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SHARP INEQUALITIES BETWEEN MEANS

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Abstract. For $p\in\mathbb{R}$ the p-th power mean $M_p(a,b)$, arithmetic mean $A(a,b)$, geometric mean $G(a,b)$, and harmonic mean $H(a,b)$ of two positive numbers a and b are defined by

$$M_p(a,b)=\begin{cases}\left(\frac{a^p+b^p}{2}\right)^{1/p}, & p\neq 0,\\ \sqrt{ab}, & p=0,\end{cases}$$

$A(a,b)=(a+b)/2$, $G(a,b)=\sqrt{ab}$, and $H(a,b)=2ab/(a+b)$, respectively.

In this paper, we answer the questions: For $\alpha\in(0,1)$, what are the greatest values p, r and m, and the least values q, s and n, such that the inequalities $M_p(a,b)\leqslant A^{\alpha}(a,b)G^{1-\alpha}(a,b)\leqslant M_q(a,b), M_r(a,b)\leqslant G^{\alpha}(a,b)H^{1-\alpha}(a,b)\leqslant M_s(a,b)$ and $M_m(a,b)\leqslant A^{\alpha}(a,b)H^{1-\alpha}(a,b)\leqslant M_n(a,b)$ hold for all $a,b>0$?

1. Introduction

For $p\in\mathbb{R}$ the p-th power mean $M_p(a,b)$ of two positive numbers a and b is defined by

$$M_p(a,b)=\begin{cases}\left(\frac{a^p+b^p}{2}\right)^{1/p}, & p\neq 0,\\ \sqrt{ab}, & p=0,\end{cases}$$

It is well-known that $M_p(a,b)$ is continuous and strictly increasing with respect to $p\in\mathbb{R}$ for fixed a and b with $a\neq b$. In the recent past, the power mean has been the subject of intensive research. In particular, many remarkable inequalities for $M_p(a,b)$ can be found in literature [1, 3, 4, 6-8, 10-15]. Let $A(a,b)=(a+b)/2$, $G(a,b)=\sqrt{ab}$, and $H(a,b)=2ab/(a+b)$ be the arithmetic, geometric, and harmonic means of two positive numbers a and b, respectively. Then

$$\begin{aligned}\min\{a,b\}\leqslant H(a,b)=M_{-1}(a,b)\leqslant G(a,b)=M_0(a,b)\\ \leqslant A(a,b)=M_1(a,b)\leqslant \max\{a,b\}.\end{aligned}$$

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In [2], Alzer and Janous established the following sharp double inequality (see also [5, p. 350]):

$$M_{\frac{\ln 2}{\ln 3}}(a,b) \leqslant \frac{2}{3}A(a,b) + \frac{1}{3}G(a,b) \leqslant M_{\frac{2}{3}}(a,b)$$

for all $a,b > 0$.

For $\alpha \in (0,1)$, Janous [9] found the greatest value p and least value q such that $M_p(a,b) \leqslant \alpha A(a,b) + (1-\alpha)G(a,b) \leqslant M_q(a,b)$ for all $a,b > 0$.

However the following problems remain: For $\alpha \in (0,1)$, what are the greatest values p, r and m, and the least values q, s and n, such that the inequalities

$$M_p(a,b) \leqslant A^{\alpha}(a,b)G^{1-\alpha}(a,b) \leqslant M_q(a,b),$$

$$M_r(a,b) \leqslant G^{\alpha}(a,b)H^{1-\alpha}(a,b) \leqslant M_s(a,b)$$

and

$$M_m(a,b) \leqslant A^{\alpha}(a,b)H^{1-\alpha}(a,b) \leqslant M_n(a,b)$$

hold for all a, $b > 0$? In this paper we solve these problems.

2. Main Results

THEOREM 2.1. *If $\alpha \in (0,1)$, then $A^{\alpha}(a,b)G^{1-\alpha}(a,b) \leqslant M_{\alpha}(a,b)$ for all $a,b > 0$, with equality if and only if $a = b$, and the constant α in $M_{\alpha}(a,b)$ cannot be improved.*

Proof. If $a = b$, clearly $A^{\alpha}(a,b)G^{1-\alpha}(a,b) = M_{\alpha}(a,b) = a$. Otherwise $a > b$ without loss of generality.

