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ON SOME GRONWALL TYPE INEQUALITIES INVOLVING ITERATED INTEGRALS

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Abstract. In this paper we consider simple inequalities involving iterated integrals in the inequality (1.1) for functions when the function u in the both side of the inequality (1.1) are replaced by the function $w(u)$ and $\varphi(u)$ for some functions w,φ and provide some retarded integral inequalities involving iterated integrals. Some applications are also given to illustrate the usefulness of our results.

1. Introduction

Let $u:[\alpha,\alpha+h]\to R$ be a continuous real-valued function satisfying the inequality

$$0\leqslant u(t)\leqslant\int_\alpha^t[a+bu(s)]\,ds,\quad \forall t\in[\alpha,\alpha+h],$$

where a,b are nonnegative constants. Then $u(t)\leqslant ahe^{bh}$ for all $t\in[\alpha,\alpha+h]$. This result was proved by Gronwall [8] in the year 1919, and is the prototype for the study of several integral inequalities of Volterra type, and also for obtaining explicit bounds of the unknown function. Among the several publications on this subject, the paper of Bellman [2] is very well known:

Let $x(t)$ and $k(t)$ be real valued nonnegative continuous functions for $t\geqslant\alpha$. If a is a constant, $a\geqslant 0$, and

$$x(t)\leqslant a+\int_\alpha^t k(s)x(s)\,ds,\quad \forall t\geqslant\alpha,$$

then

$$x(t)\leqslant a\exp\left(\int_\alpha^t k(s)\,ds\right),\quad \forall t\geqslant\alpha.$$

It is clear that Bellman's result contains that of Gronwall. This is the reason why inequalities of this type were called "Gronwall-Bellman inequalities" or "Inequalities of

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Gronwall type". The Gronwall type integral inequalities provide a necessary tool for the study of the theory of differential equations, integral equations and inequalities of various types (see Gronwall [8] and Guiliano [9]). Some applications of this result to the study of stability of the solution of linear and nonlinear differential equations may be found in Bellman [1, 2, 13]. Some applications to existence and uniqueness theory of differential equations may be found in Nemyckii-Stepanov [12], Bihari [3] and Langenhop [10]. During the past few years several authors [4–7, 11, 14–18] (see references below and some of the references cited therein) have established several Gronwall type integral inequalities in two or more independent real variables. Of course, such results have application in the theory of some differential and integral equations.

Ráb proved the following interesting integral inequality, which appear in [1, p. 100]:

THEOREM 1.1. *Let $u(t), a(t)$ and $b(t)$ be nonnegative continuous functions in $J=[\alpha,\beta]$, and suppose that*

$$u(t) \leqslant a(t)+b(t)\left[\int_\alpha^t k_1(t,t_1)u(t_1)\,dt_1+\cdots + \int_\alpha^t \left(\int_\alpha^{t_1}\cdots\left(\int_\alpha^{t_{n-1}} k_n(t,t_1,\ldots,t_n)u(t_n)\,dt_n\right)\ldots\right)dt_1\right] \tag{1.1}$$

for all $t\in J$, where $k_i(t,t_1,\cdots,t_i)$ are nonnegative continuous functions in J_{i+1} for all $i=1,2,\cdots,n$. Suppose the partial derivatives $\frac{\partial k_i}{\partial t}(t,t_1,\ldots,t_i)$ exist and are nonnegative and continuous in J_{i+1} for all $i=1,2,\cdots,n$. Then, for all $t\in J$,

$$u(t)\leqslant a(t)+b(t)\int_\alpha^t (R[a]+Q[a])(s)\exp\left(\int_s^t (R[b]+Q[b])(\tau)\,d\tau\right)ds,$$

$$R[w](t) = k_1(t,t)w(t)+\int_\alpha^t k_2(t,t,t_2)w(t_2)dt_2+ + \sum_{i=3}^n \int_\alpha^t\left(\int_\alpha^{t_2}\cdots\left(\int_\alpha^{t_{i-1}} k_i(t,t,t_2,\ldots,t_i)w(t_i)\,dt_i\right)\cdots\right)dt_2,$$

$$Q[w](t) = \int_\alpha^t \frac{\partial k_1}{\partial t}(t,t_1)w(t_1)\,dt_1+ + \sum_{i=2}^n \int_\alpha^t\left(\int_\alpha^{t_1}\cdots\left(\int_\alpha^{t_{i-1}} \frac{\partial k_i}{\partial t}(t,t_1,\ldots,t_i)w(t_i)\,dt_i\right)\cdots\right)dt_1$$

for each continuous function $w(t)$ in J.

In this paper, we consider simple inequalities involving iterated integrals in the inequality (1.1) for functions when the function u in the both side of the inequality

(1.1) are replaced by the function $w(u)$ and $\varphi(u)$ for some functions w,φ. We also provide some integral inequalities involving iterated integrals and some applications for the main results.

2. Main results

In this section, we state and prove some new nonlinear integral inequalities involving iterated integrals. Throughout the paper, all the functions which appear in the inequalities are assumed to be real-valued.

We shall introduce some notation, R denotes the set of real numbers and $R_+ = [0,\infty)$ be the given subset of R. Let $J = [\alpha,\beta], \alpha < \beta$ and set $J_i = \{(t_1,t_2,\cdots,t_i) \in R^i : \alpha \leqslant t_i \leqslant \cdots \leqslant t_1 \leqslant \beta\}$ for all $i = 1,2,\cdots,n$. Denote by $C^i(M,N)$ the class of all i-times continuously differentiable functions defined on set M to the set N for all $i = 1,2,\cdots$ and $C^0(M,N) = C(M,N)$. Given a continuous functions $a,b : J \to R_+$, we write

$$\hat{a}(t) = \max\{a(s) : \alpha \leqslant s \leqslant t\}, \quad \hat{b}(t) = \max\{b(s) : \alpha \leqslant s \leqslant t\}, \quad \forall t \in J. \tag{2.1}$$

THEOREM 2.1. *Let $u(t), a(t)$, $b(t)$ are nonnegative continuous functions for all $t \in J$, and $w(u)$ be a nondecreasing continuous function for all $u \in R_+$ with $w(u) > 0$ for all $u > 0$. Let $\varphi \in C(R_+,R_+)$ be an increasing function with $\varphi(\infty) = \infty$.*

