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ON MEAN VALUES OF DIRICHLET POLYNOMIALS

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Abstract. We show the following general lower bound valid for any positive integer q, and arbitrary reals $\varphi_1,\dots,\varphi_N$ and non-negative reals $a_1,\dots,a_N$,

$$c_q\Big(\sum_{n=1}^N a_n^2\Big)^q \leqslant \frac{1}{2T}\int_{|t|\leqslant T}\Big|\sum_{n=1}^N a_n e^{it\varphi_n}\Big|^{2q} dt.$$

1. Main Result

The object of this short Note is to prove the following lower bound

THEOREM 1. *For any positive integer q, there exists a constant c_q, such that for any reals $\varphi_1,\dots,\varphi_N$, any non-negative reals $a_1,\dots,a_N$, and any $T>0$,*

$$c_q\Big(\sum_{n=1}^N a_n^2\Big)^q \leqslant \frac{1}{2T}\int_{|t|\leqslant T}\Big|\sum_{n=1}^N a_n e^{it\varphi_n}\Big|^{2q} dt.$$

The result is no longer true for arbitrary reals $a_1,\dots,a_N$ as yields the case $\varphi_1=\dots=\varphi_N$. It also follows that

$$c\Big(\sum_{n=1}^N a_n^2\Big)^{1/2} \leqslant \sup_{t\in\mathbf{R}}\Big|\sum_{n=1}^N a_n e^{it\varphi_n}\Big|. \tag{1}$$

In the case $\varphi_n=\log n$, it is known from [5] and [8] that for any (a_n)

$$\sup_{t\in\mathbf{R}}\Big|\sum_{n=0}^{N-1} a_n n^{it}\Big| \geqslant \alpha_1\frac{e^{\beta_1\sqrt{\log N\log\log N}}}{\sqrt{N}}\Big(\sum_{n=0}^{N-1}|a_n|\Big) \tag{2}$$

and for some (a_n)

$$\sup_{t\in\mathbf{R}}\Big|\sum_{n=0}^{N-1} a_n n^{it}\Big| \leqslant \alpha_2\frac{e^{\beta_2\sqrt{\log N\log\log N}}}{\sqrt{N}}\Big(\sum_{n=0}^{N-1}|a_n|\Big), \tag{3}$$

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with some universal constants $\alpha_1,\alpha_2,\beta_1,\beta_2$. Then (1) is better than (2) if for instance $a_n = n^{-\alpha}$, $\alpha > 1/2$, since

$$\frac{e^{\beta_1\sqrt{\log N\log\log N}}}{\sqrt{N}}\Big(\sum_{n=0}^{N-1}|a_n|\Big)\sim e^{\beta_1\sqrt{\log N\log\log N}})N^{\frac{1}{2}-\alpha}=o(1)\ll\Big(\sum_{n=1}^{N}a_n^2\Big)^{1/2}.$$

The L^1-case is related to well-known Ingham's inequality [2]. We state the sharper form due to Mordell [7]: let $0<\varphi_1<\ldots<\varphi_N$ and let γ be such that $\min_{1<n\leqslant N}\varphi_n-\varphi_{n-1}\geqslant \gamma>0$. Then

$$\sup_{n=1}^{N}|a_n|\leqslant\frac{K}{T}\int_{-T}^{T}\Big|\sum_{n=1}^{N}a_ne^{it\varphi_n}\Big|dt\qquad\text{with } T=\frac{\pi}{\gamma},\tag{4}$$

where $K\leqslant 1$.

Further with no restriction, one always have

$$\sup_{n=1}^{N}|a_n|\leqslant\limsup_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\Big|\sum_{n=1}^{N}a_ne^{it\varphi_n}\Big|dt\leqslant\sup_{t\in\mathbf{R}}\Big|\sum_{n=1}^{N}a_ne^{it\varphi_n}\Big|,\tag{5}$$

a very familiar inequality in the theory of uniformly almost periodic functions. See also [1] where the more complicated inequality is established:

$$|a_n|\leqslant\frac{1}{\prod_{j=0}^{n-1}\cos(\frac{\pi\varphi_j}{2\varphi_n})\cdot\prod_{n+1}^{N}\cos(\frac{\pi\varphi_n}{2\varphi_j})}\cdot\sup_{|t|\leqslant\frac{\pi}{2}(\frac{n}{\varphi_n}+\sum_{j=n+1}^{N}\frac{1}{\varphi_j})}\Big|\sum_{n=1}^{N}a_ne^{it\varphi_n}\Big|.\tag{6}$$

In particular, if $\varphi_1,\ldots,\varphi_N$ are linearly independent, and T is large enough, then

$$b_q\Big(\sum_{n=1}^{N}a_n^2\Big)^q\leqslant\frac{1}{2T}\int_{|t|\leqslant T}\Big|\sum_{n=1}^{N}a_ne^{it\varphi_n}\Big|^{2q}dt\leqslant B_q\Big(\sum_{n=1}^{N}a_n^2\Big)^q,\tag{7}$$

holds for any nonnegative reals $a_1,\ldots,a_N$ and b_q, B_q depend on q only.

