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CHARACTERIZATIONS OF THE CONVERGENCE OF HARMONIC AVERAGES OF DOUBLE NUMERICAL SEQUENCES

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Abstract. In recent years, the almost sure central limit theorem has attracted widespread attention in Probability Theory. It involves the harmonic (also called logarithmic) averages of a certain numerical sequence formed from a sequence of independent, identically distributed random variables. The convergence behavior of the sequence of harmonic averages of a given numerical sequence was studied in [3] by the third author. Our main goal in this paper is to extend these characterization results from single to double numerical sequences of complex numbers.

Among others, the following Theorem 2^* is proved. Let $\{x_{ij} : i, j = 1,2,\dots\}$ be a double sequence of complex numbers. Necessary and sufficient condition for the existence of the bounded limit relation

$$\mathrm{b}-\lim_{k,\ell\to\infty} \frac{1}{(\ln k)(\ln \ell)} \sum_{i=1}^{k} \sum_{j=1}^{\ell} \frac{x_{ij}}{ij} = \xi$$

is that

$$\mathrm{b}-\lim_{m,n\to\infty} \frac{1}{2^{m+n}} \max_{k\in J_m,\ell\in J_n} \left| \sum_{i=\mu_{m-1}+1}^{k} \sum_{j=\mu_{n-1}+1}^{\ell} \frac{x_{ij}-\xi}{ij} \right| = 0,$$

where

$$J_m := \{\mu_{m-1}+1, \mu_{m-1}+2, \dots, \mu_m\}, \quad \mu_m := 2^{2^m}, \quad m = 0,1,\dots.$$

Background in Probability Theory

In the framework of Kolmogorov's axiomatic treatment of probability, one of the fundamental questions is the relationship between probability and relative frequency. The results of this investigation are called the *laws of large numbers*. Similarly, the relationship between expectation of a random variable and sample mean can also be studied by using the laws of large numbers. For example, the celebrated Kolmogorov strong law of large numbers reads as follows. Let $\{X_i : i = 1,2,\dots\}$ be a sequence of independent, identically distributed random variables, in abbreviation: i.i.d.r.v.'s on a probability space $(\Omega, \mathscr{F}, P)$. Then the *arithmetic averages*

$$\frac{1}{k}\sum_{i=1}^{k} X_i, \quad k = 1,2,\dots,$$

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converge almost surely to some constant c if and only if the expectation EX_1 exists, in which case $c = EX_1$; that is,

$$P\Big[\lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}X_i(\omega) = EX_1\Big] = 1.$$

The interested reader may consult [3] to get broad perspective of this topic.

In recent years, the so-called *almost sure central limit theorem* has attracted widespread attention in Probability Theory, initiated by Brosamler [1]; and as to further papers see in [2, References]. To be more specific, let $\{X_i : i = 1,2,\ldots\}$ be i.i.d.r.v.'s with $EX_1 = 0$ and $EX_1^2 = 1$. In this case, the almost sure central limit theorem says that

$$P\Big[\lim_{k\to\infty}\frac{1}{\ln k}\sum_{i=1}^{k}\frac{1}{i}I_A\Big(\frac{X_1(\omega)+\ldots+X_i(\omega)}{\sqrt{i}}\Big) = \frac{1}{\sqrt{2\pi}}\int_A e^{-u^2/2}du\Big] = 1$$

for each Borel set $A \subset \mathbb{R}$ whose boundary is of zero (Lebesgue) measure, where $I_A(\cdot)$ is the indicator function of the set A. Here and in the sequel, the logarithm is to the natural base e.

Now, the ratio on the left-hand side in the above equality is the *harmonic average* of the sequence

$$\Big\{I_A\Big(\frac{X_1(\omega)+\ldots+X_i(\omega)}{\sqrt{i}}\Big) :\ i = 1,2,\ldots\Big\},$$

whose terms are nonnegative real numbers.

Our primary aim in this paper is to study the convergence behavior of the double sequence of harmonic averages of a given double numerical sequence from the viewpoint of Summability Theory. Meanwhile, we summarize briefly the analogous results involving double sequences of arithmetic averages. These results may be of use in the study of strong laws of large numbers as well as the almost sure central limit theorems for random fields $\{X_{ij} : i,j = 1,2,\ldots\}$ defined on a probability space $(\Omega,\mathscr{F},P)$.

1. Known results for single sequences

Given a sequence $\{x_i : i = 1,2,\ldots\}$ of complex numbers, its *arithmetic averages* σ_k are defined by

$$\sigma_k := \frac{1}{k}\sum_{i=1}^{k}x_i, \quad k = 1,2,\ldots.$$

It is well known that the ordinary convergence of $\{x_i\}$ is a sufficient condition for the convergence of $\{\sigma_k\}$ to the same limit. On the other hand, a necessary condition for the convergenc of $\{\sigma_k\}$ is that

$$\lim_{k\to\infty}\frac{x_k}{k} = 0.$$

For brevity in writing, we introduce the notation

$$I_0 := \{1\},\ I_m := \{2^{m-1}+1, 2^{m-1}+2,\ldots,2^m\},\ m = 1,2,\ldots. \tag{1.1}$$

The next three theorems are folklore.

THEOREM A. *Necessary and sufficient condition for the existence of the finite limit*

$$\lim_{m\to\infty} \sigma_{2^m} = \xi$$

is that

$$\lim_{m\to\infty} \frac{1}{2^{m-1}} \sum_{i\in I_m} x_i = \xi.$$

The ratio on the left is called the *moving arithmetic awerage* of the sequence $\{x_i\}$.

THEOREM B. *Necessary and sufficient condition for*

$$\lim_{k\to\infty} \sigma_k = \xi \tag{1.2}$$

is that

$$\lim_{m\to\infty} \frac{1}{2^{m-1}} \max_{k\in I_m} \Big| \sum_{i=2^{m-1}+1}^{k} (x_i - \xi) \Big| = 0.$$

The ratio on the left may be called the *moving maximal arithmetic average* of the sequence $\{x_i - \xi\}$.

THEOREM C. *Necessary and sufficient condition for*

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} |x_i - \xi| = 0 \tag{1.3}$$

is that

$$\lim_{m\to\infty} \frac{1}{2^{m-1}} \sum_{i\in I_m} |x_i - \xi| = 0.$$

We note that a sequence $\{x_i\}$ satisfying (1.3) is called *strongly arithmetically summable* to ξ. Clearly, (1.3) implies (1.2), but the converse implication is not true in general.

We recall that the *harmonic averages* τ_k of a sequence $\{x_i\}$ are defined by

$$\tau_k := \frac{1}{\lambda_k} \sum_{i=1}^{k} \frac{x_i}{i}, \quad \text{where} \quad \lambda_k := \sum_{i=1}^{k} \frac{1}{i}, \quad k = 1, 2, \ldots. \tag{1.4}$$

Since

$$\lim_{k\to\infty} \frac{\lambda_k}{\ln k} = 1, \tag{1.5}$$

The sequences $\{\tau_k\}$ and $\{\tau_k^*\}$ are equiconvergent to the same limit (if one of them exists), where

$$\tau_k^* := \frac{1}{\ln k} \sum_{i=1}^{k} \frac{x_i}{i}, \quad k = 2, 3, \ldots.$$

Therefore, these τ_k^* are sometimes called *logarithmic averages.* However, we prefer the usage of τ_k rather than that of τ_k^*, partly due to the fact that the definition of τ_k^* makes no sense in case $k = 1$.

