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ORDER AMONG POWER OPERATOR MEANS WITH CONDITION ON SPECTRA

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Abstract. As a continuation of our previous considerations about the order among power operator means [Linear Algebra Appl. 434 (2011), 1228–1237], we observe this order for other cases.

1. Introduction

We recall some notations and definitions. Let $\mathscr{B}(H)$ be the C*-algebra of all bounded linear operators on a Hilbert space H and 1_H stands for the identity operator. We define bounds of an operator $A \in \mathscr{B}(H)$

$$m_A = \inf_{\|x\|=1} \langle Ax, x\rangle \quad \text{and} \quad M_A = \sup_{\|x\|=1} \langle Ax, x\rangle. \tag{1}$$

If $\mathrm{Sp}(A)$ denotes the spectrum of a self-adjoint operator A, then it is well known that $\mathrm{Sp}(A)$ is real and $\mathrm{Sp}(A) \subseteq [m_A, M_A]$.

B. Mond and J. Pečarić in [7] proved the following version of Jensen's operator inequality:

$$f\left(\sum_{i=1}^{n} w_i \Phi_i(A_i)\right) \leqslant \sum_{i=1}^{n} w_i \Phi_i f(A_i), \tag{2}$$

where f is an operator convex functions defined on an interval I, $\Phi_i : \mathscr{B}(H) \to \mathscr{B}(K)$, $i = 1,\dots,n$, are unital positive linear mappings, $A_1,\dots,A_n$ are self-adjoint operators with spectra in I and $w_1,\dots,w_n$ are non-negative real numbers with $\sum_{i=1}^{n} w_i = 1$.

F. Hansen, J. Pečarić and I. Perić in [2] gave a generalization of (2) for a unital field of positive linear mappings. Recently, J.Mićić, J.Pečarić and Y.Seo in [6] gave a generalization of this results for field of positive linear mappings such that the field $t \to \Phi_t(1_H)$ is integrable with $\int_T \Phi_t(1_H)\,d\mu(t) = k1_K$ for some positive scalar k.

Very recently, J. Mićić, Z. Pavić and J. Pečarić [3, Theorem 1] gave the following version of Jensen's operator inequality without operator convexity:

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THEOREM A. *Let $(A_1,\ldots,A_n)$ be an $n-$tuple of self-adjoint operators $A_i \in \mathscr{B}(H)$ with bounds m_i and M_i, $m_i \leqslant M_i$, $i=1,\ldots,n$. Let $(\Phi_1,\ldots,\Phi_n)$ be an $n-$tuple of positive linear mappings $\Phi_i : \mathscr{B}(H) \to \mathscr{B}(K)$, $i=1,\ldots,n$, such that $\sum_{i=1}^n \Phi_i(1_H) = 1_K$. If*

$$(m_A, M_A) \cap [m_i, M_i] = \emptyset \quad \text{for } i=1,\ldots,n,$$

where m_A and M_A, $m_A \leqslant M_A$, are bounds of the self-adjoint operator $A = \sum_{i=1}^n \Phi_i(A_i)$, then

$$f\left(\sum_{i=1}^n \Phi_i(A_i)\right) \leqslant \sum_{i=1}^n \Phi_i\left(f(A_i)\right) \tag{3}$$

holds for every continuous convex function $f : I \to \mathbb{R}$ provided that the interval I contains all m_i, M_i.

If $f : I \to \mathbb{R}$ is concave, then the reverse inequality is valid in (3).

In the same paper [3], we observe the operator power mean

$$M_n^{[r]}(\mathbf{A},\boldsymbol{\Phi}) = \begin{cases} \left(\sum_{i=1}^n \Phi_i\left(A_i^r\right)\right)^{1/r}, & r \in \mathbb{R}\backslash\{0\}, \\ \exp\left(\sum_{i=1}^n \Phi_i\left(\ln\left(A_i\right)\right)\right), & r = 0. \end{cases} \tag{4}$$

where $(A_1,\ldots,A_n)$ is an $n-$tuple of strictly positive operators in $\mathscr{B}(H)$ and $(\Phi_1,\ldots,\Phi_n)$ is an $n-$tuple of positive linear mappings $\Phi_i : \mathscr{B}(H) \to \mathscr{B}(K)$ such that $\sum_{i=1}^n \Phi_i(1_H) = 1_K$.