Let $t = a/b > 1$ and $f(t) = \log M_{\alpha}(a,b) - \log[A^{\alpha}(a,b)G^{1-\alpha}(a,b)]$. Then we have

$$f(t) = \frac{1}{\alpha}\log\frac{1+t^{\alpha}}{2} - \alpha\log\frac{1+t}{2} - \frac{1-\alpha}{2}\log t,$$

$$f(1) = 0, \tag{2.1}$$

$$f'(t) = \frac{1}{2t(1+t)(1+t^{\alpha})}g(t), \tag{2.2}$$

where $g(t) = (1-\alpha)t^{\alpha+1} + (1+\alpha)t^{\alpha} - (\alpha+1)t - (1-\alpha)$,

$$g(1) = 0, \tag{2.3}$$

$$g'(t) = \frac{1+\alpha}{t}h(t), \tag{2.4}$$

where $h(t) = (1-\alpha)t^{\alpha+1} + \alpha t^{\alpha} - t$,

$$h(1) = 0, \tag{2.5}$$

$$h'(t) = (1-\alpha^2)t^\alpha + \alpha^2 t^{\alpha-1} - 1, \tag{2.6}$$

$$h'(1) = 0 \tag{2.7}$$

and

$$h''(t) = \alpha(1-\alpha)t^{\alpha-2}[(1+\alpha)t-\alpha] > 0 \tag{2.8}$$

for $t \in (1,+\infty)$.

From (2.1)–(2.8) we get $f(t) > 0$ for $t \in (1,+\infty)$.

Next, we prove that the constant α in $M_\alpha(a,b)$ cannot be improved.

For any $\varepsilon \in (0,\alpha)$ and $x \in (0,1)$ one has

$$\begin{aligned}
&[M_{\alpha-\varepsilon}(1+x,1)]^{\alpha-\varepsilon} - [A^\alpha(1+x,1)G^{1-\alpha}(1+x,1)]^{\alpha-\varepsilon} \\
=&\frac{1}{2}\left[1+(1+x)^{\alpha-\varepsilon}\right] - \left(1+\frac{x}{2}\right)^{\alpha(\alpha-\varepsilon)}(1+x)^{\frac{(1-\alpha)(\alpha-\varepsilon)}{2}} \\
=&\left[1+\frac{\alpha-\varepsilon}{2}x + \frac{(\alpha-\varepsilon)(\alpha-\varepsilon-1)}{4}x^2 + o(x^2)\right] - \left\{1+\frac{\alpha(\alpha-\varepsilon)}{2}x\right. \\
&\left. + \frac{\alpha(\alpha-\varepsilon)[\alpha(\alpha-\varepsilon)-1]}{8}x^2 + o(x^2)\right\}\left\{1+\frac{(1-\alpha)(\alpha-\varepsilon)}{2}x\right. \\
&\left. + \frac{(1-\alpha)(\alpha-\varepsilon)}{4}\left[\frac{(1-\alpha)(\alpha-\varepsilon)}{2}-1\right]x^2 + o(x^2)\right\} \\
=&-\frac{1}{8}\varepsilon(\alpha-\varepsilon)x^2 + o(x^2) \quad (x \to 0).
\end{aligned} \tag{2.9}$$

Equation (2.9) implies that for any $\varepsilon \in (0,\alpha)$, there exists $\delta = \delta(\varepsilon,\alpha) > 0$, such that $M_{\alpha-\varepsilon}(1+x,1) < A^\alpha(1+x,1)G^{(1-\alpha)}(1+x,1)$ for $x \in (0,\delta)$. □

REMARK 2.2. If $\alpha \in (0,1)$, then $A^\alpha(a,b)G^{1-\alpha}(a,b) \geqslant M_0(a,b)$ for all $a,b > 0$, with equality if and only if $a = b$, and the constant 0 in $M_0(a,b)$ cannot be improved. In fact, if $a = b$, then clearly $A^\alpha(a,b)G^{1-\alpha}(a,b) = M_0(a,b) = a$. Otherwise $a \neq b$ and

$$\frac{A^\alpha(a,b)G^{1-\alpha}(a,b)}{M_0(a,b)} = \left(\frac{a+b}{2\sqrt{ab}}\right)^\alpha > 1.$$

Next, we show that the constant 0 in $M_0(a,b)$ cannot be improved.