It is well known that (1.2) is a sufficient condition for the convergence of τ_k to the same limit ξ. On the other hand, a necessary condition for the convergence of $\{\tau_k\}$ is that

$$\lim_{k\to\infty} \frac{x_k}{k\ln k} = 0.$$

For brevity in writing, we introduce the notations

$$\mu_m := 2^{2^m} \text{ and } J_m := \{\mu_{m-1}+1, \mu_{m-1}+2, \ldots, \mu_m\}, \quad m = 0,1,2,\ldots; \ \mu_{-1} := 0. \quad (1.6)$$

The next three theorems were proved in [2] by the third author.

THEOREM D. *Necessary and sufficient condition for*

$$\lim_{m\to\infty} \tau_{\mu_m} = \xi$$

is that

$$\lim_{m\to\infty} \frac{1}{\lambda_{\mu_m} - \lambda_{\mu_{m-1}}} \sum_{i\in J_m} \frac{x_i}{i} = \xi.$$

The ratio on the left is called the *moving harmonic average* of the sequence $\{x_i\}$. We note that by (1.5) and (1.6), we have

$$\lim_{m\to\infty} \frac{\lambda_{\mu_m}}{2^m} = \lim_{m\to\infty} \frac{\lambda_{\mu_m} - \lambda_{\mu_{m-1}}}{2^{m-1}} = \ln 2. \quad (1.7)$$

THEOREM E. *Necessary and sufficient condition for*

$$\lim_{k\to\infty} \tau_k = \xi \quad (1.8)$$

is that

$$\lim_{m\to\infty} \frac{1}{\lambda_{\mu_m} - \lambda_{\mu_{m-1}}} \max_{k\in J_m} \left| \sum_{i=\mu_{m-1}+1}^{k} \frac{x_i - \xi}{i} \right| = 0.$$

The ratio on the left may be called the *moving maximal harmonic average* of the sequence $\{x_i - \xi\}$.

THEOREM F. *Necessary and sufficient condition for*

$$\lim_{k\to\infty} \frac{1}{\lambda_k} \sum_{i=1}^{k} \frac{|x_i - \xi|}{i} = 0 \quad (1.9)$$

is that

$$\lim_{m\to\infty} \frac{1}{\lambda_{\mu_m} - \lambda_{\mu_{m-1}}} \sum_{i\in J_m} \frac{|x_i - \xi|}{i} = 0.$$

We note that a sequence $\{x_i\}$ satisfying (1.9) is called *strongly harmonically summable* to ξ. Clearly (1.9) implies (1.8), but the converse implication is not true in general.

One may define harmonic averages of higher order, too. For example, given a sequence $\{x_i\}$, its *harmonic averages* $\tau_k(2)$ *of second order* are defined by

$$\tau_k(2) := \frac{1}{\lambda_k(2)} \sum_{i=1}^{k} \frac{x_i}{i\lambda_i}, \quad \text{where} \quad \lambda_k(2) := \sum_{i=1}^{k} \frac{1}{i\lambda_i},\ k = 1,2,\ldots. \tag{1.10}$$

By (1.4) and (1.5), we have

$$\lim_{k\to\infty} \frac{\lambda_k(2)}{\ln\ln k} = 1. \tag{1.11}$$

Therefore, the sequences $\{\tau_k(2)\}$ and $\{\tau_k^*(2)\}$ are equiconvergent with the same limit (if one of them exists), where

$$\tau_k^*(2) := \frac{1}{\ln\ln k} \sum_{i=1}^{k} \frac{x_i}{i\ln(i+1)}, \quad k = 3,4,\ldots.$$

Again the usage of $\tau_k(2)$ instead of $\tau_k^*(2)$ is explained partly by the fact that the definition of $\tau_k^*(2)$ makes no sense for $k = 1$ and 2.

The next three theorems were indicated in [2] by the third author. For brevity in writing, we introduce the notation

$$K_0 := \{1,2,3,4\},\ K_m := \{2^{\mu_{m-1}}+1,\ 2^{\mu_{m-1}}+2,\ldots,2^{\mu_m}\},\ m = 1,2\ldots. \tag{1.12}$$

THEOREM G. *Necessary and sufficient condition for*

$$\lim_{m\to\infty} \tau_{2^{\mu_m}}(2) = \xi$$

is that

$$\lim_{m\to\infty} \frac{1}{\lambda_{2^{\mu_m}}(2) - \lambda_{2^{\mu_{m-1}}}(2)} \sum_{i\in K_m} \frac{x_i}{i\lambda_i} = \xi.$$

We note that by (1.5), (1.6) and (1.10), we have

$$\lim_{m\to\infty} \frac{\lambda_{2^{\mu_m}}(2)}{2^m} = \lim_{m\to\infty} \frac{\lambda_{2^{\mu_m}}(2) - \lambda_{2^{\mu_{m-1}}}(2)}{2^{m-1}} = \ln 2. \tag{1.13}$$

THEOREM H. *Necessary and sufficient condition for*

$$\lim_{m\to\infty} \tau_m(2) = \xi$$

is that

$$\lim_{m\to\infty} \frac{1}{\lambda_{2^{\mu_m}}(2) - \lambda_{2^{\mu_{m-1}}}(2)} \max_{k\in K_m} \left| \sum_{i=2^{\mu_{m-1}}+1}^{k} \frac{x_i - \xi}{i\lambda_i} \right| = 0.$$

THEOREM I. *Necessary and sufficient condition for*

$$\lim_{k\to\infty} \frac{1}{\lambda_k(2)} \sum_{i=1}^{k} \frac{|x_i - \xi|}{i\lambda_i} = 0$$

is that

$$\lim_{m\to\infty} \frac{1}{\lambda_{2^{\mu_m}}(2) - \lambda_{2^{\mu_{m-1}}}(2)} \sum_{i\in K_m} \frac{|x_i - \xi|}{i\lambda_i} = 0.$$

2. New results for double sequences

Given a double sequence $\{x_{ij} : i,j = 1,2,\ldots\}$ of complex numbers, its *harmonic averages* $\tau_{k\ell}$ are defined by

$$\tau_{k\ell} := \frac{1}{\lambda_k \lambda_\ell} \sum_{i=1}^{k} \sum_{j=1}^{\ell} \frac{x_{ij}}{ij}, \quad k,\ell = 1,2,\ldots;$$

where λ_k is defined in (1.4).