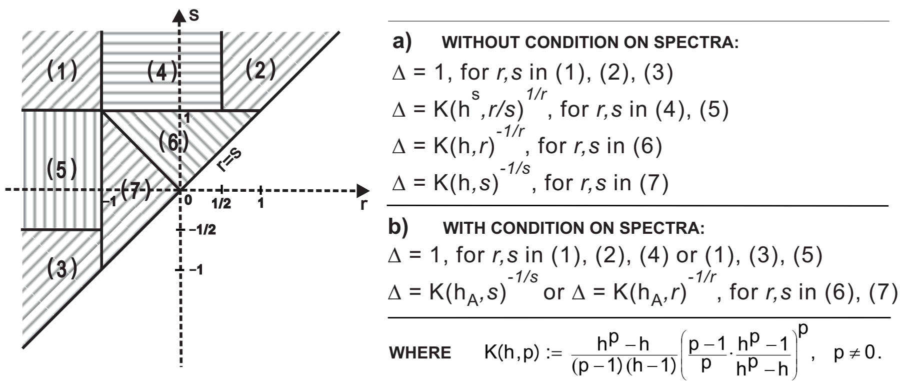

Figure 1: Regions for the order $M_n^{[r]}(\mathbf{A},\boldsymbol{\Phi}) \leqslant \Delta M_n^{[s]}(\mathbf{A},\boldsymbol{\Phi})$

Figure 1 shows regions (1),(2),(3) in which the monotonicity of the power mean

$$M_n^{[r]}(\mathbf{A},\boldsymbol{\Phi}) \leqslant M_n^{[s]}(\mathbf{A},\boldsymbol{\Phi})$$

holds true [3, Corollary 6], also Figure 1 shows regions (1)-(5) which this holds true with condition on spectra [3, Corollary 7]. We show in [3, Example 2] that the order among power means does not hold generally without a condition on spectra in regions (4),(5) in Figure 1.

In this paper we observe the order among power operator means with condition on spectra in the triangle $r,s \in [-1,1]$, $r \leqslant s$.

2. Results

As a continuation of our previous considerations [3, Corollary 7], we shall give the order among power means (4) with condition on spectra in regions (6) and (7) in Figure 1.

First, we recall that without condition on spectra we have the order between power means given in the following corollary. This is part of [4, Theorem 11], see also [6].

COROLLARY B. *Let* $(A_1,\dots,A_n)$ *and* $(\Phi_1,\dots,\Phi_n)$ *be as in the definition of the power mean* (4). *Let* $\mathrm{Sp}(A_i) \subseteq [m,M]$ *for some* $0<m<M$.

(i) If either $-s \leqslant r < s/2$, $r \neq 0$, $0 < s \leqslant 1$, *then*

$$M_n^{[r]}(\mathbf{A},\mathbf{\Phi}) \leqslant \min\{\Delta(h,r,1),\Delta(h,s,1)\Delta(h,r,s)\}\, M_n^{[s]}(\mathbf{A},\mathbf{\Phi}) \tag{5}$$

(ii) If either $r/2 < s \leqslant -r$, $s \neq 0$, $-1 \leqslant r < 0$ *or* $r \leqslant s \leqslant 2r$, $0 < s \leqslant 1$ *or* $2s \leqslant r \leqslant s$, $-1 \leqslant r < 0$, *then*

$$M_n^{[r]}(\mathbf{A},\mathbf{\Phi}) \leqslant \Delta(h,s,1)\, M_n^{[s]}(\mathbf{A},\mathbf{\Phi}), \tag{6}$$

where the constant $\Delta(h,r,s)$ *is defined as follows*

$$\Delta(h,r,s) = \left\{\frac{r(h^s-h^r)}{(s-r)(h^r-1)}\right\}^{1/s} \left\{\frac{s(h^r-h^s)}{(r-s)(h^s-1)}\right\}^{-1/r}, \qquad h=\frac{M}{m}. \tag{7}$$

Applying Theorem A, we obtain new bounds.