The proof of Theorem 1 relies upon the following lemma, which just generalizes a useful majorization argument ([6], p.131) to arbitrary even powers.

LEMMA 2. *Let q be any positive integer. Let $c_1,\ldots,c_N$ be complex numbers and nonnegative reals $a_1,\ldots,a_N$ such that $|c_n|\leqslant a_n$, $n=1,\ldots,N$. Then for any reals T,T_0 with $T>0$*

$$\int_{|t-T_0|\leqslant T}\Big|\sum_{n=1}^{N}c_ne^{it\varphi_n}\Big|^{2q}dt\leqslant 3\int_{|t|\leqslant T}\Big|\sum_{n=1}^{N}a_ne^{it\varphi_n}\Big|^{2q}dt.$$

Proof. Let

$$K_T(t)=K_T(|t|)=\big(1-|t|/T\big)\chi_{\{|t|\leqslant T\}}$$

Observe that for any reals t,H

$$a)\qquad K_T(t-H)=\big(1-|t-H|/T\big)\chi_{\{|t-H|\leqslant T\}}$$

$$b) \qquad \chi_{\{|t-H|\leqslant T\}} \leqslant K_T(t-H)+K_T(t-H+T)+K_T(t-H-T)$$

$$c) \qquad \widehat{K}_T(u)=\frac{1}{T}\Big(\frac{\sin Tu}{u}\Big)^2 \geqslant 0, \qquad \text{for all real } u.$$

Suppose that $|c_n| \leqslant a_n$ for $n=1,\dots,N$. From

$$\Big(\sum_{n=1}^{N} c_n e^{it\varphi_n}\Big)^q = \sum_{k_1+\dots+k_N=q} \Big(\frac{q!}{k_1!\dots k_N!}\Big) \prod_{n=1}^{N} c_n^{k_n} e^{itk_n\varphi_n}. \tag{8}$$

and

$$\Big|\sum_{n=1}^{N} c_n e^{it\varphi_n}\Big|^{2q} = \sum_{\substack{k_1+\dots+k_N=q\\ h_1+\dots+h_N=q}} \Big(\frac{(q!)^2}{k_1!h_1!\dots k_N!h_N!}\Big) \prod_{n=1}^{N} c_n^{k_n}\overline{c}_n^{\,h_n} e^{it(k_n-h_n)\varphi_n}$$

we get

$$\begin{aligned}
&\int_{\mathbf{R}} K_T(t-H)\Big|\sum_{n=1}^{N} c_n e^{it\varphi_n}\Big|^{2q} dt\\
&= \sum_{\substack{k_1+\dots+k_N=q\\ h_1+\dots+h_N=q}} \frac{(q!)^2}{k_1!h_1!\dots k_N!h_N!} \prod_{n=1}^{N} c_n^{k_n}\overline{c}_n^{\,h_n} \int_{\mathbf{R}} K_T(t-H) e^{it\sum_{n=1}^{N}(k_n-h_n)\varphi_n} dt\\
&= \sum_{\substack{k_1+\dots+k_N=q\\ h_1+\dots+h_N=q}} \frac{(q!)^2}{k_1!h_1!\dots k_N!h_N!} \prod_{n=1}^{N} c_n^{k_n}\overline{c}_n^{\,h_n} \int_{\mathbf{R}} K_T(s) e^{i(s+H)\sum_{n=1}^{N}(k_n-h_n)\varphi_n} ds\\
&= \sum_{\substack{k_1+\dots+k_N=q\\ h_1+\dots+h_N=q}} \frac{(q!)^2}{k_1!h_1!\dots k_N!h_N!} \prod_{n=1}^{N} (c_n e^{iH\varphi_n})^{k_n}\overline{(c_n e^{iH\varphi_n})}^{h_n} \widehat{K}_T\Big(\sum_{n=1}^{N}(k_n-h_n)\varphi_n\Big)\\
&\leqslant \sum_{\substack{k_1+\dots+k_N=q\\ h_1+\dots+h_N=q}} \frac{(q!)^2}{k_1!h_1!\dots k_N!h_N!} \prod_{n=1}^{N} a_n^{k_n+h_n} \widehat{K}_T\Big(\sum_{n=1}^{N}(k_n-h_n)\varphi_n\Big)\\
&= \int_{\mathbf{R}} K_T(t)\Big[\sum_{\substack{k_1+\dots+k_N=q\\ h_1+\dots+h_N=q}} \frac{(q!)^2}{k_1!h_1!\dots k_N!h_N!} \prod_{n=1}^{N} a_n^{k_n+h_n} e^{it\sum_{n=1}^{N}(k_n-h_n)\varphi_n}\Big] dt\\
&= \int_{\mathbf{R}} K_T(t)\Big|\sum_{n=1}^{N} a_n e^{it\varphi_n}\Big|^{2q} dt.
\end{aligned}$$

