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SAITOH'S INEQUALITY AND OPIAL'S INEQUALITY

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Abstract. We prove an elementary integral inequality which extends a norm inequality of Saitoh concerning absolutely continuous functions on an interval of the real axis. Our inequality immediately yields well-known Opial-type inequalities.

1. Introduction

In 1995 S. Saitoh showed that natural norm inequalities hold for a wide class of nonlinear maps between reproducing kernel Hilbert spaces [11] by means of the theory of reproducing kernels (cf. [1], [9], [10]). His method has proved to be very important for applications such as identifications of nonlinear systems [12]. As examples of his norm inequalities he presented several concrete inequalities, among which the most beautiful is the following: For a real-valued function $f \in \mathrm{AC}[0,1]$ with $f(0)=0$ and $\int_0^1 f'(x)^2 dx < 1$, we have

$$\int_0^1 \left(\frac{f(x)}{1-f(x)}\right)'^2 (1-x)^2 dx \leqslant \frac{\int_0^1 f'^2(x)\,dx}{1-\int_0^1 f'^2(x)\,dx}, \tag{1.1}$$

where $\mathrm{AC}[a,b]$ denotes the space of real-valued absolutely continuous functions on $[a,b]$. Equality holds in (1.1) if there exists $y \in [0,1)$ such that $f(x) = \min\{x,y\}$, $x \in [0,1]$.

On the other hand, the following Opial's inequality [8] is very famous and there are many papers extending it: For $f \in \mathrm{AC}[0,a]$ with $f(0)=0$, we have

$$\int_0^a |f(x)f'(x)|\,dx \leqslant \frac{a}{2}\int_0^a |f'(x)|^2\,dx.$$

For Opial-type inequalities see e.g. [7], [2], [3].

A function $f(x)$ positive and continuous on an interval $(0,R)$ is called *geometrically convex* if f satisfies the inequality, for all $x,y \in (0,R)$

$$f(\sqrt{xy}) \leqslant \sqrt{f(x)f(y)},$$

that is, $\log \circ f \circ \exp$ is convex on $(-\infty, \log R)$, (cf. [6]).

The aim of this paper is to extend Saitoh's norm inequality (1.1) by using geometrically convex functions (Theorem 2.1), and as an application we show that our inequality immediately yields basic Opial-type inequalities (Theorem 4.1). Our main tool is Hölder's inequality, and so the proof is elementary.

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2. Main inequality

THEOREM 2.1. *Let $G(x)$ be a function of class C^1 on an interval $(-R,R)$ $(0 < R \leqslant +\infty)$ satisfying the conditions*

(i) $G(0) = 0$,

(ii) $|G'(x)| \leqslant G'(|x|)$, *for all* $x \in (-R,R)$, *and*

(iii) *if* $x^2 \leqslant yz$, $(0 < x,y,z < R)$, *then* $0 < G'^2(x) \leqslant G'(y)G'(z)$.

Assume that functions F, $f \in \mathrm{AC}[a,b]$ *with* $F(a) = f(a) = 0$ *satisfy*

(iv) $F'(x) > 0$ *a.e. on* $[a,b]$, *and*

(v) $F(b) \leqslant R$, *and* $\int_a^b |f'(t)|^p / F'(t)^{p-1}\, dt < R$ *for some* $p > 1$.

Then,

$$\int_a^b \frac{|(G\circ f)'(x)|^p}{(G\circ F)'(x)^{p-1}}\, dx \leqslant G\Big(\int_a^b \frac{|f'(x)|^p}{F'(x)^{p-1}}\, dx\Big). \tag{2.1}$$

If $f(x) = C\cdot F(\min\{x,y\})$ $(a < y \leqslant b,\ C = 0,1)$, *then equality holds in* (2.1).

Proof. Since G' is continuous, we remark that from (ii) and (iii) the function $G'(x)$ is positive, monotone increasing and geometrically convex on the interval $(0,R)$. Applying Hölder's inequality with conjugate exponent $1/p + 1/q = 1$ to the identity $f(x) = \int_a^x f'(t)\, dt$, we have

$$|f(x)| \leqslant F(x)^{1/q}\Big(\int_a^x \frac{|f'(t)|^p}{F'(t)^{p-1}}\, dt\Big)^{1/p}.$$

From the above remark on G', we see that for $0 \leqslant x,y,z < R$,

$$x \leqslant y^{1/p} z^{1/q} \implies G'(x) \leqslant G'(y)^{1/p} G'(z)^{1/q}. \tag{2.2}$$

Hence, by (iv) and (v), we obtain for $a \leqslant x < b$

$$G'(|f(x)|) \leqslant G'(F(x))^{1/q} G'\Big(\int_a^x \frac{|f'(t)|^p}{F'(t)^{p-1}}\, dt\Big)^{1/p}.$$