(1) *Let $\phi \in C^1(J,J)$ be increasing with $\phi(t) \leqslant t$ on J. Suppose that*

$$\begin{aligned}\varphi(u(t)) \leqslant a(t) + b(t)\Bigg[&\int_{\phi(\alpha)}^{\phi(t)} k_1(t,t_1)w(u(t_1))\,dt_1 + \cdots \\ &+ \int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{\phi(t_1)} \cdots \left(\int_{\phi(\alpha)}^{\phi(t_{n-1})} k_n(t,t_1,\cdots,t_n)w(u(t_n))\,dt_n\right)\cdots\right)dt_1\Bigg]\end{aligned} \tag{2.2}$$

for all $t \in J$, where $k_i(t,t_1,\cdots,t_i)$ are nonnegative, continuous functions in J_{i+1} for all $i = 1,2,\cdots,n$. Suppose that the partial derivative $\frac{\partial k_i}{\partial t}(t,t_1,\cdots,t_i)$ exists and are nonnegative, continuous in J_{i+1} for all $i = 1,2,\cdots,n$. Then

$$u(t) \leqslant \varphi^{-1}\left\{G^{-1}\left[G(\hat{a}(t)) + \hat{b}(t)\left(\int_{\phi(\alpha)}^{\phi(t)} R[1](\phi^{-1}(\sigma),\sigma)\,d\sigma + \int_\alpha^t Q[1](\sigma)\,d\sigma\right)\right]\right\} \tag{2.3}$$

for all $t \in [\alpha,T]$, where $\hat{a}(t)$ and $\hat{b}(t)$ are defined in (2.1), $T \in J$ is chosen so that

$$G(\hat{a}(t)) + \hat{b}(t)\left(\int_{\phi(\alpha)}^{\phi(t)} R[1](\phi^{-1}(\sigma),\sigma)\,d\sigma + \int_\alpha^t Q[1](\sigma)\,d\sigma\right) \in Dom(G^{-1}),$$

the function G is defined by

$$G(r) = \int_{r_0}^{r} \frac{ds}{w(\varphi^{-1}(s))}, \quad \forall r \geqslant r_0 > 0, \tag{2.4}$$

G^{-1} denotes the inverse function of G, and

$$R[x](t,s) = k_1(t,s)x(s) + \int_{\phi(\alpha)}^{\phi(s)} k_2(t,s,t_2)x(t_2)dt_2 + \sum_{i=3}^{n} \int_{\phi(\alpha)}^{\phi(s)} \left(\int_{\phi(\alpha)}^{\phi(t_2)} \cdots \left(\int_{\phi(\alpha)}^{\phi(t_{i-1})} k_i(t,s,t_2,\cdots,t_i)x(t_i)\,dt_i \right) \cdots \right) dt_2, \tag{2.5}$$

$$Q[x](t) = \int_{\phi(\alpha)}^{\phi(t)} \frac{\partial k_1}{\partial t}(t,t_1)x(t_1)\,dt_1 + \sum_{i=2}^{n} \int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{\phi(t_1)} \cdots \left(\int_{\phi(\alpha)}^{\phi(t_{i-1})} \frac{\partial k_i}{\partial t}(t,t_1,\ldots,t_i)x(t_i)\,dt_i \right) \cdots \right) dt_1 \tag{2.6}$$

for all $x(t) \in C(J,J)$ and $t,s \in J$.

(2) *Let $\phi \in C^1(J,J)$ be nondecreasing with $\phi(t) \leqslant t$ on J. Suppose that*

$$\varphi(u(t)) \leqslant a(t) + b(t)\left[\int_{\phi(\alpha)}^{\phi(t)} k_1(t,t_1)w(u(t_1))\,dt_1 + \cdots + \int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{\phi(t_1)} \cdots \left(\int_{\phi(\alpha)}^{\phi(t_{n-1})} k_n(t,t_1,\cdots,t_n)w(u(t_n))\,dt_n \right) \cdots \right) dt_1 \right] \tag{2.7}$$

for all $t \in J$, where $k_i(t,t_1,\cdots,t_i)$ are nonnegative, continuous functions in J_{i+1} for all $i = 1,2,\cdots,n$, which are nondecreasing in $t \in J$ for all fixed $(t_1,\cdots,t_i) \in J_i$ for all $i = 1,2,\cdots,n$. Then we have

$$u(t) \leqslant \varphi^{-1}\left[G^{-1}\left(G(\hat{a}(t)) + \hat{b}(t) \int_{\phi(\alpha)}^{\phi(t)} R[1](t,\sigma)\,d\sigma \right) \right] \tag{2.8}$$

for all $t \in [\alpha,T_1]$, where $\hat{a}(t)$ and $\hat{b}(t)$ are defined in (2.1), $T_1 \in J$ is chosen so that

$$G(\hat{a}(t)) + \hat{b}(t) \int_{\phi(\alpha)}^{\phi(t)} R[1](t,\sigma)\,d\sigma \in Dom(G^{-1}),$$

the functions G is defined in (2.4), G^{-1} is the inverse function of G, and $R[x](t,s)$ is defined in (2.5).