Hence, if $|c_n| \leqslant a_n$ for $n=1,\dots,N$

$$\int_{\mathbf{R}} K_T(t-H)\Big|\sum_{n=1}^{N} c_n e^{it\varphi_n}\Big|^{2q} dt \leqslant \int_{\mathbf{R}} K_T(t)\Big|\sum_{n=1}^{N} a_n e^{it\varphi_n}\Big|^{2q} dt. \tag{9}$$

By applying Lemma 2 with $H = 0, T_0, -T_0$, and using b), we get

$$\int_{|t-T_0|\leqslant T} \Big|\sum_{n=1}^{N} c_n e^{it\varphi_n}\Big|^{2q} dt \tag{10}$$

$$\leqslant \int_{\mathbf{R}} \Big(K_T(t-T_0)+K_T(t-T_0+T)+K_T(t-T_0-T)\Big)\Big|\sum_{n=1}^{N} c_n e^{it\varphi_n}\Big|^{2q} dt \tag{11}$$

$$\leqslant 3\int_{\mathbf{R}} K_T(t)\Big|\sum_{n=1}^{N} a_n e^{it\varphi_n}\Big|^{2q} dt \leqslant 3\int_{|t|\leqslant T} \Big|\sum_{n=1}^{N} a_n e^{it\varphi_n}\Big|^{2q} dt. \qquad \square \tag{12}$$

The proof of Theorem 1 is now achieved as follows. First recall the Khintchin-Kahane inequalities [4]. Let $\{\varepsilon_i, 1\leqslant i\leqslant N\}$ be independent Rademacher random variables, thus satisfying $\mathbf{P}\{\varepsilon_i=\pm1\}=1/2$, if $(\Omega,\mathscr{A},\mathbf{P})$ denotes the underlying basic probability. Then for any $0<p<\infty$, there exist positive finite constants c_p, C_p depending on p only, such that for any sequence $\{a_i, 1\leqslant i\leqslant N\}$ of real numbers

$$c_p\Big(\sum_{i=1}^{N} a_i^2\Big)^{1/2} \leqslant \Big\|\sum_{i=1}^{N} a_i\varepsilon_i\Big\|_p \leqslant C_p\Big(\sum_{i=1}^{N} a_i^2\Big)^{1/2}. \tag{13}$$

This remains true for complex a_n. If $a_n=\alpha_n+i\beta_n$, then

$$\Big\|\sum_{j=1}^{N} a_j\varepsilon_j\Big\|_p^p = \mathbf{E}\Big|\sum_{j=1}^{N}\alpha_j\varepsilon_j + i\sum_{j=1}^{N}\beta_j\varepsilon_j\Big|^p = \mathbf{E}\Big(\Big|\sum_{j=1}^{N}\alpha_j\varepsilon_j\Big|^2+\Big|\sum_{j=1}^{N}\beta_j\varepsilon_j\Big|^2\Big)^{p/2}$$
$$\leqslant 2^{(p/2)-1}\Big(\mathbf{E}\Big|\sum_{j=1}^{N}\alpha_j\varepsilon_j\Big|^p+\mathbf{E}\Big|\sum_{j=1}^{N}\beta_j\varepsilon_j\Big|^p\Big),$$

where we have denoted by $\mathbf{E}$ the corresponding expectation symbol. Thus, since $\sqrt{A}+\sqrt{B}\leqslant\sqrt{2(A+B)}$, $A,B\geqslant 0$,

$$\Big\|\sum_{j=1}^{N} a_j\varepsilon_j\Big\|_p \leqslant 2^{1/2-1/p}C_p\Big[\big(\sum_{j=1}^{N}\alpha_j^2\big)^{1/2}+\big(\sum_{j=1}^{N}\beta_j^2\big)^{1/2}\Big]$$
$$\leqslant 2^{1-1/p}C_p\Big(\sum_{j=1}^{N}(\alpha_j^2+\beta_j^2)\Big)^{1/2} = C'_p\Big(\sum_{j=1}^{N}|a_j|^2\Big)^{1/2}.$$