Multiplying $|f'(x)|^p/(G\circ F)'(x)^{p-1}$ $(\geqslant 0)$ to the p-th power of the above inequality, and using (ii) we have

$$\frac{|(G\circ f)'(x)|^p}{(G\circ F)'(x)^{p-1}} \leqslant \frac{d}{dx}\Big\{G\Big(\int_a^x \frac{|f'(t)|^p}{F'(t)^{p-1}}\, dt\Big)\Big\}.$$

Integrating both sides of this inequality on the interval $[a,b]$, we obtain the desired inequality (2.1) in view of (i). When $f(x)$ is of the form $F(\min\{x,y\})$, $y \in [a,b]$, it is obvious that equality holds. □

REMARK 2.2. By setting $G(x) = x/(1-x)$, $F(x) = x$, $p = 2$, $a = 0$ and $b = R = 1$ in Theorem 2.1, we easily obtain Saitoh's inequality (1.1). Note that $G'(x) = (1-x)^{-2}$ is geometrically convex on $(0,1)$. Equality holds if and only if $f(x) = \min\{x,y\}$, $y \in [0,1)$, which is seen from Theorem 3.1 in the next section.

REMARK 2.3. Let H be a real vector space of functions $f \in \mathrm{AC}[a,b]$ with $f(a) = 0$. Then H becomes a reproducing kernel Hilbert space (RKHS) if we assign $f \in H$ the norm

$$\left(\int_a^b |f'(t)|^2 \rho(t)\,dt\right)^{1/2},$$

where the weight $\rho(t)$ is a positive continuous function on $[a,b]$. The reproducing kernel k of H is given by

$$k(x,y) = F(\min\{x,y\})$$

with $F(x) = \int_a^x \rho(t)^{-1} dt$. Since a reproducing kernel Hilbert space is uniquely determined by its kernel function, denote this RKHS by H_k and the above norm by $\|f\|_k$. Then, for $p = 2$ we can rewrite formally the inequality (2.1) of Theorem 2.1 as

$$\|G \circ f\|_{G(k)}^2 \leqslant G(\|f\|_k^2) \quad \text{for any } f \in H_k.$$

Thus, it may seem that our inequality (2.1) is merely an example of general norm inequalities for RKHSs (cf. [10], [11]). This, however, is not the case, since our inequality does not require the real analyticity of the function G, while this must be assumed in general norm inequalities for RKHSs.

3. Equality condition

For most cases equality in Main Inequality is attained only for functions stated in Theorem 2.1, i.e. the function 0 or $F(\min\{x,y\})$. We are able to show this by adding further assumptions on the function $G(x)$.

THEOREM 3.1. *Under the same hypothesis as in Theorem* 2.1*, assume, moreover, that*

(vi) G' *is strictly monotone increasing on* $(0,R)$.

Then, if equality holds in inequality (2.1)*, then there exist constants* C *and* y $(a < y \leqslant b)$ *such that*

$$f(x) = C \cdot F(\min\{x,y\}). \tag{3.1}$$

If, in addition, we assume that

(vii) $|G'(-x)| \neq G'(x)$ *for some* $x \in (0, F(y))$*, and*

(viii) *there exist no constants* $\alpha > 0$ *and* $\beta \geqslant 0$ *such that* $G'(x) = \alpha x^\beta$ *on* $(0, F(y))$*,*

then equality holds in inequality (2.1) *if and only if* f *is of the form* (3.1) *with* C *either 0 or 1.*

Proof. First, we remark that equality occurs on the right-hand side inequality of (2.2) if and only if $x = y^{1/p}z^{1/q}$. For, since G' is strictly monotone increasing, we have

$$G'(x) \leqslant G'(y^{1/p}z^{1/q}) \leqslant G'(y)^{1/p}G'(z)^{1/q} = G'(x),$$

which implies $x = y^{1/p}z^{1/q}$.

When $f = 0$ it suffices to take $C = 0$. Hence, we may assume $f \neq 0$. Putting $y = \operatorname{ess\,sup}\{x\colon f'(x) \neq 0,\ a \leqslant x < b\}$, we have $a < y \leqslant b$. If equality holds in (2.1), then y must be a cluster point of the set $\{x;\ f'(x) \neq 0\}$. Hence, by continuity

$$G'(|f(y)|) = G'\big(F(y)\big)^{1/q} G'\Big(\int_a^y \frac{|f'(t)|^p}{F'(t)^{p-1}}\,dt\Big)^{1/p},$$

and from the remark above we obtain

$$|f(y)| = \big(F(y)\big)^{1/q}\Big\{\int_a^y \frac{|f'(t)|^p}{F'(t)^{p-1}}\,dt\Big\}^{1/p}.$$