THEOREM 1. *Let* $(A_1,\dots,A_n)$ *and* $(\Phi_1,\dots,\Phi_n)$ *be as in the definition of the power mean* (4). *Let* m_i *and* M_i, $0 < m_i \leqslant M_i$ *be bounds of* A_i, $i=1,\dots,n$. *Let* $r,s \in (-1,1)$, $r \leqslant s$ *(Figure 1. (6),(7)).*

(i) If

$$\left(m^{[r]}, M^{[r]}\right) \cap [m_i, M_i] = \emptyset, \qquad i=1,\dots,n,$$

where $m^{[r]}$ *and* $M^{[r]}$, $m^{[r]} \leqslant M^{[r]}$ *are bounds of* $M_n^{[r]}(\mathbf{A},\mathbf{\Phi})$, *then*

$$M_n^{[r]}(\mathbf{A},\mathbf{\Phi}) \leqslant C(h^{[r]},s)\, M_n^{[s]}(\mathbf{A},\mathbf{\Phi}), \qquad h^{[r]} = M^{[r]}/m^{[r]}. \tag{8}$$

(ii) If

$$\left(m^{[s]}, M^{[s]}\right) \cap [m_i, M_i] = \emptyset, \qquad i = 1, \ldots, n,$$

where $m^{[s]}$ *and* $M^{[s]}$, $m^{[s]} \leqslant M^{[s]}$ *are bounds of* $M_n^{[s]}(\mathbf{A}, \mathbf{\Phi})$, *then*

$$M_n^{[r]}(\mathbf{A}, \mathbf{\Phi}) \leqslant C(h^{[s]}, r) M_n^{[s]}(\mathbf{A}, \mathbf{\Phi}), \qquad h^{[s]} = M^{[s]}/m^{[s]}. \tag{9}$$

The constant $C(h,p)$ *for* $h > 0$ *is defined as follows*

$$C(h,p) := \begin{cases} \frac{p(h-h^p)}{(1-p)(h^p-1)} \left(\frac{(p-1)(h-1)}{h^p-h}\right)^{\frac{1}{p}}, & \text{for } p \neq 0 \text{ and } h \neq 1, \\ \frac{(h-1)h^{\frac{1}{h-1}}}{e \ln h}, & \text{for } p = 0 \text{ and } h \neq 1, \\ 1, & \text{for } h = 1. \end{cases}$$

In order to prove Theorem 1, we need the operator order given in the following theorems.

THEOREM C. [5, Corollary 6.5], [8, Corollary 2.5] *If* $A, B \in \mathscr{B}(H)$, $A \geqslant B > 0$ *such that* $\mathsf{Sp}(A) \subseteq [m_A, M_A]$ *for some scalars* $0 < m_A < M_A$ *(resp.* $\mathsf{Sp}(B) \subseteq [m_B, M_B]$ *for some scalars* $0 < m_B < M_B$*), then*

$$K(m_A, M_A, p) A^p \geqslant B^p > 0 \qquad \text{for all} \quad p \geqslant 1, \tag{10}$$

$$(\text{resp. } K(m_B, M_B, p) A^p \geqslant B^p > 0 \qquad \text{for all} \quad p \geqslant 1,) \tag{11}$$

$$K(m_A, M_A, p) B^p \geqslant A^p > 0 \qquad \text{for all} \quad p \leqslant -1, \tag{12}$$

$$(\text{resp. } K(m_B, M_B, p) B^p \geqslant A^p > 0 \qquad \text{for all} \quad p \leqslant -1,) \tag{13}$$

where $K(m,M,p)$ *is the Kantorovich constant* [1, §2.7] *defined as follows*

$$(*) \quad K(m,M,p) := \frac{mM^p - Mm^p}{(p-1)(M-m)} \left(\frac{p-1}{p} \frac{M^p - m^p}{mM^p - Mm^p}\right)^p, \; p \in \mathbb{R}.$$

