)}\left(\frac{v_i}{1+M}\right) < 1$ and $\rho_{L^{\tilde{p}(\cdot)}(Q_+)}\left(\frac{|D^2v_i|}{1+M}\right) < 1$. Therefore, from *step 1*, we can write

$$\int_{Q_+}\left|\frac{Dv_i(y)}{1+M}\right|^{\tilde{p}(y)} dy \leqslant c(n,\tilde{p}(\cdot),Q_+)\Big[\varepsilon^{\tilde{p}_-}\int_{Q_+}\left|\frac{D^2v_i(y)}{1+M}\right|^{\tilde{p}(y)} dy + \varepsilon^{-\tilde{p}_+}\int_{Q_+}\left|\frac{v_i(y)}{1+M}\right|^{\tilde{p}(y)} dy + \varepsilon^{-\tilde{p}_+}||S_{\tilde{p}}||_1\Big],$$

that is,

$$\int_{Q_+}\left|\frac{Du(H_i^{-1}(y))}{1+M}\right|^{\tilde{p}(y)} dy \leqslant c(n,\tilde{p}(\cdot),Q_+)\Big[\varepsilon^{\tilde{p}_-}\int_{Q_+}\left|\frac{D^2u(H_i^{-1}(y))}{1+M}\right|^{\tilde{p}(y)} dy + \varepsilon^{-\tilde{p}_+}\int_{Q_+}\left|\frac{u(H_i^{-1}(y))}{1+M}\right|^{\tilde{p}(y)} dy + \varepsilon^{-\tilde{p}_+}||S_{\tilde{p}}||_1\Big].$$

Setting $x = H_i^{-1}(y)$,

$$\int_{\Omega_i}\left|\frac{Du(x)}{1+M}\right|^{p(x)} |\mathrm{Jac}\, H_i(x)|\, dx$$

$$\leqslant c(n,p(\cdot),\Omega_i)\Big[\varepsilon^{p_-}\int_{\Omega_i}\left|\frac{D^2u(x)}{1+M}\right|^{p(x)} |\mathrm{Jac}\, H_i(x)|\, dx + \varepsilon^{-p_+}\int_{\Omega_i}\left|\frac{u(x)}{1+M}\right|^{p(x)} |\mathrm{Jac}\, H_i(x)|\, dx + \varepsilon^{-p_+}\int_{\Omega_i} S(x)|\mathrm{Jac}\, H_i(x)|\, dx\Big],$$

and therefore

$$\frac{m}{(1+M)^{p_+}}\int_{\Omega_i}|Du(x)|^{p(x)}dx \leqslant c(n,p(\cdot),\Omega_i)\Big[\varepsilon^{p_-}\frac{M}{(1+M)^{p_-}}\int_{\Omega_i}|D^2u(x)|^{p(x)}dx + \varepsilon^{-p_+}\frac{M}{(1+M)^{p_-}}\int_{\Omega_i}|u(x)|^{p(x)}dx + \varepsilon^{-p_+}M||S||_1\Big].$$

Hence

$$\int_{\Omega_i}|Du(x)|^{p(x)}dx \leqslant c(n,p(\cdot),\Omega_i)\Big[\varepsilon^{p_-}\int_{\Omega_i}|D^2u(x)|^{p(x)}dx + \varepsilon^{-p_+}\int_{\Omega_i}|u(x)|^{p(x)}dx + \varepsilon^{-p_+}||S||_1\Big].$$

Since Ω_0 can be covered by a finite number of cubes, an analogous inequality holds in it. Summing on these cubes and patches, we find for ε sufficiently small that

$$\int_{\Omega}|Du(x)|^{p(x)}dx \leqslant c(n,p(\cdot),\Omega)\Big[\varepsilon^{p_-}\int_{\Omega}|D^2u(x)|^{p(x)}\, dx + \varepsilon^{-p_+}\int_{\Omega}|u(x)|^{p(x)}dx + \varepsilon^{-p_+}||S||_1\Big] \quad (3.7)$$

and therefore, using the expression of $S(x)$ in (3.5),

$$\int_\Omega |Du(x)|^{p(x)}dx \leqslant c\left(\varepsilon^{p_-}\int_\Omega |D^2u(x)|^{p(x)}dx + \varepsilon^{-p_+}\int_\Omega |u(x)|^{p(x)}dx \right.$$
$$\left. +\varepsilon^{-p_+}\max\{(p_+-p_-)^{p_+},(p_+-p_-)^{p_-}\}\right).$$

Finally, in virtue of Lemma 2.1 and by using the fact that the convergence in norm implies the convergence of the modulars (see [17], [15], [8]), Theorem 1.2 is proved for all $u\in W^{2,p(\cdot)}(\Omega)$. Observe that the assumptions $\rho_{L^{p(\cdot)}(\Omega)}(u)<1$ and $\rho_{L^{p(\cdot)}(\Omega)}(|D^2u|)<1$ are equivalent to the conditions $||u||_{L^{p(\cdot)}(\Omega)}<1$ and $||\,|D^2u|\,||_{L^{p(\cdot)}(\Omega)}<1$, see [15]. Therefore, if $u_n\to u$ in $W^{2,p(\cdot)}(\Omega)$, and hence $u_n\to u$ in $L^{p(\cdot)}(\Omega)$ and $|D^2u_n|\to|D^2u|$ in $L^{p(\cdot)}(\Omega)$, then $||u_n||_{L^{p(\cdot)}(\Omega)}<1$ and $||\,|D^2u_n|\,||_{L^{p(\cdot)}(\Omega)}<1$ definitely. □

Now denote by $\tilde\rho(u)=\rho_{L^{p(\cdot)}(\Omega)}(u)+\rho_{L^{p(\cdot)}(\Omega)}(|D^2u|)$ and observe that for any $u\in W^{2,p(\cdot)}(\Omega)$, it is $\rho_{L^{p(\cdot)}(\Omega)}\left(\frac{u}{1+\tilde\rho(u)}\right)<1$ and $\rho_{L^{p(\cdot)}(\Omega)}\left(\frac{|D^2u|}{1+\tilde\rho(u)}\right)<1$. As a consequence, we can replace u by $\dfrac{u}{1+\tilde\rho(u)}$ in Theorem 1.2 and obtain the following

COROLLARY 3.4. *Let $\Omega\subset\mathbb{R}^n$, $n\geqslant 1$, a domain satisfying the uniform C^2-regularity property and let u be a real valued mapping in $W^{2,p(\cdot)}(\Omega)$. There exists $c=c(n,p(\cdot),\Omega)$ such that for ε sufficiently small*

$$\int_\Omega \left|\frac{Du(x)}{1+\tilde\rho(u)}\right|^{p(x)}dx \leqslant c\left(\varepsilon^{p_-}\int_\Omega \left|\frac{D^2u(x)}{1+\tilde\rho(u)}\right|^{p(x)}dx + \varepsilon^{-p_+}\int_\Omega \left|\frac{u(x)}{1+\tilde\rho(u)}\right|^{p(x)}dx\right.$$
$$\left. +\varepsilon^{-p_+}\max\{(p_+-p_-)^{p_+},(p_+-p_-)^{p_-}\}\right).$$

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