Mathematical
Inequalities
& Applications
Volume 14, Number 3 (2011), 633–645

# SOME NONLINEAR DYNAMIC INEQUALITIES ON TIME SCALES

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(*Communicated by M. Bohner*)

*Abstract.* Our aim in this paper is to establish some explicit bounds of solutions of a certain class of nonlinear dynamic inequalities on a time scale $\mathbb{T}$ which is unbounded above. Also, we establish some sufficient conditions for global existence and an estimate of the rate of decay of the solutions. These on the one hand generalizes and on the other hand furnish a handy tool for the study of qualitative as well as quantitative properties of solutions of dynamic equations on time scales.

## 1. Introduction

It is well known that the dynamic inequalities play important roles in the development of the qualitative theory of dynamic equations on time scales. During the past decade a number of dynamic inequalities has been established by some authors which are motivated by some applications, for example, when studying the behavior of solutions of certain class of dynamic equations on time scales, the bounds provided by earlier inequalities are inadequate in applications and we need some new and specific type of dynamic inequalities in time scales. For contributions, we refer the reader to [1], [2], [3], [4], [5], [9], [10], [14] and the references cited therein. In [4, Theorem 6.1] it is proved that if $y$, $a$ and $p \in C_{rd}$ and $p \in \mathscr{R}^+$, then

$$y^{\Delta}(t) \leqslant f(t) + p(t)y(t), \ \text{ for all } t \in [t_0, \infty)_{\mathbb{T}}, \tag{1.1}$$

implies

$$y(t) \leqslant y(t_0)e_p(t,t_0) + \int_{t_0}^{t} e_p(t,\sigma(s))f(s)\Delta s, \text{ for all } t \in [t_0, \infty)_{\mathbb{T}}, \tag{1.2}$$

where $\mathscr{R}^+ := \{a \in \mathscr{R} : 1 + \mu(t)a(t) > 0, \ t \in \mathbb{T}\}$ and $\mathscr{R}$ is the class of rd-continuous and regressive functions. Using this comparison result, the authors in [4, Theorem 6.4] proved a new Gronwall dynamic inequality which proves that: If $y$, $a$ and $p \in C_{rd}$ and $p \in \mathscr{R}^+$, then

$$y(t) \leqslant a(t) + \int_{t_0}^{t} p(s)y(s)\Delta s, \ \ \text{for all } t \in [t_0, \infty)_{\mathbb{T}}, \tag{1.3}$$

*Mathematics subject classification* (2010): 26D15, 26D20, 39A12.
*Keywords and phrases*: Dynamic equations, time scale, integral inequality.


Paper MIA-14-54

implies that

$$y(t) \leqslant a(t) + \int_{t_0}^{t} e_p(t, \sigma(s)) a(s) p(s) \Delta s, \quad \text{for all } t \in [t_0, \infty)_{\mathbb{T}}. \tag{1.4}$$

Since (1.4) provides an explicit bound to the unknown function $y(t)$ and a tool to the study of many qualitative as well as quantitative properties of solutions of dynamic equations, it has become one of the very few classic and most influential results in the theory and applications of inequalities. Because of its fundamental importance, over the years, many generalizations and analogous results of (1.4) have been established.

In this paper, we consider the nonlinear dynamic inequality

$$(u^p(t))^{\Delta} \leqslant \beta(t) + \alpha(t) u^q(t) - \varphi(t) u^p(\sigma(t)), \text{ for all } t \in [t_0, \infty)_{\mathbb{T}}, \tag{1.5}$$

and its integral form on a time scale $\mathbb{T}$ where $t_0 \geqslant 0$ is a fixed number. One of our aims in this paper is establish some sufficient conditions for global existence and an estimate of the rate of decay of solutions. Note that the dynamic inequality (1.1) is a special case of (1.5) with $\varphi(t) = 0$ and $q = 1$. For (1.5) we will assume the following hypotheses:

$$(H_1) \quad \begin{cases} \beta,\ \varphi,\ \alpha \text{ and } u \text{ are rd-continuous positive functions defined on } [t_0,\infty)_{\mathbb{T}}, \\ p \text{ and } q \text{ are positive constants such that } q \geqslant 1 \text{ and } p \geqslant 1. \end{cases}$$

The inequality (1.5) is of interest in the study of the continuous and discrete dynamical systems and nonlinear evolution equations as well as in oscillation theory of dynamic equations on time scales. Also, it can be applied in studying the global existence of solutions of nonlinear partial dynamic equations on time scales.

Also in this paper, we are concerned with the nonlinear dynamic inequality

$$u^{\gamma}(t) \leqslant a(t) + b(t) \int_{t_0}^{t} \left[ f(s) u^{\delta}(s) + g(s) u^{\alpha}(s) \right] \Delta s, \text{ for all } t \in [t_0, \infty)_{\mathbb{T}}, \tag{1.6}$$

and investigate new nonlinear dynamic inequalities which provide some explicit bounds on the unknown function $u(t)$.