Proof. (1) First, we note that $R[w]$ and $Q[w]$ are linear functional,

$$R[w_1] \leqslant R[w_2], \quad Q[w_1] \leqslant Q[w_2]$$

if $w_1(t) \leqslant w_2(t)$, for all $t \in J$, and

$$R[w_1w_2] \leqslant R[w_1]w_2, \quad Q[w_1w_2] \leqslant Q[w_1]w_2$$

if $w_1(t)$ is nonnegative in J and $w_2(t)$ is nondecreasing in J. For any fixed $T \in (\alpha,\beta]$ and $\alpha \leqslant t \leqslant T$, we define a function $v(t)$ by

$$v(t) = \hat{a}(T)+\hat{b}(T)\left[\int_{\phi(\alpha)}^{\phi(t)} k_1(t,t_1)w(u(t_1))\,dt_1+\cdots + \int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{\phi(t_1)}\cdots\left(\int_{\phi(\alpha)}^{\phi(t_{n-1})} k_n(t,t_1,\cdots,t_n)w(u(t_n))\,dt_n\right)\cdots\right)dt_1\right]. \tag{2.9}$$

Then $v(t)$ is nondecreasing and continuous function, $v(\alpha)=\hat{a}(T)$ and $u(t)\leqslant \varphi^{-1}(v(t))$. Taking the derivative to $v(t)$, we have

$$\begin{aligned} v'(t) &= \hat{b}(T)[R[w(u)](t,\phi(t))\phi'(t)+Q[w(u)](t)] \\ &\leqslant \hat{b}(T)[R[1](t,\phi(t))\phi'(t)+Q[1](t)]w(\varphi^{-1}(v(t))) \end{aligned}$$

or

$$\frac{v'(t)}{w(\varphi^{-1}(v(t)))} \leqslant \hat{b}(T)[R[1](t,\phi(t))\phi'(t)+Q[1](t)]. \tag{2.10}$$

By taking $t=s$ in (2.10) and then integrating it from α to any $t\in J$, changing the variables to the right-hand side and using the definition of the function G, one get the inequality

$$G(v(t)) \leqslant G(v(\alpha))+\hat{b}(T)\left[\int_{\phi(\alpha)}^{\phi(t)} R[1](\phi^{-1}(\sigma),\sigma)\,d\sigma+\int_{\alpha}^{t} Q[1](\sigma)\,d\sigma\right]$$

or

$$v(t) \leqslant G^{-1}\left[G(\hat{a}(T))+\hat{b}(T)\left(\int_{\phi(\alpha)}^{\phi(t)} R[1](\phi^{-1}(\sigma),\sigma)\,d\sigma+\int_{\alpha}^{t} Q[1](\sigma)\,d\sigma\right)\right] \tag{2.11}$$

for all $t\in[\alpha,T]$, where T is chosen so that he quality in the braces of (2.11) in the range of G^{-1}. Now, for $T=t$, we find the desired inequality in (2.3) follows by the inequality $u(t)\leqslant\varphi^{-1}(v(t))$.

(2) For any fixed $T\in(\alpha,\beta]$ and $\alpha\leqslant t\leqslant T$, we define a function $v(t)$ by

$$v(t) = \hat{a}(T)+\hat{b}(T)\left[\int_{\phi(\alpha)}^{\phi(t)} k_1(T,t_1)w(u(t_1))\,dt_1+\cdots + \int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{\phi(t_1)}\cdots\left(\int_{\phi(\alpha)}^{\phi(t_{n-1})} k_n(T,t_1,\cdots,t_n)w(u(t_n))\,dt_n\right)\cdots\right)dt_1\right]. \tag{2.12}$$

Then $v(\alpha)=\hat{a}(T)$, the function $v(t)$ is nondecreasing continuous and $u(t)\leqslant\varphi^{-1}(v(t))$. Since $\frac{\partial k_i}{\partial t}(T,t_1,\cdots,t_i)=0$ for all $i=1,2,\cdots,n$ and $t\in J=[\alpha,\beta]$, we have

$$v'(t)=\hat{b}(T)[R[w(u)](T,\phi(t))]\phi'(t). \tag{2.13}$$

The equality (2.13) implies the estimate

$$v'(t) \leqslant \hat{b}(T)[R[1](T,\phi(t))]\phi'(t)w(\varphi^{-1}(v(t))). \tag{2.14}$$

From the equality (2.14), we derive the equation

$$\frac{v'(t)}{w(\varphi^{-1}(v(t)))} \leqslant \hat{b}(T)[R[1](T,\phi(t))]\phi'(t). \tag{2.15}$$

By taking $t=s$ in (2.15) and then integrating it from α to any $t \in J$, changing the variables to the right-hand side and using the definition of the function G, one get the inequality

$$G(v(t)) \leqslant G(v(\alpha)) + \hat{b}(T)\int_{\phi(\alpha)}^{\phi(t)} R[1](T,\sigma)\,d\sigma$$

or

$$v(t) \leqslant G^{-1}\left(G(\hat{a}(T)) + \hat{b}(T)\int_{\phi(\alpha)}^{\phi(t)} R[1](T,\sigma)\,d\sigma\right) \tag{2.16}$$

for all $t \in [\alpha, T_1]$, where T_1 is chosen so that he quality in the braces of (2.16) in the range of G. In particular, for $T=t$, we find the desired inequality in (2.8) follows by the inequality $u(t) \leqslant \varphi^{-1}(v(t))$. This completes the proof. □

Let $\varphi(u) = u^p$ ($p>0, p \neq 1$ is a constant) in Theorem 2.1, we get the following retarded integral inequality with iterated integrals immediately.

COROLLARY 2.2. *Let $u(t), a(t), b(t)$ and $w(u)$ be as in Theorem 2.1 and $p > 0, p \neq 1$, be a constant.*

(1) Let $\phi \in C^1(J,J)$ be increasing with $\phi(t) \leqslant t$ on J. Suppose that

$$u^p(t) \leqslant a(t) + b(t)\left[\int_{\phi(\alpha)}^{\phi(t)} k_1(t,t_1)w(u(t_1))\,dt_1 + \cdots \right. \tag{2.17}$$
$$\left. + \int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{\phi(t_1)} \cdots \left(\int_{\phi(\alpha)}^{\phi(t_{n-1})} k_n(t,t_1,\cdots,t_n)w(u(t_n))\,dt_n\right)\cdots\right)dt_1\right]$$

for all $t \in J$, where $k_i(t,t_1,\cdots,t_i)$ are nonnegative, continuous functions in J_{i+1} for all $i=1,2,\cdots,n$. Suppose that the partial derivative $\frac{\partial k_i}{\partial t}(t,t_1,\cdots,t_i)$ exists and are nonnegative, continuous in J_{i+1} for all $i=1,2,\cdots,n$. Then

$$u(t) \leqslant \left\{G_p^{-1}\left[G_p(\hat{a}(t)) + \hat{b}(t)\left(\int_{\phi(\alpha)}^{\phi(t)} R[1](\phi^{-1}(\sigma),\sigma)\,d\sigma + \int_\alpha^t Q[1](\sigma)\,d\sigma\right)\right]\right\}^{\frac{1}{p}} \tag{2.18}$$

for all $t \in [\alpha, T_2]$, where $T_2 \in J$ is chosen so that

$$\left[G_p(\hat{a}(t)) + \hat{b}(t)\left(\int_{\phi(\alpha)}^{\phi(t)} R[1](\phi^{-1}(\sigma),\sigma)\,d\sigma + \int_\alpha^t Q[1](\sigma)\,d\sigma\right)\right] \in Dom(G_p^{-1}),$$

the function G_p is defined by

$$G_p(r) = \int_{r_0}^{r} \frac{ds}{w(s^{1/p})}, \quad \forall r \geqslant r_0 > 0, \tag{2.19}$$