We recall that the double sequence $\{\tau_{k\ell}\}$ is said to *converge in Pringsheim's sense* to the finite limit ξ if $\tau_{k\ell}$ converges to ξ as both k and ℓ tend to infinity independently of one another. That is, for every $\varepsilon > 0$ there exists a natural number $k_0 = k_0(\varepsilon)$ such that

$$|\tau_{k\ell} - \xi| < \varepsilon \quad \text{whenever} \quad \min\{k,\ell\} > k_0.$$

By (1.5), the double sequences $\{\tau_{k\ell}\}$ and $\{\tau^*_{k\ell}\}$ are equiconvergent to the same limit (if one them exists), where

$$\tau^*_{k\ell} := \frac{1}{(\ln k)(\ln \ell)} \sum_{i=1}^{k} \sum_{j=1}^{\ell} \frac{x_{ij}}{ij}, \quad k,\ell = 2,3,\ldots.$$

The usage of $\tau_{k\ell}$ enjoys some technical advantege in the proofs of Theorems 1-3 below (see in Section 4) over $\tau^*_{k\ell}$, partly due to the fact that the definition of $\tau^*_{k\ell}$ makes no sense in case $\min\{k,\ell\} = 1$.

It is easy to see that a necessary condition for the convergence of $\{\tau_{k\ell}\}$ is that

$$\lim_{k,\ell \to \infty} \frac{x_{k\ell}}{k(\ln k)\ell(\ln \ell)} = 0.$$

This follows from the obvious equality

$$\frac{x_{k\ell}}{k\ell} = \lambda_k \lambda_\ell \tau_{k\ell} - \lambda_{k-1}\lambda_\ell \tau_{k-1,\ell} - \lambda_k \lambda_{\ell-1} \tau_{k,\ell-1} + \lambda_{k-1}\lambda_{\ell-1}\tau_{k-1,\ell-1}; \tag{2.1}$$

with the agreement that

$$\lambda_0 = \tau_{00} = \tau_{k0} = \tau_{0\ell} := 0 \quad \text{for} \quad k,\ell = 1,2,\ldots.$$

It is also known that if a bounded sequence $\{x_{ij}\}$ converges in Pringsheim's sense to a finite limit ξ, then the double sequence $\{\tau_{mn}\}$ of its harmonic averages is also bounded and converges to the same limit ξ. The restriction to bounded sequences is justified by the fact that convergence in Pringsheim's sense of a double sequence does not imply the boundedness of its terms in general.

In the sequel, we agree to use the term of *bounded convergence*, in symbols:

$$\text{b} - \lim_{i,j \to \infty} x_{ij} = \xi, \tag{2.2}$$

to indicate that the double sequence $\{x_{ij}\}$ converges to ξ in Pringsheim's sense and the terms x_{ij} are uniformly bounded, that is,

$$\sup_{i,j\geqslant 1} |x_{ij}| < \infty.$$

Our main goal is to extend Theorems D,E and F in Section 1 from single to double numerical sequences. These extensions are formulated in the next three theorems.

THEOREM 1. *Necessary and sufficient condition for*

$$\mathrm{b}-\lim_{m,n\to\infty} \tau_{\mu_m,\mu_n} = \xi \tag{2.3}$$

is that

$$\mathrm{b}-\lim_{m,n\to\infty} \frac{1}{(\lambda_{\mu_m}-\lambda_{\mu_{m-1}})(\lambda_{\mu_n}-\lambda_{\mu_{n-1}})} \sum_{i\in J_m}\sum_{j\in J_n} \frac{x_{ij}}{ij} = \xi, \tag{2.4}$$

where μ_m and J_m are defined in (1.6).

The ratio on the left-hand side of (2.4) may be called the *moving rectangular harmonic average* of the sequence $\{x_{ij}\}$.

THEOREM 2. *Necessary and sufficient condition for*

$$\mathrm{b}-\lim_{k,\ell\to\infty} \tau_{k\ell} = \xi \tag{2.5}$$

is that

$$\mathrm{b}-\lim_{m,n\to\infty} \frac{1}{(\lambda_{\mu_m}-\lambda_{\mu_{m-1}})(\lambda_{\mu_n}-\lambda_{\mu_{n-1}})} \times \tag{2.6}$$

$$\times \max_{k\in J_m,\ell\in J_n} \left| \sum_{i=\mu_{m-1}+1}^{k} \sum_{j=\mu_{n-1}+1}^{\ell} \frac{x_{ij}-\xi}{ij} \right| = 0.$$

We may call the expression on the left in (2.6) the *moving maximal rectangular harmonic average* of the sequence $\{x_{ij}-\xi\}$.

THEOREM 3. *Necessary and sufficient condition for*

$$\mathrm{b}-\lim_{k,\ell\to\infty} \frac{1}{\lambda_k\lambda_\ell} \sum_{i=1}^{k}\sum_{j=1}^{\ell} \frac{|x_{ij}-\xi|}{ij} = 0 \tag{2.7}$$

is that

$$\mathrm{b}-\lim_{m,n\to\infty} \frac{1}{(\lambda_{\mu_m}-\lambda_{\mu_{m-1}})(\lambda_{\mu_n}-\lambda_{\mu_{n-1}})} \sum_{i\in J_m}\sum_{j\in J_n} \frac{|x_{ij}-\xi|}{ij} = 0. \tag{2.8}$$

Observe that the maximal average is not involved in Theorem 3, in contrast to Theorem 2. Analogously to the notion of strong harmonic summability used in the case of a single sequence $\{x_i\}$ (defined in [2]), a double sequence $\{x_{ij}\}$ satisfying (2.7) is called *strongly harmonically summable* to ξ. Clearly, (2.7) implies (2.5), but the converse implication is not true in general.

REMARK 1. By (1.7), the denominator

$$(\lambda_{\mu_m} - \lambda_{\mu_{m-1}})(\lambda_{\mu_n} - \lambda_{\mu_{n-1}})$$

in (2.4), (2.6) and (2.8) can be equivalently replaced by 2^{m+n}. But in the proofs of Theorems 1-3, the usage of the notation in terms of λ 's and μ 's is more convenient to us. On the other hand, the following reformulation of Theorem 2 may be more appropriate for possible application in Probability Theory.

THEOREM 2*. *Necessary and sufficient condition for*

$$\mathrm{b}-\lim_{k,\ell\to\infty} \frac{1}{(\ln k)(\ln \ell)} \sum_{i=1}^{k} \sum_{j=1}^{\ell} \frac{x_{ij}}{ij} = \xi$$

is that

$$\mathrm{b}-\lim_{m,n\to\infty} \frac{1}{2^{m+n}} \max_{k\in J_m,\ell\in J_n} \Big| \sum_{i=\mu_{m-1}+1}^{k} \sum_{j=\mu_{n-1}+1}^{\ell} \frac{x_{ij}-\xi}{ij} \Big| = 0,$$

where J_m and μ_m are defined in (1.6).

3. Auxiliary results

Two results in Summability Theory will play crucial roles in the proofs of Theorems 1-3. To this effect, let

$$A = (a_{mk} : k = 0, 1, \ldots, m; m = 0, 1, \ldots) \tag{3.1}$$

be an infinite triangular matrix of real or complex numbers. With every sequence $s = \{s_k : k = 0, 1, \ldots\}$ of numbers we associate the sequence $t = \{t_m : m = 0, 1, \ldots\}$ given by

$$t_m = t_m(A, s) := \sum_{k=0}^{m} a_{mk} s_k, \quad m = 0, 1, \ldots.$$

The so-called summability matrix A is called *regular* if for every convergent sequence $s = \{s_k\}$, the sequence $t = \{t_m\}$ also converges to the same limit.