For (1.6) we will assume the following hypotheses:

$$(H_2) \quad \begin{cases} u,\ a,\ b,\ f \text{ and } g \text{ are rd-continuous positive functions defined on } [t_0,\infty)_{\mathbb{T}}, \\ \alpha,\ \delta \text{ and } \gamma \text{ are positive constants such that } \gamma \geqslant 1 \text{ and } \delta,\ \alpha \leqslant \gamma. \end{cases}$$

Our motivations for considering the inequality (1.6) are the results obtained by Ou-Iang [12], Dafermos [6] and Pachpatte [13] when $\mathbb{T} = \mathbb{R}$. Ou-Iang [12] in his study of the boundedness of certain second order differential equations established the following result which is generally known as Ou-Iang's inequality: If $u$ and $f$ are non-negative functions defined on $[0, \infty)$ such that

$$u^2(t) \leqslant k^2 + 2 \int_0^t f(s) u(s) ds, \text{ or all } t \in [0, \infty), \tag{1.7a}$$

where $k \geqslant 0$ is a constant, then $u(t) \leqslant k + \int_0^t f(s)ds$, for all $t \in [0,\infty)$.

Dafermos [6] established a generalization of Ou-Iang's inequality in the process of investigation the connection between stability and the second law of thermodynamics. He proved that if $u \in L^\infty[0,r]$ and $f \in L^1[0,r]$ are non-negative functions satisfying

$$u^2(t) \leqslant M^2u^2(0) + 2\int_0^t [Nf(s)u(s) + Ku^2(s)]ds, \quad \text{for all } t \in [0,r], \tag{1.8}$$

where $M,\ N,\ K$ are non-negative constants, then $u(t) \leqslant \left[Mu(0) + N\int_0^t f(s)ds\right] e^{Kt}$.

Pachpatte [13] established the following further generalizations of the result of Dafermos [6] and proved that: If $u,\ f,\ g$ are continuous non-negative functions on $[0,\infty)$ satisfying

$$u^2(t) \leqslant k^2 + 2\int_0^t [f(s)u(s) + g(s)u^2(s)]ds, \quad \text{for all } t \in [0,\infty), \tag{1.9}$$

where $k \geqslant 0$ is a constant, then $u(t) \leqslant \left(k + \int_0^t f(s)ds\right) \exp\left(\int_0^t g(s)ds\right)$, for all $t \in [0,\infty)$.

Our aim in this paper, is to establish some sufficient conditions for global existence and an estimate of the rate of decay of solutions of (1.5) and give an explicit bound of the unknown function of its integral from. Also, we give an explicit bound of the unknown function of the inequality (1.6). The results will be different from the results established by Ou-Iang [12], Dafermos [6] and Pachpatte [13]. The results will be proved in Section 2 by employing the Bernoulli inequality.

The study of dynamic equations and inequalities on time scales, which goes back to its founder Stefan Hilger [7], is an area of mathematics that has recently received a lot of attention. The general idea is to prove a result for a dynamic equation or a dynamic inequality where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which may be an arbitrary closed subset of the real numbers $\mathbb{R}$. Since we are interested in the asymptotic behavior of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[t_0,\infty)_\mathbb{T}$ by $[t_0,\infty)_\mathbb{T} := [t_0,\infty) \cap \mathbb{T}$. The book on the subject of time scale by Bohner and Peterson [4] summarizes and organizes much of time scale calculus. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus (see Kac and Cheung [8]), i.e, when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ where $q > 1$. There are applications of dynamic equations on time scales to quantum mechanics, electrical engineering, neural networks, heat transfer, and combinatorics. A recent cover story article in New Scientist [15] discusses several possible applications.

For completeness in the following, we recall the following concepts related to the notion of time scales. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right–dense continuous (rd–continuous) provided $g$ is continuous at right–dense points and at left–dense points in

$\mathbb{T}$, left hand limits exist and are finite. The set of all such rd-continuous functions is denoted by $C_{rd}(\mathbb{T})$. The graininess function $\mu$ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$ where $\sigma(t)$ is the forward jump operator defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$. We will make use of the following product and quotient rules for the derivative of the product $fg$ and the quotient $f/g$ (where $gg^{\sigma} \neq 0$, here $g^{\sigma} = g \circ \sigma$) of two differentiable function $f$ and $g$

$$(fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta} = fg^{\Delta} + f^{\Delta}g^{\sigma}, \text{ and } \left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta}g - fg^{\Delta}}{gg^{\sigma}}. \tag{1.10}$$

We say that a function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided $1 + \mu(t)p(t) \neq 0,\ t \in \mathbb{T}$. The set of all regressive functions on a time scale $\mathbb{T}$ forms an Abelian group under the addition $\oplus$ defined by $p \oplus q := p + q + \mu pq$.