G_p^{-1} denotes the inverse function of G_p, $R[x](t,s)$ and $Q[x](t)$ are defined in Theorem 2.1.

(2) Let $\phi \in C^1(J,J)$ be nondecreasing with $\phi(t) \leqslant t$ on J such that (2.17) is satisfied for any $t \in J$, where $k_i(t,t_1,\cdots,t_i)$ are nonnegative, continuous functions in J_{i+1} for all $i = 1,2,\cdots,n$, which are nondecreasing in $t \in J$ for all fixed $(t_1,\cdots,t_i) \in J_i$ for all $i = 1,\cdots,n$. Then

$$u(t) \leqslant \left[G_p^{-1}\left(G_p(\hat{a}(t)) + \hat{b}(t) \int_{\phi(\alpha)}^{\phi(t)} R[1](t,\sigma)\, d\sigma \right) \right]^{\frac{1}{p}}$$

for all $t \in [\alpha, T_2]$, where $T_2 \in J$ is chosen so that

$$\left(G_p(\hat{a}(t)) + \hat{b}(t) \int_{\phi(\alpha)}^{\phi(t)} R[1](t,\sigma)\, d\sigma \right) \in Dom(G_p^{-1}),$$

the function G_p is defined in (2.19), G_p^{-1} denotes the inverse function of G_p, and $R[x](t,s)$ is defined in Theorem 2.1.

Proof. The proof follows by an argument similar to that in the proof of Theorem 2.1 with suitable modification. We omit the details here. □

COROLLARY 2.3. *If, under the conditions of (1) of Theorem 2.1, the functions $a(t)$ and $b(t)$ are also nondecreasing in J, then*

$$u(t) \leqslant \varphi^{-1}\left\{ G^{-1}\left[G(a(t)) + b(t) \left(\int_{\phi(\alpha)}^{\phi(t)} R[1](\phi^{-1}(\sigma),\sigma)\, d\sigma + \int_{\alpha}^{t} Q[1](\sigma)\, d\sigma \right) \right] \right\} \tag{2.20}$$

for all $t \in [\alpha, T]$, where $T \in J$ is chosen so that

$$G(a(t)) + b(t) \left(\int_{\phi(\alpha)}^{\phi(t)} R[1](\phi^{-1}(\sigma),\sigma)\, d\sigma + \int_{\alpha}^{t} Q[1](\sigma)\, d\sigma \right) \in Dom(G^{-1}).$$

If, under the conditions of (2) of Theorem 2.1, the functions $a(t)$ and $b(t)$ are also nondecreasing in J, then

$$u(t) \leqslant \varphi^{-1}\left[G^{-1}\left(G(a(t)) + b(t) \int_{\phi(\alpha)}^{\phi(t)} R[1](t,\sigma)\, d\sigma \right) \right] \tag{2.21}$$

for all $t \in [\alpha, T_1]$, where $T_1 \in J$ is chosen so that

$$G(a(t)) + b(t) \int_{\phi(\alpha)}^{\phi(t)} R[1](t,\sigma)\, d\sigma \in Dom(G^{-1}).$$

Proof. The proof follows by an argument similar to that in the proof of Theorem 2.1 with suitable modification. We omit the details here. □

REMARK 2.1. (1) When $w(u) = \varphi'(u)$ in (1) of Theorem 2.1, we get following result:

$$u(t) \leqslant \varphi^{-1}(\hat{a}(t)) + \hat{b}(t)\left(\int_{\phi(\alpha)}^{\phi(t)} R[1](\phi^{-1}(\sigma),\sigma)\,d\sigma + \int_{\alpha}^{t} Q[1](\sigma)\,d\sigma\right).$$

(2) When $w(u) = \varphi'(u)$ in (2) of Theorem 2.1, we get following result:

$$u(t) \leqslant \varphi^{-1}(\hat{a}(t)) + \hat{b}(t)\int_{\phi(\alpha)}^{\phi(t)} R[1](\phi^{-1}(\sigma),\sigma)\,d\sigma.$$

Theorem 2.1 can easily be applied to generate other useful nonlinear integral inequalities in more general situations. For example, we have the following results.

THEOREM 2.4. *Let the functions $u(t), a(t), b(t), w(u)$ and φ are as in Theorem 2.1.*

(1) *Let $\phi \in C^1(J,J)$ be increasing with $\phi(t) \leqslant t$ on J, φ' is nondecreasing and suppose*

$$\begin{aligned}\varphi(u(t)) \leqslant a(t) + b(t)\Bigg[&\int_{\phi(\alpha)}^{\phi(t)} k_1(t,t_1)\varphi'(u(t_1))w(u(t_1))\,dt_1 + \cdots \\ &+ \int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{\phi(t_1)} \cdots \left(\int_{\phi(\alpha)}^{\phi(t_{n-1})} k_n(t,t_1,\cdots,t_n)\varphi'(u(t_n))w(u(t_n))\,dt_n\right)\cdots\right)dt_1\Bigg]\end{aligned} \tag{2.22}$$

for all $t \in J$, where $k_i(t,t_1,\cdots,t_i)$ are nonnegative, continuous functions in J_{i+1} for all $i = 1,2,\cdots,n$. Suppose that the partial derivative $\frac{\partial k_i}{\partial t}(t,t_1,\cdots,t_i)$ exists and are nonnegative, continuous in J_{i+1} for all $i = 1,2,\cdots,n$. Then