Conversely, from

$$\Big\|\sum_{j=1}^{N} a_j\varepsilon_j\Big\|_p^p = \mathbf{E}\Big(\Big|\sum_{j=1}^{N}\alpha_j\varepsilon_j\Big|^2+\Big|\sum_{j=1}^{N}\beta_j\varepsilon_j\Big|^2\Big)^{p/2}$$
$$\geqslant \max\Big(\mathbf{E}\Big|\sum_{j=1}^{N}\alpha_j\varepsilon_j\Big|^p, \mathbf{E}\Big|\sum_{j=1}^{N}\beta_j\varepsilon_j\Big|^p\Big),$$

we get

$$\Big\|\sum_{j=1}^N a_j\varepsilon_j\Big\|_p \geqslant \max\Big(\|\sum_{j=1}^N \alpha_j\varepsilon_j\|_p, \|\sum_{j=1}^N \beta_j\varepsilon_j\|_p\Big)$$
$$\geqslant c_p \max\Big(\big(\sum_{j=1}^N |\alpha_j|^2\big)^{1/2}, \big(\sum_{j=1}^N |\beta_j|^2\big)^{1/2}\Big)$$
$$\geqslant \frac{c_p}{2}\Big(\sum_{j=1}^N (|\alpha_j|^2+|\beta_j|^2)\Big)^{1/2} = c_p'\Big(\sum_{j=1}^N |a_j|^2\Big)^{1/2}.$$

Now choose $c_n = \varepsilon_n a_n$. Taking expectation in inequality of Lemma 2.1, and using Fubini's Theorem, gives

$$\int_{|t|\leqslant T} \mathbf{E}\Big|\sum_{n=1}^N \varepsilon_n a_n e^{it\varphi_n}\Big|^{2q} \leqslant 3\int_{|t|\leqslant T}\Big|\sum_{n=1}^N a_n e^{it\varphi_n}\Big|^{2q} dt. \tag{14}$$

By (13) we have

$$c_q\Big(\sum_{n=1}^N a_n^2\Big)^q \leqslant \mathbf{E}\Big|\sum_{n=1}^N \varepsilon_n a_n e^{it\varphi_n}\Big|^{2q} \leqslant C_q\Big(\sum_{n=1}^N a_n^2\Big)^q. \tag{15}$$

By reporting

$$2Tc_q\Big(\sum_{n=1}^N a_n^2\Big)^q \leqslant 3\int_{|t|\leqslant T}\Big|\sum_{n=1}^N a_n e^{it\varphi_n}\Big|^{2q} dt, \tag{16}$$

which proves our claim. □

2. Application

We shall deduce from Theorem 1 the following lower bound.

COROLLARY 3. *For every N, T and ν*

$$c_\nu \log^{\nu^2} N \leqslant \frac{1}{2T}\int_{|t|\leqslant T}\Big|\sum_{n=1}^N \frac{1}{n^{\frac12+it}}\Big|^{2\nu} dt.$$

In relation with this is Ramachandra's well-known lower bound (see [3] section 9.5, to which we also refer for the estimates used in the proof)

$$c_\nu(\log T)^{\nu^2} \leqslant \frac{1}{2T}\int_{|t|\leqslant T}\big|\zeta(\tfrac12+it)\big|^{2\nu} dt. \tag{17}$$

Proof. Apply Theorem 1 with $q=2$ to the sum

$$\Big(\sum_{n=1}^N \frac{1}{n^{\frac12+it}}\Big)^\nu := \sum_{m=1}^{N^\nu} \frac{b_m}{m^{\frac12+it}},$$

where

$$b_m = \#\{(n_j)_{j\leqslant\nu}\,;\, n_j \leqslant N : m = \prod_{j\leqslant\nu} n_j\}.$$

Thus for all N and T

$$c_\nu \sum_{m=1}^{N^\nu} \frac{b_m^2}{m} \leqslant \frac{1}{2T}\int_{|t|\leqslant T}\Big|\sum_{n=1}^{N}\frac{1}{n^{\frac12+it}}\Big|^{2\nu} dt.$$

But if $m \leqslant N$, $b_m = d_\nu(m)$ where $d_\nu(m)$ denotes the number of representations of m as a product of ν factors, and we know that

$$\sum_{m\leqslant x}\frac{d_\nu^2(m)}{m} = (C_\nu + o(1))\log^{\nu^2} x.$$

Thus

$$\sum_{m=1}^{N^\nu}\frac{b_m^2}{m} \geqslant \sum_{m=1}^{N}\frac{b_m^2}{m} \geqslant c_\nu \log^{\nu^2} N$$

Henceforth

$$c_\nu \log^{\nu^2} N \leqslant \frac{1}{2T}\int_{|t|\leqslant T}\Big|\sum_{n=1}^{N}\frac{1}{n^{\frac12+it}}\Big|^{2\nu} dt. \qquad \square$$

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