We denote the set of all $f : \mathbb{T} \to \mathbb{R}$ which are rd-continuous and regressive by $\mathscr{R}$. If $p \in \mathscr{R}$, then we can define the exponential function by $e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right)$, for $t \in \mathbb{T},\ s \in \mathbb{T}^k$, where $\xi_h(z)$ is the cylinder transformation, which is given by

$$\xi_h(z) = \begin{cases} \frac{\log(1+hz)}{h}, & h \neq 0, \\ z, & h = 0. \end{cases}$$

Alternatively, for $p \in \mathscr{R}$ one can define the exponential function $e_p(\cdot,t_0)$, to be the unique solution of the IVP $x^{\Delta} = p(t)x$, with $x(t_0) = 1$. We define $\mathscr{R}^{+} := \{f \in \mathscr{R} : 1 + \mu(t)f(t) > 0, t \in \mathbb{T}\}$. Form the properties of the exponential function, see Bohner and Peterson [4], we will use the following properties $e_p^{\Delta}(t,t_0) = p(t)e_p(t,t_0)$ and

$$e_p(\sigma(t),t_0) = [1 + \mu(t)p(t)]e_p(t,t_0).$$

Also if $p \in \mathscr{R}$, then $e_p(t,s)$ is real-valued and nonzero on $\mathbb{T}$. If $p \in \mathscr{R}^{+}$, then $e_p(t,t_0)$ is always positive, $e_p(t,t) = 1$ and $e_0(t,s) = 1$. Note that If $\mathbb{T} = \mathbb{R}$, then $e_p(t,t_0) = \exp(\int_{t_0}^t p(s)ds)$, if $\mathbb{T} = \mathbb{N}$, then $e_p(t,t_0) = \prod_{s=t_0}^{t-1}(1 + p(s))$, and if $\mathbb{T} = q^{\mathbb{N}_0}$, then $e_p(t,t_0) = \prod_{s=t_0}^{t-1}(1 + (q-1)sp(s))$.

## 2. Main Results

Before we state and prove the main results we present some Lemmas which play important roles in the proofs of the main results.

LEMMA 2.1. ([11, Bernoulli's inequality]) *Let* $0 < \gamma \leqslant 1$ *and* $x > -1$. *Then* $(1+x)^{\gamma} \leqslant 1 + \gamma x$.

LEMMA 2.2. ([3, 5]) *Let* $\mathbb{T}$ *be an unbounded time scale with* $t_0$ *and* $t \in \mathbb{T}$. *Suppose that* $y,\ a,\ b,\ p \in C_{rd}$ *and* $b,\ p \geqslant 0$. *If*

$$y(t) \leqslant a(t) + b(t)\int_{t_0}^{t} p(s)y(s)\Delta s, \ \textit{for all } t \in [t_0,\infty)_{\mathbb{T}}, \tag{2.1}$$

*then*

$$y(t) < a(t) + b(t)\int_{t_0}^{t} a(s)p(s)e_{bp}(t,\sigma(s))\Delta s, \quad \textit{for all } t \in [t_0,\infty)_{\mathbb{T}}. \tag{2.2}$$

In the following, we consider the nonlinear dynamic inequality (1.5) and establish some sufficient conditions for global existence and an estimate of the rate of decay of its solutions.

THEOREM 2.1. *Assume that* $(H_1)$ *holds and there exists a positive* rd*-continuous function* $\pi(t)$, $\pi \in C_r^1[t_0,\infty)_{\mathbb{T}}$ *such that*

$$\beta(t) + \frac{\alpha(t)}{\pi^q(t)} \leqslant \frac{1}{(\pi^\sigma(t))^p}\left[\varphi(t) - \frac{(\pi^p)^\Delta(t)}{\pi^p(t)}\right]. \tag{2.3}$$

*Let* $u(t) \geqslant 0$ *is a solution of the inequality (*1.5*) such that*

$$\pi^p(t_0)u(t_0) < 1. \tag{2.4}$$

*Then* $u(t)$ *exists globally and the following estimate holds:*

$$0 \leqslant u(t) < \frac{1}{\pi(t)}, \quad \textit{for } t \geqslant t_0.$$

*Consequently, if* $\lim_{t\to\infty}\pi(t) = \infty$*, then* $\lim_{t\to\infty}u(t) = 0$.