$$u(t) \leqslant G_1^{-1}\left[G_1(\varphi^{-1}(\hat{a}(t))) + \hat{b}(t)\left(\int_{\phi(\alpha)}^{\phi(t)} R[1](\phi^{-1}(s),s)\,ds + \int_{\alpha}^{t} Q[1](s)\,ds\right)\right] \tag{2.23}$$

for all $t \in [\alpha, T_3]$, where $\hat{a}(t)$ and $\hat{b}(t)$ are defined in (2.1), $T_3 \in J$ is chosen so that

$$G_1(\varphi^{-1}(\hat{a}(t))) + \hat{b}(t)\left(\int_{\phi(\alpha)}^{\phi(t)} R[1](\phi^{-1}(s),s)\,ds + \int_{\alpha}^{t} Q[1](s)\,ds\right) \in Dom(G_1^{-1}),$$

the function G_1 is defined by

$$G_1(r) = \int_{r_0}^{r} \frac{ds}{w(s)}, \quad \forall r \geqslant r_0 > 0, \tag{2.24}$$

G_1^{-1} denotes the inverse function of G_1, the functions $R[x](t,s)$ and $Q[x](t)$ are defined in (2.5) and (2.6), respectively.

(2) *Let $\phi \in C^1(J,J)$ be nondecreasing with $\phi(t) \leqslant t$ on J, φ' is nondecreasing and such that (2.22) is satisfied for any $t \in J$, where $k_i(t,t_1,\cdots,t_i)$ are nonnegative, continuous functions in J_{i+1} for all $i = 1,2,\cdots,n$, which are nondecreasing in $t \in J$ for all fixed $(t_1,\cdots,t_i) \in J_i$ for all $i = 1,2,\cdots,n$, then*

$$u(t) \leqslant G_1^{-1}\left[G_1(\varphi^{-1}(\hat{a}(t))) + \hat{b}(t)\int_{\phi(\alpha)}^{\phi(t)} R[1](\phi^{-1}(s),s)\,ds\right] \tag{2.25}$$

for all $t \in [\alpha, T_4]$, where $\hat{a}(t)$ and $\hat{b}(t)$ are defined in (2.1), $T_4 \in J$ is chosen so that

$$G_1(\varphi^{-1}(\hat{a}(t))) + \hat{b}(t)\int_{\phi(\alpha)}^{\phi(t)} R[1](\phi^{-1}(s),s)\,ds \in Dom(G_1^{-1}),$$

the function G_1 is defined in (2.24) and G_1^{-1} is the inverse function of G_1, $R[x](t,s)$ is defined in (2.5).

Proof. (1) For any fixed $T \in (\alpha,\beta]$ and $\alpha \leqslant t \leqslant T$, we define a function $v(t)$ by

$$\begin{aligned} v(t) = \hat{a}(T) + \hat{b}(T)\bigg[&\int_{\phi(\alpha)}^{\phi(t)} k_1(t,t_1)\varphi'(u(t_1))w(u(t_1))\,dt_1 + \cdots \\ &+ \int_{\phi(\alpha)}^{\phi(t)}\bigg(\int_{\phi(\alpha)}^{\phi(t_1)}\cdots\bigg(\int_{\phi(\alpha)}^{\phi(t_{n-1})} k_n(t,t_1,\cdots,t_n)\varphi'(u(t_n)) \\ &\times w(u(t_n))\,dt_n\bigg)\cdots\bigg)dt_1\bigg]. \end{aligned} \tag{2.26}$$

Then $v(\alpha) = \hat{a}(T)$, the function $v(t)$ is nondecreasing continuous and $u(t) \leqslant \varphi^{-1}(v(t))$. Taking derivative to $v(t)$, we have

$$\begin{aligned} v'(t) &= \hat{b}(T)[R[\varphi'(u)w(u)](t,\phi(t))\phi'(t) + Q[\varphi'(u)w(u)(t)] \\ &\leqslant \hat{b}(T)\{R[w(\varphi^{-1}(v(t)))](t,\phi(t))\phi'(t) + Q[w(\varphi^{-1}(v(t)))](t)\}\varphi'(\varphi^{-1}(v(t))) \end{aligned}$$

or

$$\frac{v'(t)}{\varphi'(\varphi^{-1}(v(t)))} \leqslant \hat{b}(T)\{R[w(\varphi^{-1}(v))](t,\phi(t))\phi'(t) + Q[w(\varphi^{-1}(v))](t)\}. \tag{2.27}$$

By taking $t = s$ in (2.27) and then integrating it from α to any $t \in J$ and changing the variables to the right-hand side, one get the inequality

$$\begin{aligned} \varphi^{-1}(v(t)) &\leqslant \varphi^{-1}(v(\alpha)) \\ &\quad + \hat{b}(T)\left[\int_{\alpha}^{t}\{R[w(\varphi^{-1}(v))](s,\phi(s))\phi'(s) + Q[w(\varphi^{-1}(v))](s)\}\,ds\right]. \end{aligned} \tag{2.28}$$

We denote the right-hand side of (2.28) by $p(t)$. Then $p(\alpha) = \varphi^{-1}(\hat{a}(T))$, the function $p(t)$ is positive and nondecreasing in $t \in [\alpha,\beta]$, $u(t) \leqslant \varphi^{-1}(v(t)) \leqslant p(t)$ and

$$p'(t) \leqslant \hat{b}(T)\{R[w(\varphi^{-1}(v))](t,\phi(t))\phi'(t) + Q[w(\varphi^{-1}(v))](t)\}. \tag{2.29}$$

The inequality (3.29) implies the estimate

$$p'(t) \leqslant \hat{b}(T)\{R[1](t,\phi(t))\phi'(t) + Q[1](t)\}w(p(t)). \tag{2.30}$$

From the inequality (2.30), we derive the equation

$$\frac{p'(t)}{w(p(t))} \leqslant \hat{b}(T)\{R[1](t,\phi(t))\phi'(t) + Q[1](t)\}. \tag{2.31}$$