The following *Toeplitz theorem* characterizes regularity of summability matrices applied for single sequences.

LEMMA 1. (See, e.g., [5, pp. 74–75 and 168].) *The triangular summability matrix $A = (a_{mk})$ given in (3.1) is regular if and only if the following three conditions are satisfied:*

(i) $$\lim_{m\to\infty} a_{mk} = 0 \quad for \quad k = 0, 1, \ldots;$$

(ii) $$\sup_{m=0,1,\ldots} \sum_{k=0}^{m} |a_{mk}| < \infty,$$

(iii) $$\lim_{m\to\infty}\sum_{k=0}^{m} a_{mk} = 1.$$

Condition (iii) is not needed in the case when

$$\lim_{k\to\infty} s_k = 0.$$

Next, let

$$\mathscr{A} = (a_{k\ell}^{mn} : k = 0,1,\ldots,m; \ell = 0,1,\ldots,n; m,n = 0,1,\ldots) \tag{3.2}$$

be matrices of real or complex numbers for all $m,n = 0,1,\ldots$. With every double sequence $s = \{s_{k\ell} : k,\ell = 0,1,\ldots\}$ of numbers we associate the sequence $t = \{t^{mn} : m,n = 0,1,\ldots\}$ given by

$$t^{mn} = t^{mn}(\mathscr{A}, s) := \sum_{k=0}^{m}\sum_{\ell=0}^{n} a_{k\ell}^{mn} s_{k\ell}, \quad m,n = 0,1,\ldots.$$

The summability matrix $\mathscr{A} = (a_{k\ell}^{mn})$ is called *bounded-regular* if for every double sequence $\{s_{k\ell}\}$ with

$$\mathrm{b}-\lim_{k,\ell\to\infty} s_{k\ell} = \xi,$$

we have

$$\mathrm{b}-\lim_{m,n\to\infty} t^{mn}(\mathscr{A}, s) = \xi.$$

The following characterization of the bounded-regularity of summability matrices applied for double sequences is due to Robison [4].

LEMMA 2. *The triangular summability matrix $\mathscr{A} = (a_{k\ell}^{mn})$ given in (3.2) is bounded-regular if and only if the following four conditions are satisfied:*

(i) $$\lim_{m,n\to\infty}\sum_{k=0}^{m} |a_{k\ell}^{mn}| = 0 \quad for \quad \ell = 0,1,\ldots;$$

(ii) $$\lim_{m,n\to\infty}\sum_{\ell=0}^{n} |a_{k\ell}^{mn}| = 0 \quad for \quad k = 0,1,\ldots;$$

(iii) $$\sup_{m,n=0,1,\ldots}\sum_{k=0}^{m}\sum_{\ell=0}^{n} |a_{k\ell}^{mn}| < \infty,$$

(iv) $$\lim_{m,n\to\infty}\sum_{k=0}^{m}\sum_{\ell=0}^{n} a_{k\ell}^{mn} = 1.$$

Condition (iv) is not needed in the case when

$$\mathrm{b}-\lim_{k,\ell\to\infty} s_{k\ell} = 0.$$

We note that both Lemmas 1 and 2 are valid in the more general setting when the summability matrices $A=(a_{mk})$ and $\mathscr{A}=(a^{mn}_{k\ell})$ are not triangular. But in this paper, we do not need these more general formulations.

We also note that the limit relation

$$\lim_{m\to\infty}\frac{\lambda_{\mu_{m-1}}}{\lambda_{\mu_m}}=\frac{1}{2} \tag{3.3}$$

will be frequently used in Section 4, where λ_k and μ_m are defined in (1.4) and (1.6), respectively; and (3.3) is an immediate consequence of (1.5). In particular, it follows from (3.3) that

$$\lim_{m\to\infty}\frac{\lambda_{\mu_m}}{\lambda_{\mu_m}-\lambda_{\mu_{m-1}}}=2 \quad\text{and}\quad \lim_{m\to\infty}\frac{\lambda_{\mu_{m-1}}}{\lambda_{\mu_m}-\lambda_{\mu_{m-1}}}=1.$$

4. Proofs of Theorems 1–3

Proof of Theorem 1. Necessity. Assume (2.3). By this and (3.3), we obtain

$$\begin{aligned}
&\frac{1}{(\lambda_{\mu_m}-\lambda_{\mu_{m-1}})(\lambda_{\mu_n}-\lambda_{\mu_{n-1}})}\sum_{i\in J_m}\sum_{j\in J_n}\frac{x_{ij}}{ij}\\
&=\frac{1}{(\lambda_{\mu_m}-\lambda_{\mu_{m-1}})(\lambda_{\mu_n}-\lambda_{\mu_{n-1}})}(\lambda_{\mu_m}\lambda_{\mu_n}\tau_{\mu_m,\mu_n}-\lambda_{\mu_{m-1}}\lambda_{\mu_n}\tau_{\mu_{m-1},\mu_n}\\
&\qquad-\lambda_{\mu_m}\lambda_{\mu_{n-1}}\tau_{\mu_m,\mu_{n-1}}+\lambda_{\mu_{m-1}}\lambda_{\mu_{n-1}}\tau_{\mu_{m-1},\mu_{n-1}})\\
&\qquad\to 4\xi-2\xi-2\xi+\xi=\xi \quad\text{as}\quad m,n\to\infty.
\end{aligned}$$

This proves (2.4).

Sufficiency. Assume (2.4). By (3.3), we find that (2.4) is equivalent to the following bounded convergence:

$$\phi_{mn}:=\frac{1}{\lambda_{\mu_m}\lambda_{\mu_n}}\sum_{i\in J_m}\sum_{j\in J_n}\frac{x_{ij}-\xi}{ij}\to 0 \quad\text{as}\quad m,n\to\infty. \tag{4.1}$$

For all $m,n\geqslant 1$, we may write that

$$\begin{aligned}
\tau_{\mu_m,\mu_n}-\xi&=\frac{1}{\lambda_{\mu_m}\lambda_{\mu_n}}\sum_{i=1}^{\mu_m}\sum_{j=1}^{\mu_n}\frac{x_{ij}-\xi}{ij}\\
&=\frac{1}{\lambda_{\mu_m}\lambda_{\mu_n}}\sum_{k=0}^{m}\sum_{\ell=0}^{n}\sum_{i\in J_k}\sum_{j\in J_\ell}\frac{x_{ij}-\xi}{ij}\\
&=\sum_{k=0}^{m}\sum_{\ell=0}^{n}a^{mn}_{k\ell}\phi_{k\ell},\quad\text{where}\quad a^{mn}_{k\ell}:=\frac{\lambda_{\mu_k}\lambda_{\mu_\ell}}{\lambda_{\mu_m}\lambda_{\mu_n}}.
\end{aligned}$$

It is easy to check that in this case the summability matrix $\mathscr{A}=(a^{mn}_{k\ell})$ of form (3.2) satisfies the conditions (i)-(iii) of Lemma 2. Taking into account that now the limit in (4.1) equals 0, so we can conclude (2.3) to be proved. □

Proof of Theorem 2. We begin with the observation that the bounded convergence in (2.6) is equivalent to the following one:

$$\mathrm{b}-\lim_{m,n\to\infty}\frac{1}{\lambda_{\mu_m}\lambda_{\mu_n}}\max_{k\in J_m,\ell\in J_n}\Big|\sum_{i=\mu_{m-1}+1}^{k}\ \sum_{j=\mu_{n-1}+1}^{\ell}\frac{x_{ij}-\xi}{ij}\Big|=0, \tag{4.2}$$

again due to (3.3).