*Proof.* Let

$$w(t) := u^p(t)e_\varphi(t,t_0).$$

Using the product rule in (1.10), we have

$$w^\Delta(t) = (u^p(t))^\Delta e_\varphi(t,t_0) + \varphi(t)\,(u^\sigma)^p\, e_\varphi(t,t_0).$$

This and (1.5) imply that

$$\begin{aligned} w^\Delta(t) &\leqslant e_\varphi(t,t_0)\left[\beta(t) - \varphi(t)\,(u^\sigma(t))^p + \alpha(t)u^q(t)\right] + \varphi(t)u^p(\sigma(t))e_\varphi(t,t_0) \\ &= e_\varphi(t,t_0)\beta(t) - \varphi(t)e_\varphi(t,t_0)u^p(\sigma(t)) \\ &\quad + \alpha(t)e_\varphi(t,t_0)u^q(t) + \varphi(t)u^p(\sigma(t))e_\varphi(t,t_0) \\ &= e_\varphi(t,t_0)\beta(t) + \alpha(t)e_\varphi(t,t_0)u^q(t) \\ &= b(t) + a(t)w^\lambda(t), \end{aligned} \tag{2.5}$$

where

$$\lambda = \frac{q}{p}, \quad b(t) := e_\varphi(t,t_0)\beta(t) \quad \text{and} \quad a(t) := \alpha(t)\left(e_\varphi(t,t_0)\right)^{1-\lambda} > 0. \tag{2.6}$$

Define

$$\eta(t) := \frac{e_\varphi(t,t_0)}{\pi^p(t)}. \tag{2.7}$$

From (2.4) and (2.7), we have

$$w(t_0) = u(t_0) < \frac{1}{\pi^p(t_0)} = \eta(t_0), \tag{2.8}$$

where $e_\varphi(t_0,t_0) = 1$. It follows from the inequalities (2.3), (2.5) and (2.8) that

$$\begin{aligned} w^\Delta(t_0) &\leqslant \beta(t_0) + \alpha(t_0)u^q(t_0) \leqslant \beta(t_0) + \frac{\alpha(t_0)}{\pi^q(t_0)} \\ &\leqslant \frac{1}{(\pi^\sigma(t_0))^p}\left[\varphi(t_0) - \frac{(\pi^p)^\Delta(t_0)}{\pi^p(t_0)}\right] \\ &= \frac{e_\varphi(t_0,t_0)}{(\pi^\sigma(t_0))^p}\left[\varphi(t_0) - \frac{(\pi^p)^\Delta(t_0)}{\pi^p(t_0)}\right]. \end{aligned} \tag{2.9}$$

Using the quotient rule in (1.10), we note that

$$\begin{aligned} \frac{e_\varphi(t,t_0)}{(\pi^\sigma(t))^p}\left[\varphi(t) - \frac{(\pi^p)^\Delta(t)}{\pi^p(t)}\right] &= \frac{\varphi(t)e_\varphi(t,t_0)}{(\pi^\sigma(t))^p} - \frac{e_\varphi(t,t_0)(\pi^p)^\Delta(t)}{\pi^p(t)\,(\pi^\sigma(t))^p} \\ &= \frac{\varphi\pi^p(t)e_\varphi(t,t_0) - e_\varphi(t,t_0)(\pi^p)^\Delta(t)}{\pi^p(t)\,(\pi^\sigma(t))^p} \\ &= \left(\frac{e_\varphi(t,t_0)}{\pi^p(t)}\right)^\Delta. \end{aligned} \tag{2.10}$$

This, (2.7) and (2.9) imply that

$$w^\Delta(t_0) \leqslant \left.\left(\frac{e_\varphi(t,t_0)}{\pi^p(t)}\right)^\Delta\right|_{t=t_0} = \eta^\Delta(t_0). \tag{2.11}$$

From (2.8) and (2.11), it follows that there exists $\varepsilon > 0$, such that

$$w(t) \leqslant \eta(t), \quad \text{for } t_0 \leqslant t \leqslant T, \tag{2.12}$$

where $\varepsilon$ is chosen so that $T = t + \varepsilon \in \mathbb{T}$. Now, we prove that if (2.12) holds, then

$$w^\Delta(t) \leqslant \eta^\Delta(t), \quad \text{for } t \in [t_0,T_1], \quad \text{for } T_1 > t_0. \tag{2.13}$$

From (2.5), (2.6) and (2.12), we see that

$$\begin{aligned} w^\Delta(t) &\leqslant e_\varphi(t,t_0)\beta(t) + \alpha(t)\left(e_\varphi(t,t_0)\right)^{1-\lambda} w^\lambda(t) \\ &\leqslant e_\varphi(t,t_0)\beta(t) + \alpha(t)\left(e_\varphi(t,t_0)\right)^{1-\lambda} \eta^\lambda(t) \\ &= e_\varphi(t,t_0)\beta(t) + \alpha(t)\left(e_\varphi(t,t_0)\right)^{1-\lambda}\left(\frac{e_\varphi(t,t_0)}{\pi^p(t)}\right)^\lambda \\ &= e_\varphi(t,t_0)\beta(t) + \frac{\alpha(t)e_\varphi(t,t_0)}{\pi^q(t)}, \quad \text{where } \lambda = \frac{q}{p}. \end{aligned}$$