THEOREM D. [5, Remark 6.8] *Self-adjoint operators* $A, B \in \mathscr{B}(H)$ *with* $Sp(A) \subseteq [m_A, M_A]$ *where* $0 < m_A < M_A$ *satisfy the following implication:*

$$A \leqslant B \qquad \Longrightarrow \qquad e^A \leqslant S(e^{M_A - m_A}, 1) e^B$$

where $S(h,1)$ *is the Specht ratio* [1, §2.7] *defined as follows*

$$(**) \qquad S(h,1) := \frac{(h-1)h^{\frac{1}{h-1}}}{e \ln h}, \; h > 0.$$

Proof of Theorem 1. First we prove the case (i).
a) Let $m^{[r]} < M^{[r]}$.

Suppose that $\mathbf{0 < r \leqslant s \leqslant 1}$. Since $m_i 1_H \leqslant A_i \leqslant M_i 1_H$, $i = 1, \ldots, n$, and $m^{[r]} 1_K \leqslant M_n^{[r]}(\mathbf{A}, \mathbf{\Phi}) \leqslant M^{[r]} 1_K$, then

$$m_i^r 1_H \leqslant A_i^r \leqslant M_i^r 1_H, \quad i = 1, \ldots, n, \tag{14}$$

$$(m^{[r]})^r 1_K \leqslant \textstyle\sum_{i=1}^n \Phi_i(A_i^r) \leqslant (M^{[r]})^r 1_K. \tag{15}$$

Then,

$$\left(m^{[r]}, M^{[r]}\right) \cap [m_i, M_i] = \emptyset \quad \text{for } i = 1, \ldots, n$$

implies

$$\left((m^{[r]})^r, (M^{[r]})^r\right) \cap [m_i^r, M_i^r] = \emptyset \quad \text{for } i = 1, \ldots, n. \tag{16}$$

Putting $f(t) = t^{s/r}$, which is convex, in Theorem A and replacing A_i by A_i^r, we obtain

$$\left(\sum_{i=1}^{n} \Phi_i(A_i^r)\right)^{s/r} \leqslant \sum_{i=1}^{n} \Phi_i(A_i^s). \tag{17}$$

Now, applying (11) for $p = \frac{1}{s} \geqslant 1$ and using that

$$(m^{[r]})^s 1_K \leqslant \left(\sum_{i=1}^{n} \Phi_i(A_i^r)\right)^{s/r} \leqslant (M^{[r]})^s 1_K, \tag{18}$$

then we obtain

$$M_n^{[r]}(\mathbf{A}, \mathbf{\Phi}) \leqslant K\left((m^{[r]})^s, (M^{[r]})^s, 1/s\right) M_n^{[s]}(\mathbf{A}, \mathbf{\Phi}), \tag{19}$$

which gives the desired inequality by using $K\left((m^{[r]})^s, (M^{[r]})^s, 1/s\right) = C\left(h^{[r]}, s\right)$ (see [1, (2.97)]).

Next, suppose that $-1 \leqslant r < 0 < s \leqslant 1$. Then the reverse inequality is valid in (14) and (15). It follows that

$$\left((M^{[r]})^r, (m^{[r]})^r\right) \cap [M_i^r, m_i^r] = \emptyset \quad \text{for } i = 1, \ldots, n \tag{20}$$

holds. Putting $f(t) = t^{s/r}$, which is convex, in Theorem A and replacing A_i by A_i^r, we again obtain (17). Now, applying (11) for $p = \frac{1}{s} \geqslant 1$ and since (18) holds, then we obtain again (19).