By taking $t = s$ in (2.31) and then integrating it from α to any $t \in J$, changing the variables to the right-hand side and using the definition of the function G_1, one get the inequality

$$G_1(p(t)) \leqslant G_1(p(\alpha)) + \hat{b}(T)\left(\int_{\phi(\alpha)}^{\phi(t)} R[1](\phi^{-1}(s),s)\,ds + \int_{\alpha}^{t} Q[1](s)\,ds\right),$$

or

$$p(t) \leqslant G_1^{-1}\left[G_1(\varphi^{-1}(\hat{a}(T))) + \hat{b}(T)\left(\int_{\phi(\alpha)}^{\phi(t)} R[1](\phi^{-1}(s),s)\,ds + \int_{\alpha}^{t} Q[1](s)\,ds\right)\right]. \tag{2.32}$$

for all $t \in [\alpha, T_2]$, where T_2 is chosen so that he quality in the braces of (2.32) in the range of G_1. In particular, for $T = t$, we find the desired inequality in (2.23) follows by the inequality $u(t) \leqslant \varphi^{-1}(v(t)) \leqslant p(t)$.

(2) The proof of the remaining inequality can be completed by combining at the proofs of the (2) of Theorem 2.1 and (1) of Theorem 2.4 with suitable modifications. This completes the proof. $\square$

Let $\varphi(u) = u^p$ ($p \geqslant 1$ is a constant) in Theorem 2.4, we get the following retarded integral inequality with iterated integrals immediately.

COROLLARY 2.5. *Let the functions $u(t), a(t), b(t)$, and $w(u)$ are as in Theorem 2.4 and $p \geqslant 1$ is a constant.*

(1) *Let $\phi \in C^1(J,J)$ be increasing with $\phi(t) \leqslant t$ on J, and suppose*

$$\begin{aligned} u^p(t) \leqslant a(t) + pb(t)\Bigg[&\int_{\phi(\alpha)}^{\phi(t)} k_1(t,t_1)u^{p-1}(t_1)w(u(t_1))\,dt_1 + \cdots \\ &+ \int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{\phi(t_1)} \cdots \left(\int_{\phi(\alpha)}^{\phi(t_{n-1})} k_n(t,t_1,\cdots,t_n)u^{p-1}(t_n)w(u(t_n))\,dt_n\right)\cdots\right)dt_1\Bigg] \end{aligned} \tag{2.33}$$

for all $t \in J$, where $k_i(t,t_1,\cdots,t_i)$ are nonnegative, continuous functions in J_{i+1} for all $i = 1,2,\cdots,n$. Suppose that the partial derivative $\frac{\partial k_i}{\partial t}(t,t_1,\cdots,t_i)$ exists and are nonnegative, continuous in J_{i+1} for all $i = 1,2,\cdots,n$. Then

$$u(t) \leqslant G_1^{-1}\left[G_1(\hat{a}^{\frac{1}{p}}(t)) + \hat{b}(t)\left(\int_{\phi(\alpha)}^{\phi(t)} R[1](\phi^{-1}(s),s)\,ds + \int_{\alpha}^{t} Q[1](s)\,ds\right)\right] \tag{2.34}$$

for all $t \in [\alpha, T_3]$, *where* $\hat{a}(t)$ *and* $\hat{b}(t)$ *are defined in (2.1),* $T_3 \in J$ *is chosen so that*

$$G_1(\hat{a}^{\frac{1}{p}}(t)) + \hat{b}(t)\left(\int_{\phi(\alpha)}^{\phi(t)} R[1](\phi^{-1}(s), s)\,ds + \int_{\alpha}^{t} Q[1](s)\,ds\right) \in Dom(G_1^{-1}),$$

the functions $R[x](t,s)$, $Q[x](t)$ *and* G_1 *are defined in (2.5), (2.6) and (2.24), respectively.*

(2) *Let* $\phi \in C^1(J,J)$ *be nondecreasing with* $\phi(t) \leqslant t$ *on* J, φ' *is nondecreasing and such that (2.33) is satisfied for any* $t \in J$, *where* $k_i(t, t_1, \cdots, t_i)$ *are nonnegative, continuous functions in* J_{i+1}, *for all* $i = 1, 2, \cdots, n$, *which are nondecreasing in* $t \in J$ *for all fixed* $(t_1, \cdots, t_i) \in J_i$ *for all* $i = 1, 2, \cdots, n$. *Then*

$$u(t) \leqslant G_1^{-1}\left[G_1(\hat{a}^{\frac{1}{p}}(t)) + \hat{b}(t)\int_{\phi(\alpha)}^{\phi(t)} R[1](\phi^{-1}(s), s)\,ds\right] \tag{2.35}$$

for all $t \in [\alpha, T_4]$, *where* $\hat{a}(t)$ *and* $\hat{b}(t)$ *are defined in (2.1),* $T_4 \in J$ *is chosen so that*

$$G_1(\hat{a}^{\frac{1}{p}}(t)) + \hat{b}(t)\int_{\phi(\alpha)}^{\phi(t)} R[1](\phi^{-1}(s), s)\,ds \in Dom(G_1^{-1}).$$

Proof. The proof follows by an argument similar to that in the proof of Theorem 2.4 with suitable modification. We omit the details here. □

Let $w(u) = u^p$ ($p > 0, p \neq 1$ is a constant) in Theorem 2.4, we get the following retarded integral inequality with iterated integrals immediately.