This, (2.10) and (2.3) imply that

$$w^{\Delta}(t) \leqslant e_{\varphi}(t,t_0)\left[\beta(t)+\frac{\alpha(t)}{\pi^{q}(t)}\right] \leqslant \frac{e_{\varphi}(t,t_0)}{(\pi^{\sigma}(t))^{p}}\left[\varphi(t)-\frac{(\pi^{p})^{\Delta}(t)}{\pi^{p}(t)}\right]$$
$$=\left(\frac{e_{\varphi}(t,t_0)}{\pi^{p}(t)}\right)^{\Delta}=\eta^{\Delta}(t), \quad \text{for } t \geqslant T_1.$$

Denote

$$T_1 := \sup\{\delta \in \mathbb{R} : w(t) < \eta(t), \quad \text{for } t \in [t_0, t_0+\delta]_{\mathbb{T}}\}.$$

Now, we claim that $T_1 = \infty$, which says that every nonnegative solution $u(t)$ to inequality (1.6) satisfying assumption (2.4) is defined globally. Assume the contrary, i.e., $T_1 < \infty$. From the definition of $T_1$, one gets $w(T_1) < \eta(T_1)$. It follows from this and (2.13) that

$$w^{\Delta}(t) \leqslant \eta^{\Delta}(t), \quad \text{for } t \in [t_0, T_1]_{\mathbb{T}}. \tag{2.14}$$

This implies, after integrating from $t_0$ to $T_1$, that

$$w(T_1)-w(t_0)=\int_{t_0}^{T_1} w^{\Delta}(s)\Delta s \leqslant \int_{t_0}^{T_1} \eta^{\Delta}(s)\Delta s=\eta(T_1)-\eta(t_0).$$

Since $w(t_0) < \eta(t_0)$ by assumption (2.4), we see that

$$w(T_1) < \eta(T_1). \tag{2.15}$$

It follows from (2.14) and (2.15), as above with $t_0 = T_1$, there exists $\varepsilon_1 > 0$ such that $w(t) < \eta(t)$, for $T_1 \leqslant t \leqslant T_1+\varepsilon_1$, where $\varepsilon$ is chosen so that $T_1+\varepsilon \in \mathbb{T}$. This contradicts the definition of $T_1$ and the contradiction proves the desired conclusion $T_1 = \infty$. It follows from the definitions of $w(t)$, $\eta(t)$, and from the relation $T_1 = \infty$, that

$$u(t)=\left(\frac{w(t)}{e_{\varphi}(t,t_0)}\right)^{\frac{1}{p}} < \left(\frac{\eta(t)}{e_{\varphi}(t,t_0)}\right)^{\frac{1}{p}}=\frac{1}{\pi(t)}, \quad \text{for } t \geqslant t_0.$$

From this we see that if $\lim_{t\to\infty}\pi(t)=0$, then $\lim_{t\to\infty}u(t)=0$. The proof is complete. □

REMARK 2.1. In Theorem 2.1, if we assume that there exists a positive function $f(t)$ such that

$$\varphi(t)-\frac{(\pi^{p})^{\Delta}(t)}{\pi^{p}(t)}=f(t),$$

then $\pi^{p}(t)=e_{\varphi-f}(t,t_0)$, with $\pi(t_0)=1$. Using this in Theorem 2.1, we get the following result.

THEOREM 2.2. *Assume that* $(H_1)$ *holds and there exists a positive rd-continuous function* $f(t)$ *such that*

$$\beta(t)+\frac{\alpha(t)}{\left(e_{\varphi-f}(t,t_0)\right)^{\frac{q}{p}}}\leqslant\frac{f(t)}{\left(e_{\varphi-f}(\sigma(t),t_0)\right)}.$$

*Let* $u(t)\geqslant 0$ *is a solution of the inequality* (1.6) *such that* $u(t_0)<1$. *Then* $u(t)$ *exists globally and the following estimate holds:*

$$0\leqslant u(t)<\frac{1}{e_{\varphi-f}(t,t_0)},\quad for\ t\geqslant t_0.$$

*Consequently, if* $\varphi(t)>f(t)$, *then* $\lim_{t\to\infty}\ u(t)=0$.

With an appropriate choice of the functions $\alpha$, $\beta$, $\varphi$, $f$ and $\pi$ one can derive from Theorem 2.1 a number of results. For example, one can derive the following result.

COROLLARY 2.1. *Assume that* $(H_1)$ *holds and there exists a positive rd-continuous function* $\pi(t)>0$, $\pi\in C_r^1[t_0,\infty)_{\mathbb{T}}$, *and* $\theta\in(0,1)$ *such that*

$$\alpha(t)\leqslant\frac{\theta}{(\pi^{\sigma}(t))^{p}}\left[\varphi(t)-\frac{(\pi^{p})^{\Delta}(t)}{\pi^{p}(t)}\right],\ \beta(t)\leqslant\frac{1-\theta}{(\pi^{\sigma}(t))^{p}}\left[\varphi(t)-\frac{(\pi^{p})^{\Delta}(t)}{\pi^{p}(t)}\right].$$

*Let* $u(t)\geqslant 0$ *is a solution of the inequality* (1.6) *such that* $\pi(t_0)u(t_0)<1$. *Then* $u(t)$ *exists globally and* $0\leqslant u(t)<1/\pi(t)$, *for* $t\geqslant t_0$. *Consequently, if* $\lim_{t\to\infty}\pi(t)=\infty$, *then* $\lim_{t\to\infty}\ u(t)=0$.