Necessity. Assume (2.5), by which we may apply the necessity part of Theorem 1 to obtain (2.4), or equivalently (4.2).

First, let $k\in J_m$ for some $m\geqslant 1$ and let $n\geqslant 1$ be arbitrary. We give a lower estimate as follows:

$$\begin{aligned}|\tau_{k,\mu_{n-1}}-\xi| &= \Big|\frac{1}{\lambda_k\lambda_{\mu_{n-1}}}\Big\{\sum_{i=1}^{\mu_{m-1}}\sum_{j=1}^{\mu_{n-1}}+\sum_{i=\mu_{m-1}+1}^{k}\ \sum_{j=1}^{\mu_{n-1}}\Big\}\frac{x_{ij}-\xi}{ij}\Big|\\ &\geqslant \frac{1}{\lambda_{\mu_m}\lambda_{\mu_{n-1}}}\Big(\Big|\sum_{i=\mu_{m-1}+1}^{k}\ \sum_{j=1}^{\mu_{n-1}}\frac{x_{ij}-\xi}{ij}\Big|-\Big|\sum_{i=1}^{\mu_{m-1}}\sum_{j=1}^{\mu_{n-1}}\frac{x_{ij}-\xi}{ij}\Big|\Big),\end{aligned}$$

whence we conclude that

$$\begin{aligned}&\max_{k\in J_m}|\tau_{k,\mu_{n-1}}-\xi| \qquad\qquad (4.3)\\ &\geqslant \frac{1}{\lambda_{\mu_m}\lambda_{\mu_{n-1}}}\max_{k\in J_m}\Big|\sum_{i=\mu_{m-1}+1}^{k}\ \sum_{j=1}^{\mu_{n-1}}\frac{x_{ij}-\xi}{ij}\Big|-|\tau_{\mu_{m-1},\mu_{n-1}}-\xi|.\end{aligned}$$

By (2.5), the left-hand side as well as the second term on the right-hand side of inequality (4.3) boundedly converge to 0 as $m,n\to\infty$. Therefore, the first term on the right-hand side of (4.3) must also converge boundedly to 0:

$$\mathrm{b}-\lim_{m,n\to\infty}\frac{1}{\lambda_{\mu_m}\lambda_{\mu_{n-1}}}\max_{k\in J_m}\Big|\sum_{i=\mu_{m-1}+1}^{k}\ \sum_{j=1}^{\mu_{n-1}}\frac{x_{ij}-\xi}{ij}\Big|=0. \tag{4.4}$$

Second, the symmetric counterpart of (4.4) gives

$$\mathrm{b}-\lim_{m,n\to\infty}\frac{1}{\lambda_{\mu_{m-1}}\lambda_{\mu_n}}\max_{\ell\in J_n}\Big|\sum_{i=1}^{\mu_{m-1}}\ \sum_{j=\mu_{n-1}+1}^{\ell}\frac{x_{ij}-\xi}{ij}\Big|=0. \tag{4.5}$$

Third, let $k\in J_m$ and $\ell\in J_n$ be arbitrary for some $m,n\geqslant 1$. We give a lower estimate as follows:

$$\begin{aligned}|\tau_{k\ell}-\xi| = \Big|\frac{1}{\lambda_k\lambda_\ell}\Big\{&\sum_{i=1}^{\mu_{m-1}}\sum_{j=1}^{\mu_{n-1}}+\sum_{i=\mu_{m-1}+1}^{k}\ \sum_{j=1}^{\mu_{n-1}} \qquad\qquad (4.6)\\ &+\sum_{i=1}^{\mu_{m-1}}\ \sum_{j=\mu_{n-1}+1}^{\ell}+\sum_{i=\mu_{m-1}+1}^{k}\ \sum_{j=\mu_{n-1}+1}^{\ell}\Big\}\frac{x_{ij}-\xi}{ij}\Big|\end{aligned}$$

$$\geqslant \frac{1}{\lambda_{\mu_m}\lambda_{\mu_n}}\Big(\Big|\sum_{i=\mu_{m-1}+1}^{k}\sum_{j=\mu_{n-1}+1}^{\ell}\frac{x_{ij}-\xi}{ij}\Big| - \Big|\sum_{i=1}^{\mu_{m-1}}\sum_{j=1}^{\mu_{n-1}}\frac{x_{ij}-\xi}{ij}\Big|$$

$$-\Big|\sum_{i=\mu_{m-1}+1}^{k}\sum_{j=1}^{\mu_{n-1}}\frac{x_{ij}-\xi}{ij}\Big| - \Big|\sum_{i=1}^{\mu_{m-1}}\sum_{j=\mu_{n-1}+1}^{\ell}\frac{x_{ij}-\xi}{ij}\Big|\Big).$$

Hence we conclude that

$$\max_{k\in J_m,\ell\in J_n}|\tau_{k\ell}-\xi| \tag{4.7}$$

$$\geqslant \frac{1}{\lambda_{\mu_m}\lambda_{\mu_n}}\max_{k\in J_m,\ell\in J_n}\Big|\sum_{i=\mu_{m-1}+1}^{k}\sum_{j=\mu_{n-1}+1}^{\ell}\frac{x_{ij}-\xi}{ij}\Big|$$

$$-|\tau_{\mu_{m-1},\nu_{n-1}}-\xi| - \frac{1}{\lambda_{\mu_m}\lambda_{\mu_{n-1}}}\max_{k\in J_m}\Big|\sum_{i=\mu_{m-1}+1}^{k}\sum_{j=1}^{\mu_{n-1}}\frac{x_{ij}-\xi}{ij}\Big|$$

$$-\frac{1}{\lambda_{\mu_{m-1}}\lambda_{\mu_n}}\max_{\ell\in J_n}\Big|\sum_{i=1}^{\mu_{m-1}}\sum_{j=\mu_{n-1}+1}^{\ell}\frac{x_{ij}-\xi}{ij}\Big|.$$

By (2.5), the left-hand side as well as the second term on the right-hand side of inequality (4.7) boundedly converge to 0 as $m,n\to\infty$. Furthermore, by (4.4) and (4.5), the third and fourth terms on the right-hand side of (4.7) also boundedly converge to 0 as $m,n\to\infty$. Therefore, the first term on the right-hand side of (4.7) must also boundedly converge to 0 as $m,n\to\infty$. This proves (4.2), which is equivalent to (2.6).

Sufficiency. Assume (2.6). Since condition (2.6) implies (2.4), applying the sufficiency part of Theorem 1 gives (2.3).