For $\alpha \in (0,1)$ and any $\varepsilon \in (0,\alpha)$, it is easy to see that

$$\begin{aligned}
&\lim_{t\to+\infty}[M_\varepsilon(t,1) - A^\alpha(t,1)G^{1-\alpha}(t,1)] \\
=&\lim_{t\to+\infty}\left[\left(\frac{1+t^\varepsilon}{2}\right)^{1/\varepsilon} - \left(\frac{1+t}{2}\right)^\alpha(1+t)^{(1-\alpha)/2}\right] \\
=&+\infty.
\end{aligned} \tag{2.10}$$

Equation (2.10) implies that for $\alpha \in (0,1)$ and any $\varepsilon \in (0,\alpha)$, there exists $T = T(\varepsilon,\alpha) > 1$, such that $M_\varepsilon(t,1) > A^\alpha(t,1)G^{1-\alpha}(t,1)$ for $t \in (T,+\infty)$. □

THEOREM 2.3. *If $\alpha \in (0,1)$, then $G^{\alpha}(a,b)H^{1-\alpha}(a,b) \geqslant M_{\alpha-1}(a,b)$ for all $a,b > 0$, with equality if and only if $a = b$, and the constants $\alpha - 1$ in $M_{\alpha-1}(a,b)$ cannot be improved.*

Proof. If $a = b$, clearly $G^{\alpha}(a,b)H^{1-\alpha}(a,b) = M_{\alpha-1}(a,b) = a$. Otherwise $a > b$ without loss of generality. Let $t = a/b > 1$ and $f(t) = \log[G^{\alpha}(a,b)H^{1-\alpha}(a,b)] - \log M_{\alpha-1}(a,b)$. Then we have

$$f(t) = \frac{\alpha}{2}\log t + (1-\alpha)\log\frac{2t}{1+t} - \frac{1}{\alpha-1}\log\frac{1+t^{\alpha-1}}{2},$$

$$f(1) = 0, \tag{2.11}$$

$$f'(t) = \frac{1}{t^2(1+t)(1+t^{\alpha-1})}g(t), \tag{2.12}$$

where $g(t) = \alpha t^2/2 - (1-\alpha/2)t^{1+\alpha} + (1-\alpha/2)t - \alpha t^{\alpha}/2$,

$$g(1) = 0, \tag{2.13}$$

$$g'(t) = \alpha t - (1-\frac{\alpha}{2})(1+\alpha)t^{\alpha} - \frac{\alpha^2}{2}t^{\alpha-1} + (1-\frac{\alpha}{2}),$$

$$g'(1) = 0, \tag{2.14}$$

$$g''(t) = \alpha - \alpha(1-\frac{\alpha}{2})(1+\alpha)t^{\alpha-1} - \frac{\alpha^2}{2}(\alpha-1)t^{\alpha-2},$$

$$g''(1) = 0 \tag{2.15}$$

and

$$g'''(t) = \alpha(1-\alpha)t^{\alpha-3}[(1+\frac{\alpha}{2}-\frac{\alpha^2}{2})t + \frac{\alpha^2}{2} - \alpha] > 0 \tag{2.16}$$

for $t \in (1,+\infty)$.

From (2.11)–(2.16) we get $f(t) > 0$ for $t \in (1,+\infty)$.

Next, we prove that the constant $\alpha - 1$ in $M_{\alpha-1}(a,b)$ cannot be improved.