Next, in the following, we consider the integral form of (1.6) on a time scale $\mathbb{T}$. This inequality given by

$$u^{p}(t)\leqslant a(t)+b(t)\int_{t_0}^{t}\left[f(s)u^{q}(s)-g(s)u^{p}(\sigma(s))\right]\Delta s,\ \text{for all}\ t\in[t_0,\infty)_{\mathbb{T}}.\tag{2.16}$$

Our aim is to establish an explicit bound of the unknown function $u(t)$ where $u(t)\geqslant 0$. For (2.16) we will assume the following hypotheses:

$$(H_3)\quad\begin{cases}a,\ f \text{ and } g \text{ are positive rd-continuous functions defined on } [t_0,\infty)_{\mathbb{T}},\\ u(t)\geqslant 0, \text{ for all } t\geqslant t_0, \text{ where } t_0\geqslant 0 \text{ is a fixed number},\\ p,\ q \text{ are positive constants such that } p>q\geqslant 1.\end{cases}$$

THEOREM 2.3. *Let* $\mathbb{T}$ *be an unbounded time scale with* $t_0$ *and* $t\in\mathbb{T}$. *Assume that* $(H_3)$ *holds. Then* (2.16) *implies*

$$u(t)\leqslant a^{\frac{1}{p}}(t)+\frac{q}{p}a^{\frac{1}{p}-1}(t)b(t)\left[\int_{t_0}^{t}e_{\left(a^{\frac{q}{p}}f\right)}(t,\sigma(s))f(s)a^{\frac{q}{p}-1}(s)\Delta s\right],\ \ t\in[t_0,\infty)_{\mathbb{T}}.\tag{2.17}$$

*Proof.* Define a function $z(t)$ by

$$z(t) = \int_{t_0}^{t} [f(s)u^q(s) - g(s)u^p(\sigma(s))]\,\Delta s. \tag{2.18}$$

Then $z(t_0) = 0$, and (2.16) can be written as

$$u^p(t) \leqslant a(t) + b(t)z(t).$$

From this, we get that

$$u(t) \leqslant (a(t)+b(t)z(t))^{\frac{1}{p}} = a^{\frac{1}{p}}(t)\left(1+\frac{b(t)}{a(t)}z(t)\right)^{\frac{1}{p}}. \tag{2.19}$$

This implies that

$$u^q(t) \leqslant a^{\frac{q}{p}}(t)(1+\frac{z(t)}{a(t)})^{\frac{q}{p}}. \tag{2.20}$$

Applying Lemma 2.1 on (2.19) and (2.20), we have (note that $p \geqslant 1$),

$$\begin{aligned} u(t) &\leqslant (a(t)+b(t)z(t))^{\frac{1}{p}} = a^{\frac{1}{p}}(t) + \frac{1}{p}a^{\frac{1}{p}-1}(t)b(t)z(t) \\ &\leqslant a^{\frac{1}{p}}(t) + a^{\frac{1}{p}-1}(t)b(t)z(t), \end{aligned} \tag{2.21}$$

and

$$u^q(t) \leqslant a^{\frac{q}{p}}(t) + \frac{q}{p}a^{\frac{q}{p}-1}(t)z(t).$$

From (2.18), we see that

$$z^{\Delta}(t) = f(t)u^q(t) - g(t)u^p(\sigma(t)) \leqslant f(t)u^q(t) = a^{\frac{q}{p}}(t)f(t) + \frac{q}{p}f(t)a^{\frac{q}{p}-1}(t)z(t).$$

So that

$$z^{\Delta}(t) \leqslant a^{\frac{q}{p}}(t)f(t) + \frac{q}{p}f(t)a^{\frac{q}{p}-1}(t)z(t).$$

Using the fact that $z(t_0) = 0$ and the comparison theorem [4, Theorem 6.1], we see that

$$z(t) \leqslant \frac{q}{p}\int_{t_0}^{t} e_{a^{\frac{q}{p}}f}(t,\sigma(s))f(s)a^{\frac{q}{p}-1}(s)\Delta s, \quad \text{for all } t \in [t_0,\infty)_{\mathbb{T}}. \tag{2.22}$$

Substituting (2.22) into (2.21), we have

$$u(t) \leqslant a^{\frac{1}{p}}(t) + \frac{q}{p}a^{\frac{1}{p}-1}(t)b(t)\left[\int_{t_0}^{t} e_{a^{\frac{q}{p}}f}(t,\sigma(s))f(s)a^{\frac{q}{p}-1}(s)\Delta s\right],$$

which is the desired inequality (2.17). The proof is complete. □

From Theorem 2.3, when $\mathbb{T} = \mathbb{R}$, $a(t) = k^2$, $b(t) = 2$, $g(t) = 0$, $q = 1$ and $p = 2$, we have the following result which is different from the result established by Ou-Iang [12].