COROLLARY 2.6. *Let the functions* $u(t), a(t), b(t), \varphi$ *and* φ' *are as in Theorem 2.4 and* $p \geqslant 0, p \neq 1$, *is a constant.*

(1) *Let* $\phi \in C^1(J,J)$ *be increasing with* $\phi(t) \leqslant t$ *on* J. *Suppose that*

$$\begin{aligned}\varphi(u(t)) \leqslant a(t) + b(t)\Bigg[&\int_{\phi(\alpha)}^{\phi(t)} k_1(t,t_1)\varphi'(u(t_1))u^p(t_1)\,dt_1 + \cdots \\ &+ \int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{\phi(t_1)} \cdots \left(\int_{\phi(\alpha)}^{\phi(t_{n-1})} k_n(t,t_1,\cdots,t_n)\varphi'(u(t_n))u^p(t_n)\,dt_n\right)\cdots\right)dt_1\Bigg]\end{aligned} \tag{2.36}$$

for all $t \in J$, *where* $k_i(t, t_1, \cdots, t_i)$ *are nonnegative, continuous functions in* J_{i+1} *for all* $i = 1, 2, \cdots, n$. *Suppose that the partial derivative* $\frac{\partial k_i}{\partial t}(t, t_1, \cdots, t_i)$ *exists and are nonnegative, continuous in* J_{i+1} *for all* $i = 1, 2, \cdots, n$. *Then*

$$u(t) \leqslant \left[[\varphi^{-1}(\hat{a}(t))]^{1-p} + (1-p)\hat{b}(t)\left(\int_{\phi(\alpha)}^{\phi(t)} R[1](\phi^{-1}(s), s)\,ds + \int_{\alpha}^{t} Q[1](s)\,ds\right)\right]^{\frac{1}{1-p}}$$

for all $t \in J$.

(2) *Let* $\phi \in C^1(J,J)$ *be nondecreasing with* $\phi(t) \leqslant t$ *on* J, *and such that (2.36) is satisfied for any* $t \in J$, *where* $k_i(t,t_1,\cdots,t_i)$ *are nonnegative, continuous functions in* J_{i+1} *for all* $i = 1,2,\cdots,n$, *which are nondecreasing in* $t \in J$ *for all fixed* $(t_1,\cdots,t_i) \in J_i$ *for all* $i = 1,2,\cdots,n$, *then*

$$u(t) \leqslant \left[[\varphi^{-1}(\hat{a}(t))]^{1-p} + (1-p)\hat{b}(t)\int_{\phi(\alpha)}^{\phi(t)} R[1](\phi^{-1}(s),s)\,ds\right]^{\frac{1}{1-p}}$$

for $t \in J$.

Proof. The proof follows by an argument similar to that in the proof of Theorem 2.4 with suitable modification. We omit the details here. □

REMARK 2.2. (1) If, under the conditions of (1) of Theorem 2.4, the functions $a(t)$ and $b(t)$ are also nondecreasing in J, then

$$u(t) \leqslant G_1^{-1}\left[G_1(\varphi^{-1}(a(t))) + b(t)\left(\int_{\phi(\alpha)}^{\phi(t)} R[1](\phi^{-1}(s),s)\,ds + \int_\alpha^t Q[1](s)\,ds\right)\right].$$

(2) If, under the conditions of (2) of Theorem 2.4, the functions $a(t)$ and $b(t)$ are also nondecreasing in J, then

$$u(t) \leqslant G_1^{-1}\left[G_1(\varphi^{-1}(a(t))) + b(t)\int_{\phi(\alpha)}^{\phi(t)} R[1](\phi^{-1}(s),s)\,ds\right].$$

3. Applications

In this section, we show that our results are useful in showing the global existence of solutions to certain integro-differential equations. First consider the following retarded integro-differential equation

$$\begin{aligned} &px^{p-1}(t)x'(t) \\ &\quad = F\left(t, f_1(t,x(t)), \int_\alpha^t f_2(t,t_1,x(t_1))\,dt_1, \int_\alpha^t \int_\alpha^{t_1} f_3(t,t_1,t_2,x(t_2))\,dt_2dt_1\right) \end{aligned} \tag{3.1}$$

for all $t \in J$, where $p > 0, p \neq 1$, is constant, $F \in C(J \times R^3, R)$, $f_i \in C(J_i \times R, R)$ for all $i = 1,2,3$. The following theorem deals with a bound on the solution of the problem (3.1).

THEOREM 3.1. *Assume that* $F : I \times R^2 \to R$ *is a continuous function for which there exists continuous nonnegative nondecreasing functions* $a(t)$ *such that*

$$\left\{\begin{array}{l} |F(t,u_1,u_2,u_3)| \leqslant a(t)(|u_1|+|u_2|+|u_3|), \\ |f_1(t,v)| \leqslant k_1(t)w(|v|), \\ |f_2(t,t_1,v)| \leqslant k_2(t,t_1)w(|v|), \\ |f_3(t,t_1,t_2,v)| \leqslant k_3(t,t_1,t_2)w(|v|), \end{array}\right. \tag{3.2}$$

where function $w(u)$ *be a nondecreasing continuous function for all* $u \in R_+$ *with* $w(u) > 0$ *for all* $u > 0$, *and* $k_i(t,t_1,\cdots,t_{i-1})$ *are nonnegative, continuous functions in* J_i *for* $i = 1,2,3$. *If* $x(t)$ *is any solution of the problem (3.1) with the condition (3.2), then*

$$| x(t) | \leqslant \Big\{ G_p^{-1}\Big[G_p(|x(\alpha)|^p) + \hat{a}(t)\Big(\int_\alpha^t k_1(t_1)\,dt_1 + \int_\alpha^t \int_\alpha^{t_1} k_1(t_1,t_2)\,dt_2\,dt_1 + \int_\alpha^t \int_\alpha^{t_1} \int_\alpha^{t_2} k_1(t_1,t_2,t_3)\,dt_3\,dt_2\,dt_1 \Big)\Big]\Big\}^{\frac{1}{p}},$$

where the functions G_p, G_p^{-1} *are as in Corollary 2.2 for any* $t_1,t_2,t_3 \in J$.