Similarly to the equivalence of (2.4) and (4.1), this time (2.6) is equivalent to the following bounded convergence:

$$\psi_{mn} := \frac{1}{\lambda_{\mu_{m-1}}\lambda_{\mu_{n-1}}}\max_{k\in J_m,\ell\in J_n}\Big|\sum_{i=\mu_{m-1}+1}^{k}\sum_{j=\mu_{n-1}+1}^{\ell}\frac{x_{ij}-\xi}{ij}\Big|\to 0 \quad \text{as} \quad m,n\to\infty. \tag{4.8}$$

After these preliminaries, let $k\in J_m$ and $\ell\in J_n$ for some $m,n\geqslant 1$. We give an upper estimate (cf. (4.6) and (4.7)) as follows:

$$|\tau_{k\ell}-\xi| \leqslant \frac{1}{\lambda_{\mu_{m-1}}\lambda_{\mu_{n-1}}}\Big(\Big|\sum_{i=1}^{\mu_{m-1}}\sum_{j=1}^{\mu_{n-1}}\frac{x_{ij}-\xi}{ij}\Big| + \Big|\sum_{i=\mu_{m-1}+1}^{k}\sum_{j=1}^{\mu_{n-1}}\frac{x_{ij}-\xi}{ij}\Big|$$

$$+\Big|\sum_{i=1}^{\mu_{m-1}}\sum_{j=\mu_{n-1}+1}^{\ell}\frac{x_{ij}-\xi}{ij}\Big| + \Big|\sum_{i=\mu_{m-1}+1}^{k}\sum_{j=\mu_{n-1}+1}^{\ell}\frac{x_{ij}-\xi}{ij}\Big|\Big).$$

Hence we conclude that

$$\begin{aligned}
&\max_{k\in J_m,\ell\in J_n} |\tau_{k\ell}-\xi| \qquad (4.9)\\
&\leqslant |\tau_{\mu_{m-1},\mu_{n-1}}-\xi| + \frac{1}{\lambda_{\mu_{m-1}}\lambda_{\mu_{n-1}}}\Big\{\max_{k\in J_m}\Big|\sum_{i=\mu_{m-1}+1}^{k}\sum_{j=1}^{\mu_{n-1}}\frac{x_{ij}-\xi}{ij}\Big|\\
&\quad + \max_{\ell\in J_n}\Big|\sum_{i=1}^{\mu_{m-1}}\sum_{j=\mu_{n-1}+1}^{\ell}\frac{x_{ij}-\xi}{ij}\Big| + \max_{k\in J_m,\ell\in J_n}\Big|\sum_{i=\mu_{m-1}+1}^{k}\sum_{j=\mu_{n-1}+1}^{\ell}\frac{x_{ij}-\xi}{ij}\Big|\Big\}\\
&=: |\tau_{\mu_{m-1},\mu_{n-1}}-\xi| + B_{mn}+C_{mn}+D_{mn}, \quad \text{say.}
\end{aligned}$$

As we have noted above, the first term on the right-hand side of (4.9) boundedly converges to 0 as $m,n\to\infty$. In the case of the second term on the right, we give an upper estimate as follows:

$$\begin{aligned}
B_{mn} &:= \frac{1}{\lambda_{\mu_{m-1}}\lambda_{\mu_{n-1}}}\max_{k\in J_m}\Big|\sum_{i=\mu_{m-1}+1}^{k}\sum_{j=1}^{\mu_{n-1}}\frac{x_{ij}-\xi}{ij}\Big|\\
&= \frac{1}{\lambda_{\mu_{m-1}}\lambda_{\mu_{n-1}}}\max_{k\in J_m}\Big|\sum_{i=\mu_{m-1}+1}^{k}\sum_{\ell=0}^{n-1}\sum_{j=\mu_{\ell-1}+1}^{\mu_\ell}\frac{x_{ij}-\xi}{ij}\Big|\\
&\leqslant \sum_{\ell=0}^{n-1}\frac{1}{\lambda_{\mu_{m-1}}\lambda_{\mu_{n-1}}}\max_{k\in J_m}\Big|\sum_{i=\mu_{m-1}+1}^{k}\sum_{j\in J_{\ell-1}}\frac{x_{ij}-\xi}{ij}\Big|\\
&\leqslant \sum_{\ell=0}^{n-1}\frac{\lambda_{\mu_{\ell-1}}}{\lambda_{\mu_{n-1}}}\psi_{m\ell},
\end{aligned}$$

where $\psi_{m\ell}$ is defined in (4.8). Due to (3.3), we may apply Lemma 1 to conclude that

$$\mathrm{b}-\lim_{m,n\to\infty} B_{mn} = 0. \qquad (4.10)$$

Analogously, in the case of the third term on the right-hand side of (4.9) we obtain that

$$\begin{aligned}
C_{mn} &:= \frac{1}{\lambda_{\mu_{m-1}}\lambda_{\mu_{n-1}}}\max_{\ell\in J_n}\Big|\sum_{i=1}^{\mu_{m-1}}\sum_{j=\mu_{n-1}+1}^{\ell}\frac{x_{ij}-\xi}{ij}\Big|\\
&\leqslant \sum_{k=0}^{m-1}\frac{\lambda_{\mu_{k-1}}}{\lambda_{\mu_{m-1}}}\psi_{kn},
\end{aligned}$$

and again by Lemma 1, we have

$$\mathrm{b}-\lim_{m,n\to\infty} C_{mn} = 0. \qquad (4.11)$$

Finally, in the case of the fourth term on the right-hand side of (4.9), we observe that

$$D_{mn} := \frac{1}{\lambda_{\mu_{m-1}}\lambda_{\mu_{n-1}}}\max_{k\in J_m,\ell\in J_n}\Big|\sum_{i=\mu_{m-1}+1}^{k}\sum_{j=\mu_{n-1}+1}^{\ell}\frac{x_{ij}-\xi}{ij}\Big|$$

is identical with ψ_{mn} defined in (4.8), so we have

$$\mathrm{b}-\lim_{m,n\to\infty} D_{mn}=0. \tag{4.12}$$

Putting together (4.9), (2.3), (4.10)-(4.12) yields (2.5) to be proved. □

Proof of Theorem 3. We begin with the observation that condition (2.8) is equivalent to the following bounded convergence:

$$\chi_{mn} := \frac{1}{\lambda_{\mu_m}\lambda_{\mu_n}} \sum_{i\in J_m}\sum_{j\in J_n} \frac{|x_{ij}-\xi|}{ij} \to 0 \quad \text{as} \quad m,n\to\infty. \tag{4.13}$$

This equivalence is due again to (3.3).