For any $\varepsilon \in (0, 1-\alpha)$ and $x \in (0,1)$ one has

$$\begin{aligned} &[M_{\alpha-1+\varepsilon}(1+x,1)]^{1-\alpha-\varepsilon} - [G^{\alpha}(1+x,1)H^{1-\alpha}(1+x,1)]^{1-\alpha-\varepsilon} \\ =&\frac{2(1+x)^{1-\alpha-\varepsilon}}{1+(1+x)^{1-\alpha-\varepsilon}} - \frac{(1+x)^{(1-\alpha/2)(1-\alpha-\varepsilon)}}{(1+x/2)^{(1-\alpha)(1-\alpha-\varepsilon)}} \\ =&\frac{F(x)}{[1+(1+x)^{1-\alpha-\varepsilon}][(1+x/2)^{(1-\alpha)(1-\alpha-\varepsilon)}]}, \end{aligned} \tag{2.17}$$

where

$$F(x) = 2(1+x)^{1-\alpha-\varepsilon}(1+x/2)^{(1-\alpha)(1-\alpha-\varepsilon)} - [1+(1+x)^{1-\alpha-\varepsilon}](1+x)^{(1-\alpha/2)(1-\alpha-\varepsilon)}.$$

By a Taylor series expansion

$$\begin{aligned}F(x) =&2\left[1+(1-\alpha-\varepsilon)x-\frac{1}{2}(\alpha+\varepsilon)(1-\alpha-\varepsilon)x^2+o(x^2)\right]\\&\times\Big\{1+\frac{1}{2}(1-\alpha)(1-\alpha-\varepsilon)x\\&+\frac{1}{8}(1-\alpha)(1-\alpha-\varepsilon)\big[(1-\alpha)(1-\alpha-\varepsilon)-1\big]x^2+o(x^2)\Big\}\\&-\left[2+(1-\alpha-\varepsilon)x-\frac{1}{2}(\alpha+\varepsilon)(1-\alpha-\varepsilon)x^2+o(x^2)\right]\\&\times\Big\{1+(1-\frac{\alpha}{2})(1-\alpha-\varepsilon)x\\&+\frac{1}{2}(1-\frac{\alpha}{2})(1-\alpha-\varepsilon)[(1-\frac{\alpha}{2})(1-\alpha-\varepsilon)-1]x^2+o(x^2)\Big\}\\=&\frac{1}{4}\varepsilon(1-\alpha-\varepsilon)x^2+o(x^2)\quad(x\to 0).\end{aligned}\tag{2.18}$$

Equations (2.17) and (2.18) imply that for $\alpha\in(0,1)$ and any $\varepsilon\in(0,1-\alpha)$, there exists $\delta=\delta(\alpha,\,\varepsilon)>0$, such that $M_{\alpha-1+\varepsilon}(1+x,1)>G^{\alpha}(1+x,1)H^{1-\alpha}(1+x,1)$ for $x\in(0,\delta)$. □

REMARK 2.4. If $\alpha\in(0,1)$, then $G^{\alpha}(a,b)H^{1-\alpha}(a,b)\leqslant M_0(a,b)$ for all $a,b>0$, with equality if and only if $a=b$, and the constant 0 in $M_0(a,b)$ cannot be improved. In fact, if $a=b$, then clearly $G^{\alpha}(a,b)H^{1-\alpha}(a,b)=M_0(a,b)=a$. Otherwise $a\neq b$ and

$$\frac{M_0(a,b)}{G^{\alpha}(a,b)H^{1-\alpha}(a,b)}=\left(\frac{a+b}{2\sqrt{ab}}\right)^{1-\alpha}>1.$$

Next, we show that the constant 0 in $M_0(a,b)$ cannot be improved.