Next, suppose that $-1 \leqslant r \leqslant s < 0$. Then the reverse inequality is valid in (14) and (15). It follows that (20) holds. Putting $f(t) = t^{s/r}$, which is concave, in Theorem A and replacing A_i by A_i^r, we obtain

$$\left(\sum_{i=1}^{n} \Phi_i(A_i^r)\right)^{s/r} \geqslant \sum_{i=1}^{n} \Phi_i(A_i^s). \tag{21}$$

Now, applying (12) for $p = \frac{1}{s} \leqslant -1$ and using that reverse inequalities is valid in (18), then we obtain

$$M_n^{[r]}(\mathbf{A}, \mathbf{\Phi}) \leqslant K\left((M^{[r]})^s, (m^{[r]})^s, 1/s\right) M_n^{[s]}(\mathbf{A}, \mathbf{\Phi}).$$

Since $K\left((M^{[r]})^s,(m^{[r]})^s,1/s\right)=K\left((m^{[r]})^s,(M^{[r]})^s,1/s\right)$ (see [1, Theorem 2.54]) we get again the desired inequality.

Suppose that $\mathbf{0=r<s\leqslant 1}$. Putting the operator concave function $f(t)=\frac{1}{s}\ln t$ in reverse of Jensen's operator inequality for a unital field of positive linear mappings (see [2]) and replace A_i by A_i^s, we obtain

$$\sum_{i=1}^{n}\Phi_i\left(\frac{1}{s}\ln A_i^s\right)\leqslant\frac{1}{s}\ln\left(\sum_{i=1}^{n}\Phi_i\left(A_i^s\right)\right),$$

i.e.

$$\ln\left(M_n^{[0]}(\mathbf{A},\mathbf{\Phi})\right)\leqslant\ln\left(M_n^{[s]}(\mathbf{A},\mathbf{\Phi})\right).$$

The spectrum of $\ln\left(M_n^{[0]}(\mathbf{A},\mathbf{\Phi})\right)$ is contained in $[\ln m^{[0]},\ln M^{[0]}]$, and after use Theorem D we get

$$M_n^{[0]}(\mathbf{A},\mathbf{\Phi})\leqslant S\left(\frac{M^{[0]}}{m^{[0]}},1\right)M_n^{[s]}(\mathbf{A},\mathbf{\Phi})=C\left(h^{[0]},0\right)M_n^{[s]}(\mathbf{A},\mathbf{\Phi}),$$

which is the desired inequality.

Suppose that $\mathbf{-1\leqslant r<s=0}$. Putting the operator convex function $f(t)=\frac{1}{r}\ln t$ in Jensen's operator inequality (see [2]) and replace A_i by A_i^r, we obtain

$$\frac{1}{r}\ln\left(\sum_{i=1}^{n}\Phi_i\left(A_i^r\right)\right)\leqslant\sum_{i=1}^{n}\Phi_i\left(\frac{1}{r}\ln A_i^r\right),$$

i.e.

$$\ln\left(M_n^{[r]}(\mathbf{A},\mathbf{\Phi})\right)\leqslant\ln\left(M_n^{[0]}(\mathbf{A},\mathbf{\Phi})\right).$$

The spectrum of $\ln\left(M_n^{[r]}(\mathbf{A},\mathbf{\Phi})\right)$ is contained in $[\ln m^{[r]},\ln M^{[r]}]$. Then applying Theorem D we get

$$M_n^{[r]}(\mathbf{A},\mathbf{\Phi})\leqslant S\left(\frac{M^{[r]}}{m^{[r]}},1\right)M_n^{[0]}(\mathbf{A},\mathbf{\Phi})=C\left(h^{[r]},0\right)M_n^{[0]}(\mathbf{A},\mathbf{\Phi}),$$

which is the desired inequality.