Furthermore, we claim that (2.7) is equivalent to the following bounded convergence:

$$\mathrm{b}-\lim_{m,n\to\infty} \frac{1}{\lambda_{\mu_m}\lambda_{\mu_n}} \sum_{i=1}^{\mu_m}\sum_{j=1}^{\mu_n} \frac{|x_{ij}-\xi|}{ij} = 0. \tag{4.14}$$

Indeed, this claim immediately follows from the following pair of inequalities: for any $k\in J_m$ and $\ell\in J_n$, we have

$$\begin{aligned}\frac{1}{\lambda_{\mu_m}\lambda_{\mu_n}} \sum_{i=1}^{\mu_{m-1}}\sum_{j=1}^{\mu_{n-1}} \frac{|x_{ij}-\xi|}{ij} &\leqslant \frac{1}{\lambda_k\lambda_\ell}\sum_{i=1}^{k}\sum_{j=1}^{\ell} \frac{|x_{ij}-\xi|}{ij} \\ &\leqslant \frac{1}{\lambda_{\mu_{m-1}}\lambda_{\mu_{n-1}}} \sum_{i=1}^{\mu_m}\sum_{j=1}^{\mu_n} \frac{|x_{ij}-\xi|}{ij}, \quad m,n=1,2,\ldots.\end{aligned}$$

Now, it remains to take into account (3.3) and this pair of inequalities justifies (4.14).

Necessity. Assume (2.7), then in particular we have (4.14). This latter one clearly implies (4.13), which in turn is equivalent to (2.8) to be proved.

Sufficiency. Assume (2.8), which is equivalent to (4.13). Clearly, we have

$$\begin{aligned}\frac{1}{\lambda_{\mu_m}\lambda_{\mu_n}} \sum_{i=1}^{\mu_m}\sum_{j=1}^{\mu_n} \frac{|x_{ij}-\xi|}{ij} &= \frac{1}{\lambda_{\mu_m}\lambda_{\mu_n}} \sum_{k=0}^{m}\sum_{\ell=0}^{n}\sum_{i\in J_k}\sum_{j\in J_\ell} \frac{|\xi_{ij}-\xi|}{ij} \\ &= \sum_{k=0}^{m}\sum_{\ell=0}^{n} \frac{\lambda_{\mu_k}\lambda_{\mu_\ell}}{\lambda_{\mu_m}\lambda_{\mu_n}} \chi_{k\ell},\end{aligned}$$

where $\chi_{k\ell}$ is defined in (4.13). It remains to apply Lemma 2 to conclude (4.14), which is equivalent to (2.7) to be proved. □

5. Concluding remarks

REMARK 2. We recall that the arithmetic averages $\sigma_{k\ell}$ of the double sequence $\{x_{ij} : i,j=1,2,\ldots\}$ of complex numbers are defined by

$$\sigma_{k\ell} := \frac{1}{k\ell}\sum_{i=1}^{k}\sum_{j=1}^{\ell} x_{ij}, \quad k,\ell=1,2,\ldots.$$

It is well known that (2.2) (the bounded convergence of $\{x_{ij}\}$) is a sufficient condition for the bounded convergence of $\{\sigma_{k\ell}\}$ to the same limit, in symbols:

$$\text{b}-\lim_{k,\ell\to\infty}\sigma_{k\ell}=\xi. \tag{5.1}$$

On the other hand, a necessary condition for the bounded convergence of $\{\sigma_{k\ell}\}$ is that

$$\lim_{k,\ell\to\infty}\frac{x_{k\ell}}{k\ell}=0,$$

which follows from the equality

$$x_{k\ell}=k\ell\sigma_{k\ell}-(k-1)\ell\sigma_{k-1,\ell}-k(\ell-1)\sigma_{k,\ell-1}+(k-1)(\ell-1)\sigma_{k-1,\ell-1}, \tag{5.2}$$

with the agreement that

$$\sigma_{00}=\sigma_{i0}=\sigma_{0j}=0 \quad \text{for} \quad i,j=1,2,\ldots.$$

Making use of the arguments analogous to those in the proofs of Theorems 1-3, the following three theorems can be easily proved.

THEOREM A*. *Necessary and sufficient condition for*

$$\text{b}-\lim_{m,n\to\infty}\sigma_{2^m,2^n}=\xi$$

is that

$$\text{b}-\lim_{m,n\to\infty}\frac{1}{2^{m-1}2^{n-1}}\sum_{i\in I_m}\sum_{j\in I_n}x_{ij}=\xi,$$

where I_m *is defined in (1.1).*

THEOREM B*. *Necessary and sufficient condition for (5.1) is that*

$$\text{b}-\lim_{m,n\to\infty}\frac{1}{2^{m-1}2^{n-1}}\max_{k\in I_m,\ell\in I_n}\left|\sum_{i=2^{m-1}+1}^{k}\sum_{j=2^{n-1}+1}^{\ell}(x_i-\xi)\right|=0.$$

THEOREM C*. *Necessary and sufficient condition for*

$$\text{b}-\lim_{k,\ell\to\infty}\frac{1}{k\ell}\sum_{i=1}^{k}\sum_{j=1}^{\ell}|x_{ij}-\xi|=0$$

is that

$$\text{b}-\lim_{m,n\to\infty}\frac{1}{2^{m-1}2^{n-1}}\sum_{i\in I_m}\sum_{j\in I_n}|x_{ij}-\xi|=0.$$

REMARK 3. It is easy to check that (2.2) implies (2.5). More generally, we claim that even (5.1) implies (2.5). Indeed by (5.2), we may write that

$$\tau_{k\ell}=\frac{1}{\lambda_k\lambda_\ell}\sum_{i=1}^{k}\sum_{j=1}^{\ell}\{\sigma_{ij}-\sigma_{i-1,j}-\sigma_{i,j-1}+\sigma_{i-1,j-1} \tag{5.3}$$

$$+\frac{1}{i}(\sigma_{i-1,j}-\sigma_{i-1,j-1})+\frac{1}{j}(\sigma_{i,j-1}-\sigma_{i-1,j-1})+\frac{1}{ij}\sigma_{i-1,j-1}\}.$$

The next three limit relations are consequences of the uniform boundedness of the double sequence $\{\sigma_{k\ell}\}$:

$$\mathrm{b}-\lim_{k,\ell\to\infty}\frac{1}{\lambda_k\lambda_\ell}\sum_{i=1}^{k}\sum_{j=1}^{\ell}(\sigma_{ij}-\sigma_{i-1,j}-\sigma_{i,j-1}+\sigma_{i-1,j-1})$$

$$=\mathrm{b}-\lim_{k,\ell\to\infty}\frac{\sigma_{k\ell}}{\lambda_k\lambda_\ell}=0,$$

$$\mathrm{b}-\lim_{k,\ell\to\infty}\frac{1}{\lambda_k\lambda_\ell}\sum_{i=1}^{k}\sum_{j=1}^{\ell}\frac{1}{i}(\sigma_{i-1,j}-\sigma_{i-1,j-1})$$

$$=\mathrm{b}-\lim_{k,\ell\to\infty}\frac{1}{\lambda_k\lambda_\ell}\sum_{j=1}^{\ell}\frac{\sigma_{k,j-1}}{j}=0,$$

$$\mathrm{b}-\lim_{k,\ell\to\infty}\frac{1}{\lambda_k\lambda_\ell}\sum_{i=1}^{k}\sum_{j=1}^{\ell}\frac{1}{j}(\sigma_{i,j-1}-\sigma_{i-1,j-1})$$

$$=\mathrm{b}-\lim_{k,\ell\to\infty}\frac{1}{\lambda_k\lambda_\ell}\sum_{i=1}^{k}\frac{\sigma_{i-1,\ell}}{i}=0.$$

Making use of the implication (2.2) $\Rightarrow$ (2.5) with $\sigma_{i-1,j-1}$ in place of x_{ij} gives

$$\mathrm{b}-\lim_{k,\ell\to\infty}\frac{1}{\lambda_k\lambda_\ell}\sum_{i=1}^{k}\sum_{j=1}^{\ell}\frac{\sigma_{i-1,j-1}}{ij}=\xi.$$

Putting together (5.3) and the last four limit relations yields the implication (5.1) $\Rightarrow$ (2.5), as we have claimed above.