For $\alpha\in(0,1)$ and any $\varepsilon\in(0,\alpha)$, it is easy to see that

$$\begin{aligned}&\lim_{t\to+\infty}[G^{\alpha}(t,1)H^{1-\alpha}(t,1)-M_{-\varepsilon}(t,1)]\\&=\lim_{t\to+\infty}\left[t^{\alpha/2}\left(\frac{2t}{1+t}\right)^{1-\alpha}-\left(\frac{1+t^{-\varepsilon}}{2}\right)^{-1/\varepsilon}\right]\\&=+\infty.\end{aligned}\tag{2.19}$$

Equation (2.19) implies that for $\alpha\in(0,1)$ and any $\varepsilon\in(0,\alpha)$, there exists $T=T(\varepsilon,\alpha)>1$, such that $M_{-\varepsilon}(t,1)<G^{\alpha}(t,1)H^{1-\alpha}(t,1)$ for $t\in(T,+\infty)$. □

THEOREM 2.5. *For $\alpha\in(0,1)$ and all $a,b>0$, we have*

(1) *If $\alpha=1/2$, then $A^{\alpha}(a,b)H^{1-\alpha}(a,b)=M_{2\alpha-1}(a,b)$;*

(2) *If $0<\alpha<1/2$, then $A^{\alpha}(a,b)H^{1-\alpha}(a,b)\geqslant M_{2\alpha-1}(a,b)$, with equality if and only if $a=b$, and the constant $2\alpha-1$ in $M_{2\alpha-1}(a,b)$ cannot be improved;*

(3) *If* $1/2 < \alpha < 1$, *then* $A^{\alpha}(a,b)H^{1-\alpha}(a,b) \leqslant M_{2\alpha-1}(a,b)$, *with equality if and only if* $a = b$, *and the constant* $2\alpha - 1$ *in* $M_{2\alpha-1}(a,b)$ *cannot be improved.*

Proof. (1) $\alpha = 1/2$. Then clearly $A^{\alpha}(a,b)H^{1-\alpha}(a,b) = M_0(a,b) = \sqrt{ab}$;

(2) $0 < \alpha < 1/2$. If $a = b$, clearly $A^{\alpha}(a,b)H^{1-\alpha}(a,b) = M_{2\alpha-1}(a,b) = a$. Otherwise $a > b$ without loss of generality. Let $t = a/b > 1$ and $f(t) = \log M_{2\alpha-1}(a,b) - \log[A^{\alpha}(a,b)H^{1-\alpha}(a,b)]$. Then we have

$$f(t) = \frac{1}{2\alpha-1}\log\frac{1+t^{2\alpha-1}}{2} - \alpha\log\frac{1+t}{2} - (1-\alpha)\log\frac{2t}{1+t}, \tag{2.20}$$

$$f(1) = 0, \tag{2.21}$$

$$f'(t) = \frac{1}{t^2(1+t)(1+t^{2\alpha-1})}g(t), \tag{2.22}$$

where $g(t) = (1-\alpha)t^{2\alpha+1} + \alpha t^{2\alpha} - \alpha t^2 - (1-\alpha)t$,

$$g(1) = 0, \tag{2.23}$$

$$g'(t) = (1-\alpha)(2\alpha+1)t^{2\alpha} + 2\alpha^2 t^{2\alpha-1} - 2\alpha t - (1-\alpha), \tag{2.24}$$

$$g'(1) = 0, \tag{2.25}$$

$$g''(t) = 2\alpha(1-\alpha)(2\alpha+1)t^{2\alpha-1} + 2\alpha^2(2\alpha-1)t^{2\alpha-2} - 2\alpha, \tag{2.26}$$

$$g''(1) = 0 \tag{2.27}$$

and

$$g'''(t) = 2\alpha(1-\alpha)(1-2\alpha)t^{2\alpha-3}[2\alpha - (2\alpha+1)t]. \tag{2.28}$$

From $0 < \alpha < 1/2$ and (2.28) we know that

$$g'''(t) < 0 \tag{2.29}$$

for $t \in (1, +\infty)$.

Therefore, $f(t) < 0$ for $t \in (1, +\infty)$ follows from (2.20)–(2.27) and (2.29).

Next, we prove that the constant $2\alpha - 1$ in $M_{2\alpha-1}(a,b)$ cannot be improved.