COROLLARY 2.2. *Let $\mathbb{T}=\mathbb{R}$ with $t_0$, $t\in\mathbb{R}$. Assume that $k>0$ and $f$ is a positive function. Then*

$$u^2(t)\leqslant k^2+2\int_0^t f(s)u(s)ds,\ \ or\ all\ \ t\in[0,\infty).$$

*implies*

$$u(t)\leqslant k+\int_{t_0}^t f(s)\exp\left(k\int_s^t f(\theta)d\theta\right)ds,\ \ t\geqslant t.$$

REMARK 2.2. As in Corollary 2.2, one can use Theorem 2.3 to establish some new explicit bounds for the inequalities (1.8) and (1.9). The new bounds will be different form the results that has been established by Dafermos [6] and Pachpatte [13]. The details are left to the interested reader.

Now, we consider (1.6) and give an explicit bound of the unknown function $u(t)$. We introduce the following notations:

$$\begin{aligned}A(t)\ :&=F(t)+\int_{t_0}^t F(s)G(s)e_G(t,\sigma(s))\Delta s,\\ F(t)\ :&=\int_{t_0}^t[f(s)a^{\frac{\delta}{\gamma}}(s)+g(s)a^{\frac{\alpha}{\gamma}}(s)]\Delta s,\\ G(t)\ :&=b(t)\left[\frac{\delta}{\gamma}a^{\frac{\delta}{\gamma}-1}(t)f(t)+\frac{\alpha}{\gamma}a^{\frac{\alpha}{\gamma}-1}(t)g(t)\right].\end{aligned}$$

THEOREM 2.4. *Let $\mathbb{T}$ be an unbounded time scale with $t_0$ and $t\in\mathbb{T}$. Assume that $(H_2)$ holds, and $\delta\leqslant\gamma$, and $\alpha\leqslant\gamma$. Then*

$$u^\gamma(t)\leqslant a(t)+b(t)\int_{t_0}^t\left[f(s)u^\delta(s)+g(s)u^\alpha(s)\right]\Delta s,\ \ for\ \ t\in[t_0,\infty)_{\mathbb{T}},\tag{2.24}$$

*implies that*

$$u(t)\leqslant a^{\frac{1}{\gamma}}(t)+\frac{1}{\gamma}a^{\frac{1}{\gamma}-1}(t)b(t)A(t),\ \ t\in[t_0,\infty)_{\mathbb{T}}.\tag{2.25}$$

*Proof.* Define a function $y(t)$ by

$$y(t):=\int_{t_0}^t\left[f(s)u^\delta(s)+g(s)u^\alpha(s)\right]\Delta s.$$

This reduces (2.24) to

$$u^\gamma(t)\leqslant a(t)+b(t)y(t),\ \ \text{for}\ \ t\in[t_0,\infty)_{\mathbb{T}}.\tag{2.26}$$

This implies that

$$u(t)\leqslant(a(t)+b(t)y(t))^{\frac{1}{\gamma}},\ \ \text{for}\ \ t\in[t_0,\infty)_{\mathbb{T}}.\tag{2.27}$$

Applying Lemma 2.1 on (2.27), we see that

$$u(t) \leqslant a^{\frac{1}{\gamma}}(t) + \frac{1}{\gamma} a^{\frac{1}{\gamma}-1}(t) b(t) y(t), \text{ for } t \in [t_0, \infty)_{\mathbb{T}}. \tag{2.28}$$

From (2.26) we obtain

$$u^{\alpha}(t) \leqslant a^{\frac{\alpha}{\gamma}}(t) \left[1 + \frac{b(t)y(t)}{a(t)}\right]^{\frac{\alpha}{\gamma}}, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}. \tag{2.29}$$

Applying Lemma 2.1 again on (2.29) (where $\alpha \leqslant \gamma$), we obtain

$$u^{\alpha}(t) \leqslant a^{\frac{\alpha}{\gamma}}(t) \left[1 + \frac{\alpha}{\gamma} \frac{b(t)}{a(t)} y(t)\right] = a^{\frac{\alpha}{\gamma}}(t) + \frac{\alpha}{\gamma} a^{\frac{\alpha}{\gamma}-1}(t) b(t) y(t), \ t \in [t_0, \infty)_{\mathbb{T}}. \tag{2.30}$$