Proof. It is easy to see that the solution $x(t)$ of the problem (3.1) satisfies the equivalent integral equation

$$x^p(t) = x^p(\alpha) + \int_\alpha^t F\Big(t, f_1(t,x(t)), \int_\alpha^t f_2(t,t_1,x(t_1))\,dt_1, \int_\alpha^t \int_\alpha^{t_1} f_3(t,t_1,t_2,x(t_2))\,dt_2 dt_1 \Big)\,ds.$$

From (3.2) and making the change of variables, we have

$$| x(t) |^p \leqslant | x(\alpha) |^p + a(t)\Big[\int_\alpha^t k_1(t_1) w(|x(t_1)|)\,dt_1 + \int_\alpha^t \int_\alpha^{t_1} k_1(t_1,t_2) w(|x(t_2)|)\,dt_2\,dt_1 + \int_\alpha^t \int_\alpha^{t_1} \int_\alpha^{t_2} k_1(t_1,t_2,t_3) w(|x(t_3)|)\,dt_3\,dt_2\,dt_1 \Big]. \tag{3.3}$$

Now, a suitable application of the inequality given in (2) of Corollary 2.2 to (3.3) yields the desired result. □

THEOREM 3.2. *Assume that* $F : I \times R^2 \to R$ *is a continuous function for which there exists continuous nonnegative nondecreasing functions* $a(t)$ *such that*

$$\begin{cases} |F(t,u_1,u_2,u_3)| \leqslant a(t)(|u_1| + |u_2| + |u_3|), \\ |f_1(t,v)| \leqslant k_1(t)|v|^{p-1} w(|v|), \\ |f_2(t,t_1,v)| \leqslant k_2(t,t_1)|v|^{p-1} w(|v|), \\ |f_3(t,t_1,t_2,v)| \leqslant k_3(t,t_1,t_2)|v|^{p-1} w(|v|), \end{cases} \tag{3.4}$$

where the function $w(u)$ *is a nondecreasing continuous function for all* $u \in R_+$ *with* $w(u) > 0$ *for all* $u > 0$ *and* $k_i(t,t_1,\cdots,t_{i-1})$ *are nonnegative, continuous functions in*

J_i for $i=1,2,3$ and $p \geqslant 1$ is a constant. If $x(t)$ is any solution of the problem (3.1) with the condition (3.4), then

$$|\,x(t)\,| \leqslant G_1^{-1}\Big[G_1(|x(\alpha)|^{\frac{1}{p}}) + \frac{1}{p}\hat{a}(t)\Big(\int_\alpha^t k_1(t_1)\,dt_1 + \int_\alpha^t\int_\alpha^{t_1} k_1(t_1,t_2)\,dt_2\,dt_1 + \int_\alpha^t\int_\alpha^{t_1}\int_\alpha^{t_2} k_1(t_1,t_2,t_3)\,dt_3\,dt_2\,dt_1\Big)\Big],$$

where the functions G_1, G_1^{-1} are as in Corollary 2.5 for any $t_1,t_2,t_3 \in J$.

Proof. The proof follows by an argument similar to that in the proof of Theorem 3.1 with suitable modification using the inequality given in (2) of Corollary 2.5. We omit the details here. □

We next consider the following integro-differential equation

$$\left(\frac{x(t)}{h(t)}\right)' = F\left(t, f_1(t,x(t)), \int_\alpha^t f_2(t,t_1,x(t_1))\,dt_1, \int_\alpha^t\int_\alpha^{t_1} f_3(t,t_1,t_2,x(t_2))\,dt_2dt_1\right) \quad (3.5)$$

for all $t \in J$, where $F \in C(J\times R^3, R)$, $f_i \in C(J_i \times R, R)$ for $i=1,2,3$ and $h(t) \neq 0$. The following theorem deals with a bound on the solution of the problem (3.5).

THEOREM 3.3. *Assume that $F : I\times R^2 \to R$ is a continuous function for which there exists continuous nonnegative nondecreasing functions $a(t)$ such that*

$$\begin{cases} |F(t,u_1,u_2,u_3)| \leqslant a(t)(|u_1|+|u_2|+|u_3|), \\ |f_1(t,v)| \leqslant k_1(t)w(|v|), \\ |f_2(t,t_1,v)| \leqslant k_2(t,t_1)w(|v|), \\ |f_3(t,t_1,t_2,v)| \leqslant k_3(t,t_1,t_2)w(|v|), \end{cases} \quad (3.6)$$

where the function $w(u)$ is a nondecreasing continuous function for all $u \in R_+$ with $w(u) > 0$ for all $u > 0$ and $k_i(t,t_1,\cdots,t_{i-1})$ are nonnegative, continuous functions in J_i for $i=1,2,3$. If $x(t)$ is any solution of the problem (3.5) with the condition (3.6), then

$$|\,x(t)\,| \leqslant G_1^{-1}\Big[G_1\Big(\Big|\frac{x(\alpha)}{h(\alpha)}\hat{h}(t)\Big|\Big) + |\hat{h}(t)\hat{a}(t)|\Big(\int_\alpha^t k_1(t_1)\,dt_1 + \int_\alpha^t\int_\alpha^{t_1} k_1(t_1,t_2)\,dt_2\,dt_1 + \int_\alpha^t\int_\alpha^{t_1}\int_\alpha^{t_2} k_1(t_1,t_2,t_3)\,dt_3\,dt_2\,dt_1\Big)\Big],$$

where the functions G_1, G_1^{-1} are as in Corollary 2.5 for any $t_1,t_2,t_3 \in J$.

Proof. It is easy to see that the solution $x(t)$ of the problem (3.5) satisfies the equivalent integral equation

$$x(t) = \frac{x(\alpha)}{h(\alpha)}h(t) + h(t)\int_\alpha^t F\left(t, f_1(t,x(t)), \int_\alpha^t f_2(t,t_1,x(t_1))\,dt_1, \int_\alpha^t\int_\alpha^{t_1} f_3(t,t_1,t_2,x(t_2))\,dt_2dt_1\right)ds.$$

From (3.6) and making the change of variables, we have

$$|\,x(t)\,| \leqslant \left|\frac{x(\alpha)}{h(\alpha)}h(t)\right| + |h(t)a(t)|\left[\int_\alpha^t k_1(t_1)w(|x(t_1)|)\,dt_1 + \int_\alpha^t\int_\alpha^{t_1} k_1(t_1,t_2)w(|x(t_2)|)\,dt_2\,dt_1 + \int_\alpha^t\int_\alpha^{t_1}\int_\alpha^{t_2} k_1(t_1,t_2,t_3)w(|x(t_3)|)\,dt_3\,dt_2\,dt_1\right]. \tag{3.7}$$

Now, a suitable application of the inequality given in (2) of Corollary 2.5 to (3.7) yields the desired result. □

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