For any $\varepsilon \in (0, 1-2\alpha)$ and $x \in (0,1)$ one has

$$\begin{aligned}
&[M_{2\alpha-1+\varepsilon}(1+x,1)]^{1-2\alpha-\varepsilon} - [A^{\alpha}(1+x,1)H^{1-\alpha}(1+x,1)]^{1-2\alpha-\varepsilon} \\
=&\frac{2(1+x)^{1-2\alpha-\varepsilon}}{1+(1+x)^{1-2\alpha-\varepsilon}} - \frac{(1+x)^{(1-\alpha)(1-2\alpha-\varepsilon)}}{(1+x/2)^{(1-2\alpha)(1-2\alpha-\varepsilon)}} \\
=&\frac{G(x)}{[1+(1+x)^{1-2\alpha-\varepsilon}][(1+x/2)^{(1-2\alpha)(1-2\alpha-\varepsilon)}]},
\end{aligned} \tag{2.30}$$

where

$$G(x) = 2(1+x)^{1-2\alpha-\varepsilon}(1+x/2)^{(1-2\alpha)(1-2\alpha-\varepsilon)} - [1+(1+x)^{1-2\alpha-\varepsilon}](1+x)^{(1-\alpha)(1-2\alpha-\varepsilon)}.$$

By a Taylor series expansion

$$\begin{aligned}
G(x) =&2\left[1+(1-2\alpha-\varepsilon)x-\frac{1}{2}(1-2\alpha-\varepsilon)(2\alpha+\varepsilon)x^2+o(x^2)\right]\\
&\times\Big\{1+\frac{1}{2}(1-2\alpha)(1-2\alpha-\varepsilon)x\\
&+\frac{1}{8}(1-2\alpha)(1-2\alpha-\varepsilon)[(1-2\alpha)(1-2\alpha-\varepsilon)-1]x^2+o(x^2)\Big\}\\
&-\left[2+(1-2\alpha-\varepsilon)x-\frac{1}{2}(2\alpha+\varepsilon)(1-2\alpha-\varepsilon)x^2+o(x^2)\right]\\
&\times\Big\{1+(1-\alpha)(1-2\alpha-\varepsilon)x\\
&+\frac{1}{2}(1-\alpha)(1-2\alpha-\varepsilon)[(1-\alpha)(1-2\alpha-\varepsilon)-1]x^2+o(x^2)\Big\}\\
=&\frac{1}{4}\varepsilon(1-2\alpha-\varepsilon)x^2+o(x^2)\quad (x\to 0). \qquad (2.31)
\end{aligned}$$

Equations (2.30) and (2.31) imply that for any $\alpha\in(0,1/2)$ and $\varepsilon\in(0,1-2\alpha)$, there exists $\delta=\delta(\alpha,\varepsilon)>0$, such that $M_{2\alpha-1+\varepsilon}(1+x,1)>A^{\alpha}(1+x,1)H^{1-\alpha}(1+x,1)$ for $x\in(0,\delta)$;

(3) $1/2<\alpha<1$. If $a=b$, then clearly $A^{\alpha}(a,b)H^{1-\alpha}(a,b)=M_{2\alpha-1}(a,b)=a$. Otherwise $a>b$ without loss of generality. Let $t=a/b>1$ and $f(t)=\log M_{2\alpha-1}(a,b)-\log[A^{\alpha}(a,b)H^{1-\alpha}(a,b)]$. Then from $1/2<\alpha<1$ and (2.28) we have

$$g'''(t)>0 \qquad (2.32)$$

for $t\in(1,+\infty)$.

Therefore, $f(t)>0$ for $t\in(1,+\infty)$ follows from (2.20)–(2.27) and (2.32).

Next, we prove that the constant $2\alpha-1$ in $M_{2\alpha-1}(a,b)$ cannot be improved.