b) Letting $m^{[r]}\to M^{[r]}$ in inequalities

$$M_n^{[r]}(\mathbf{A},\mathbf{\Phi})\leqslant K\left((m^{[r]})^s,(M^{[r]})^s,1/s\right)M_n^{[s]}(\mathbf{A},\mathbf{\Phi}),$$

$$\text{or } M_n^{[r]}(\mathbf{A},\mathbf{\Phi})\leqslant S\left(\frac{M^{[r]}}{m^{[r]}},1\right)M_n^{[s]}(\mathbf{A},\mathbf{\Phi})$$

we obtain

$$M_n^{[r]}(\mathbf{A},\mathbf{\Phi}) \leqslant K\left((m^{[r]})^s,(m^{[r]})^s,1/s\right) M_n^{[s]}(\mathbf{A},\mathbf{\Phi})$$
$$\text{or } M_n^{[r]}(\mathbf{A},\mathbf{\Phi}) \leqslant S(1,1)\, M_n^{[s]}(\mathbf{A},\mathbf{\Phi}).$$

Since $K(m^s,M^s,1/s) = K(m,M,s)^{-1/s}$ and $\lim_{m\to M} K(m,M,s) = 1$ for all $s \in \mathbb{R}$; $\lim_{h\to 1} S(h,1) = 1$ (see [1, Theorem 2.62]), we obtain the desired inequalities in the case $m^{[r]} = M^{[r]}$.

(ii) We put $f(t) = t^{r/s}$ and we use the same technique as in the case (i). □

REMARK 2. The constant $C(h^{[r]},s)$ in RHS of (8) in Theorem 1 is not worse than the constants in RHS of (5) and (6) in Corollary B, i.e. if $r,s \in (-1,1)$, $r \leqslant s$, then

$$C(h^{[r]},s) \leqslant \min\{\Delta(h,s,1),\Delta(h,s,1)\cdot\Delta(h,r,s),\Delta(h,r,1)\},$$

where $h^{[r]} = M^{[r]}/m^{[r]}$, $h = M/m$ and $m^{[r]}$ and $M^{[r]}$, $m^{[r]} \leqslant M^{[r]}$ are bounds of $M_n^{[r]}(\mathbf{A},\mathbf{\Phi})$, such that $\left(m^{[r]},M^{[r]}\right) \cap [m_i,M_i] = \emptyset$, $i = 1,\dots,n$ and

$$m = \min\{m_1,\dots,m_n\}, \qquad M = \max\{M_1,\dots,M_n\}.$$

Indeed, we should just use the following properties of the function $(h,s) \mapsto C(h,s) = \Delta(h,s,1)$

(r1) $C(h,s)$ is strictly increasing in the first variable for $h > 1$ and $s < 1$ (see [1, Theorem 2.62 (i)]),

(r2) $C(h,s)$ is strictly decreasing in the second variable for $h > 1$ and $s \in \mathbb{R}$ (see [4, Lemma 12]).

So, let $r,s \in (-1,1)$, $r \leqslant s$. Since $[m^{[r]},M^{[r]}] \subseteq [m,M]$, it follows by (r1) that

$$C(h^{[r]},s) = \Delta(h^{[r]},s,1) \leqslant \Delta(h,s,1);$$

since $\Delta(h,r,s) \geqslant 1$, then

$$C(h^{[r]},s) \leqslant \Delta(h,s,1)\cdot\Delta(h,r,s);$$

and it follows by (r2) that

$$C(h^{[r]},s) \leqslant C(h^{[r]},r) \leqslant \Delta(h,r,1).$$

The three inequalities above give the desired relation.

Similarly, we can observe that the constant $C(h^{[s]},r)$ in RHS of (9) in Theorem 1 is not worse than the constants in RHS of (5) and (6) in Corollary B.

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