From the equality condition of Hölder's inequality, there exists a constant $C \neq 0$ such that $f(x) = CF(x)$, $(a \leqslant x \leqslant y)$. Since $f'(x) = 0$ $(y \leqslant x \leqslant b$ a.e.$)$ by definition of y, we conclude that f is of the form

$$f(x) = C \cdot F(\min\{x,y\}).$$

Thus, (3.1) is proved. Now we prove the latter half of the assertion of Theorem. For all x with $a \leqslant x \leqslant y$ we have

$$\int_a^x \frac{|f'(t)|^p}{F'(t)^{p-1}}\,dt = |C|^p F(x),$$

and hence, the following identities must hold simultaneously: for all x, $a \leqslant x \leqslant y$,

$$|C|F(x) = F(x)^{1/q}(|C|^p F(x))^{1/p},$$
$$G'(|C|F(x)) = G'(F(x))^{1/q} G'(|C|^p F(x))^{1/p}.$$

If $|C| \neq 1$, then $F(x) \neq |C|^p F(x)$ for $x > a$. Since the equality condition for Jensen's inequality for two distinct points imply the linearity of the function on the interval between these points, one verifies easily that $G'(x)$ is of the form αx^β, $(\alpha > 0,\ \beta \geqslant 0)$ on the interval $(0, F(y))$. But this is excluded by our assumption (viii). Finally, if $C = -1$ then we must have equality in the inequality (ii) on $(-F(y), 0)$, which contradicts the condition (vii). □

REMARK 3.2. If $G(x) = \alpha|x|^\beta$ $(\alpha > 0,\ \beta > 1)$, then equality holds in inequality (2.1) for every $f(x)$ of the form $C \cdot F(\min\{x,y\})$ $(C \in \mathbb{R},\ a < y \leqslant b)$.

4. Application

Main Inequality allows us immediately to derive Opial-type inequalities (cf. [2], [3], [4], [7]). For brevity we restrict ourselves to the case that the constant R in Theorem 2.1 is infinity.

THEOREM 4.1. *Let $s(x)$, $t(x)$ be nonnegative, measurable functions on $[a,b]$ such that $\int_a^b t(x)^{-1/(p-1)}\,dx < +\infty$ for some $p>1$. Set $F(x)=\int_a^x t(\xi)^{-1/(p-1)}\,d\xi$ and assume that the functions $G(x)$, $F(x)$ and $f(x)$ satisfy the same conditions as stated in Theorem* 2.1 *with $R=+\infty$. Then, if $K<+\infty$, we have*

$$\left\{\int_a^b |(G\circ f)'(x)|^q s(x)\,dx\right\}^{1/q} \leqslant K\cdot G\Big(\int_a^b |f'(x)|^p t(x)\,dx\Big)^{1/p}, \qquad (4.1)$$

where $1/p+1/r=1/q$, $r>0$ and

$$K=\left\{\int_a^b (G\circ F)'(x)^{r(1-1/p)} s(x)^{r/q}\,dx\right\}^{1/r}.$$

If, in addition, we assume the conditions (vi), (vii) *and* (viii) *in Theorem* 3.1, *then equality holds in the inequality* (4.1) *if and only if either $f=0$ or there exist constants $C\ (\geqslant 0)$ and $y\ (a<y\leqslant b)$ such that $f(x)=F(\min\{x,y\})$ and $s(x)=C\cdot(G\circ F)'(x)^{1-q}$.*

Proof. Rewrite the integrand on the left-hand side of (4.1) as

$$|(G\circ f)'|^q s = \frac{|(G\circ f)'|^q}{(G\circ F)'^{\alpha}}\cdot (G\circ F)'^{\alpha} s, \quad \alpha=\frac{q(p-1)}{p},$$

use Hölder's inequality with conjugate exponents p/q and r/q, and apply Theorem 2.1. Equality condition is obtained immediately from Theorem 3.1. □

REMARK 4.2. The existence of the multiplicative constant K is a merit of our inequality (4.1). Cf. [4], [5]

REMARK 4.3. Let $G(x)=|x|^{p/q}$, $p>q$, $k>1$ and $k>q>0$. If $\int_a^b t(x)^{-1/(k-1)}dx < +\infty$, then from Theorem 4.1 we obtain Opial-type inequality

$$\int_a^b |f(x)|^{p-q}|f'(x)|^q s(x)\,dx \leqslant K\cdot\left\{\int_a^b |f'(x)|^k t(x)\,dx\right\}^{p/k}, \qquad (4.2)$$

where we assume that the constant

$$K=\Big(\frac{q}{p}\Big)^{q/k}\left\{\int_a^b s^{k/(k-q)} t^{-q/(k-q)}\Big(\int_a^x t^{-1/(k-1)}\,d\xi\Big)^{(p-q)(k-1)/(k-q)} dx\right\}^{(k-q)/k}$$

is finite. Note that this is the same inequality as (2.6) in [2] except for notation. Equality condition can be given easily as above.

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