For any $\varepsilon\in(0,2\alpha-1)$ and $x\in(0,1)$ one has

$$\begin{aligned}
&[A^{\alpha}(1+x,1)H^{1-\alpha}(1+x,1)]^{2\alpha-1-\varepsilon}-[M_{2\alpha-1-\varepsilon}(1+x,1)]^{2\alpha-1-\varepsilon}\\
=&\left(1+\frac{x}{2}\right)^{(2\alpha-1)(2\alpha-1-\varepsilon)}(1+x)^{(1-\alpha)(2\alpha-1-\varepsilon)}-\frac{1}{2}\left[1+(1+x)^{2\alpha-1-\varepsilon}\right]\\
=&\Big\{1+\frac{(2\alpha-1)(2\alpha-1-\varepsilon)}{2}x\\
&+\frac{(2\alpha-1)(2\alpha-1-\varepsilon)[(2\alpha-1)(2\alpha-1-\varepsilon)-1]}{8}x^2+o(x^2)\Big\}\\
&\times\Big\{1+(1-\alpha)(2\alpha-1-\varepsilon)x\\
&+\frac{1}{2}(1-\alpha)(2\alpha-1-\varepsilon)[(1-\alpha)(2\alpha-1-\varepsilon)-1]x^2+o(x^2)\Big\}
\end{aligned}$$

$$
\begin{aligned}
&-\left[1+\frac{1}{2}(2\alpha-1-\varepsilon)x+\frac{(2\alpha-1-\varepsilon)(2\alpha-\varepsilon-2)}{4}x^2+o(x^2)\right]\\
=&\frac{1}{8}\varepsilon(2\alpha-1-\varepsilon)x^2+o(x^2) \quad (x\to 0). \qquad (2.33)
\end{aligned}
$$

Equation (2.33) implies that for any $\alpha\in(1/2,1)$ and $\varepsilon\in(0,2\alpha-1)$, there exists $\delta=\delta(\varepsilon,\alpha)>0$, such that $M_{2\alpha-1-\varepsilon}(1+x,1)<A^{\alpha}(1+x,1)H^{1-\alpha}(1+x,1)$ for $x\in(0,\delta)$. $\square$

REMARK 2.6. If $\alpha\in(0,1/2)$, then $A^{\alpha}(a,b)H^{1-\alpha}(a,b)\leqslant M_0(a,b)$ for all $a,b>0$, with equality if and only if $a=b$, and the constant 0 in $M_0(a,b)$ cannot be improved. In fact, if $a=b$, then clearly $A^{\alpha}(a,b)H^{1-\alpha}(a,b)=M_0(a,b)=a$. Otherwise $a\neq b$ and

$$M_0(a,b)-A^{\alpha}(a,b)H^{1-\alpha}(a,b)=\sqrt{ab}\left[1-\left(\frac{a+b}{2\sqrt{ab}}\right)^{2\alpha-1}\right]>0.$$

Next, we prove that the constant 0 in $M_0(a,b)$ cannot be improved.

For $\alpha\in(0,1/2)$ and any $\varepsilon\in(0,1-2\alpha)$, it is easy to see that

$$
\begin{aligned}
&\lim_{t\to+\infty}[M_{-\varepsilon}(t,1)]-A^{\alpha}(t,1)H^{1-\alpha}(t,1)\\
=&\lim_{t\to+\infty}\left[\left(\frac{1+t^{-\varepsilon}}{2}\right)^{-1/\varepsilon}-\left(\frac{1+t}{2}\right)^{\alpha}\left(\frac{2t}{1+t}\right)^{1-\alpha}\right]\\
=&-\infty. \qquad (2.34)
\end{aligned}
$$

Equation (2.34) implies that for $\alpha\in(0,1/2)$ and any $\varepsilon\in(0,1-2\alpha)$, there exists $T=T(\varepsilon,\alpha)>0$, such that $M_{-\varepsilon}(t,1)<A^{\alpha}(t,1)H^{1-\alpha}(t,1)$ for $t\in(T,+\infty)$. $\square$

Similarly, we have

REMARK 2.7. If $\alpha\in(1/2,1)$, then $A^{\alpha}(a,b)H^{1-\alpha}(a,b)\geqslant M_0(a,b)$ for all $a,b>0$, with equality if and only if $a=b$, and the constant 0 in $M_0(a,b)$ cannot be improved.

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