Also from (2.26), we obtain

$$u^{\delta}(t) \leqslant a^{\frac{\delta}{\gamma}}(t) \left[1 + \frac{b(t)y(t)}{a(t)}\right]^{\frac{\delta}{\gamma}}, \qquad \text{for } t \in [t_0, \infty)_{\mathbb{T}}. \tag{2.31}$$

Applying Lemma 2.1 on (2.31) (where $\delta \leqslant \gamma$), we have

$$u^{\delta}(t) \leqslant a^{\frac{\delta}{\gamma}}(t) \left[1 + \frac{\delta}{\gamma} \frac{b(t)}{a(t)} y(t)\right] = a^{\frac{\delta}{\gamma}}(t) + \frac{\delta}{\gamma} a^{\frac{\delta}{\gamma}-1}(t) b(t) y(t), \ t \in [t_0, \infty)_{\mathbb{T}}. \tag{2.32}$$

Combining (2.28), (2.30) and (2.32), we see that

$$\begin{aligned} y(t) &= \int_{t_0}^{t} \left[f(s) u^{\delta}(s) + g(s) u^{\alpha}(s)\right] \Delta s \\ &\leqslant \int_{t_0}^{t} f(s) a^{\frac{\delta}{\gamma}}(s) \Delta s + \frac{\delta}{\gamma} \int_{t_0}^{t} f(s) a^{\frac{\delta}{\gamma}-1}(s) b(s) y(s) \Delta s \\ &\quad + \int_{t_0}^{t} g(s) a^{\frac{\alpha}{\gamma}}(s) \Delta s + \frac{\alpha}{\gamma} \int_{t_0}^{t} a^{\frac{\alpha}{\gamma}-1}(s) g(s) b(s) y(s) \Delta s \\ &= F(t) + \int_{t_0}^{t} G(s) y(s) \Delta s, \text{ for } t \in [t_0, \infty)_{\mathbb{T}}. \end{aligned}$$

Now an application of Lemma 2.2 gives us that

$$y(t) < F(t) + \int_{t_0}^{t} F(s) G(s) e_G(t, \sigma(s)) \Delta s, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}. \tag{2.33}$$

Substituting (2.33) into (2.28), we obtain the desired inequality (2.25). The proof is complete. □

From Theorem 2.4, we have the following results.

COROLLARY 2.3. *Let $\mathbb{T}$ be an unbounded time scale with $t_0$ and $t \in \mathbb{T}$. Assume that $(H_2)$ holds and $\delta = \gamma$. Then*

$$u^{\gamma}(t) \leqslant a(t) + b(t) \int_{t_0}^{t} [f(s)u^{\gamma}(s) + g(s)u^{\alpha}(s)]\, \Delta s, \ \ \textit{for all} \ \ t \in [t_0, \infty)_{\mathbb{T}}, \tag{2.34}$$

*implies*

$$u(t) \leqslant a^{\frac{1}{\gamma}}(t) + a^{\frac{1}{\gamma}-1}(t)b(t)B(t), \ \ t \in [t_0, \infty)_{\mathbb{T}}, \tag{2.35}$$

*where*

$$\begin{aligned} B(t) &:= F_1(t) + \int_{t_0}^{t} F_1(s)G_1(s)e_{G_1}(t, \sigma(s))\Delta s, \\ F_1(t) &:= \int_{t_0}^{t} [f(s)a(s) + g(s)a^{\frac{\alpha}{\gamma}}(s)]\Delta s, \\ G_1(t) &:= b(t)\left[f(t) + \frac{\alpha}{\gamma} a^{\frac{\alpha}{\gamma}-1}(t)g(t)\right]. \end{aligned}$$

COROLLARY 2.4. *Let $\mathbb{T}$ be an unbounded time scale with $t_0$, $t \in \mathbb{T}$. Assume that $(H_2)$ holds and $\delta = \gamma = \alpha$. Then*

$$u^{\gamma}(t) \leqslant a(t) + b(t) \int_{t_0}^{t} [f(s) + g(s)]\, u^{\gamma}(s)\Delta s, \ \ \textit{for all} \ \ t \in [t_0, \infty)_{\mathbb{T}},$$

*implies*

$$u(t) \leqslant a^{\frac{1}{\gamma}}(t) + a^{\frac{1}{\gamma}-1}(t)b(t)C(t), \ \ t \in [t_0, \infty)_{\mathbb{T}}.$$

*where*

$$\begin{aligned} C(t) &:= F_2(t) + \int_{t_0}^{t} F_2(s)G_2(s)e_{G_2}(t, \sigma(s))\Delta s, \\ F_2(t) &:= \int_{t_0}^{t} a(s)[f(s) + g(s)]\Delta s, \\ G_2(t) &:= b(t)\,[f(t) + g(t)]. \end{aligned}$$

*Acknowledgement.* The author is very grateful to the referee for valuable remarks and comments which significantly contributed to the quality of the paper.

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(Received June 30, 2009)

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