The converse implication, that is, (2.5) $\Rightarrow$ (5.1) is not true in general. To justify this claim, we define the double sequence $\{x_{k\ell}\}$ by means of its harmonic means $\{\tau_{k\ell}\}$ as follows: set

$$\tau_{k\ell}:=\begin{cases}(pq)^{-1/2} & \text{if}\quad k=2^p\quad\text{and}\quad \ell=2^q\quad\text{for some}\quad p,q=1,2,\ldots;\\ 0 & \text{otherwise for}\quad k,\ell=1,2,\ldots;\end{cases}$$

and define the sequence $\{x_{k\ell}\}$ by

$$x_{k\ell}:=k\ell(\lambda_k\lambda_\ell\tau_{k\ell}-\lambda_{k-1}\lambda_\ell\tau_{k-1,\ell}-\lambda_k\lambda_{\ell-1}\tau_{k,\ell-1}+\lambda_{k-1}\lambda_{\ell-1}\tau_{k-1,\ell-1})$$

(cf. (2.1)) with the agreement that

$$\tau_{00}=\tau_{k0}=\tau_{0\ell}:=0\quad\text{for}\quad k,\ell=1,2,\ldots.$$

According to [2, Example 1 on p. 377], we have

$$\sigma_{2^p,2^q}\to\infty\quad\text{as}\quad p,q\to\infty,$$

whereas

$$\mathrm{b}-\lim_{k,\ell\to\infty}\tau_{k\ell}=0.$$

REMARK 4. One may define harmonic averages of higher order for double sequences $\{x_{ij}\}$, too. For example, given $\{x_{ij}\}$, its *harmonic averages* $\tau_{k\ell}(2)$ *of second order* are defined by

$$\tau_{k\ell}(2):=\frac{1}{\lambda_k(2)\lambda_\ell(2)}\sum_{i=1}^{k}\sum_{j=1}^{\ell}\frac{x_{ij}}{i\lambda_i j\lambda_j},\quad k,\ell=1,2,\ldots,$$

where λ_i and $\lambda_k(2)$ are defined in (1.4) and (1.10), respectively.

By (1.11), the double sequences $\{\tau_{k\ell}(2)\}$ and $\{\tau^*_{k\ell}(2)\}$ are equiconvergent with the same limit (if one of them exists), where

$$\tau^*_{k\ell}(2):=\frac{1}{(\ln\ln k)(\ln\ln\ell)}\sum_{i=1}^{k}\sum_{j=1}^{\ell}\frac{x_{ij}}{i(\ln(i+1))j\ln(j+1)},\quad k,\ell=3,4,\ldots.$$

The following Theorems 4-6 are the extensions of Theorems G, H and I in Section 1 from single to double numerical sequences. Their proofs can be carried out along lines analogous to the proofs of Theorems 1-3 in Section 4.

THEOREM 4. *Necessary and sufficient condition for*

$$\mathrm{b}-\lim_{m,n\to\infty}\tau_{2^{\mu_m},2^{\mu_n}}(2)=\xi$$

is that

$$\mathrm{b}-\lim_{m,n\to\infty}\frac{1}{(\lambda_{2^{\mu_m}}(2)-\lambda_{2^{\mu_{m-1}}}(2))(\lambda_{2^{\mu_n}}(2)-\lambda_{2^{\mu_{n-1}}}(2))}\times \tag{5.4}$$

$$\times\sum_{i\in K_m}\sum_{j\in K_n}\frac{x_{ij}}{i\lambda_i j\lambda_j}=\xi,$$

where λ_i, μ_m, $\lambda_k(2)$ and K_m are defined in (1.4), (1.6), (1.10) and (1.12), respectively.

THEOREM 5. *Necessary and sufficient condition for*

$$\mathrm{b}-\lim_{m,n\to\infty}\tau_{mn}(2)=\xi$$

is that

$$\mathrm{b}-\lim_{m,n\to\infty}\frac{1}{(\lambda_{2^{\mu_m}}(2)-\lambda_{2^{\mu_{m-1}}}(2))(\lambda_{2^{\mu_n}}(2)-\lambda_{2^{\mu_{n-1}}}(2))}\times \tag{5.5}$$

$$\times\max_{k\in K_m,\ell\in K_n}\left|\sum_{i=2^{\mu_{m-1}}+1}^{k}\sum_{j=2^{\mu_{n-1}}+1}^{\ell}\frac{x_{ij}-\xi}{i\lambda_i j\lambda_j}\right|=0.$$

THEOREM 6. *Necessary and sufficient condition for*

$$\mathrm{b}-\lim_{m,n\to\infty}\frac{1}{\lambda_k(2)\lambda_\ell(2)}\sum_{i=1}^{k}\sum_{j=1}^{\ell}\frac{|x_{ij}-\xi|}{i\lambda_i j\lambda_j}=0$$

is that

$$\mathrm{b}-\lim_{m,n\to\infty}\frac{1}{(\lambda_{2^{\mu_m}}(2)-\lambda_{2^{\mu_{m-1}}}(2))(\lambda_{2^{\mu_n}}(2)-\lambda_{2^{\mu_{n-1}}}(2))}\times \tag{5.6}$$
$$\times\sum_{i\in K_m}\sum_{j\in K_n}\frac{|x_{ij}-\xi|}{i\lambda_i j\lambda_j}=0.$$

REMARK 5. By (1.13), the denominator

$$(\lambda_{2^{\mu_m}}(2)-\lambda_{2^{\mu_{m-1}}}(2))(\lambda_{2^{\mu_n}}(2)-\lambda_{2^{\mu_{n-1}}}(2))$$

in (5.4)–(5.6) can be equivalently replaced by 2^{m+n}. Thus, the following reformulation of Theorem 5 may be more appropriate for possible applications.

THEOREM 5*. *Necessary and sufficient condition for*

$$\mathrm{b}-\lim_{k,\ell\to\infty}\frac{1}{(\ln\ln k)(\ln\ln\ell)}\sum_{i=1}^{k}\sum_{j=1}^{\ell}\frac{x_{ij}}{i(\ln(i+1))j\ln(j+1)}=\xi$$

is that

$$\mathrm{b}-\lim_{m,n\to\infty}\frac{1}{2^{m+n}}\max_{k\in K_m,\ell\in K_n}\Big|\sum_{i=2^{\mu_{m-1}}+1}^{k}\sum_{j=2^{\mu_{n-1}}+1}\frac{x_{ij}-\xi}{i(\ln(i+1))j\ln(j+1)}\Big|=0,$$

where K_m and μ_m are defined in (1.12) and (1.6), respectively.

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