Mathematical
Inequalities
& Applications
Volume 14, Number 3 (2011), 737–745

OPTIMAL EMBEDDINGS OF GENERALIZED INHOMOGENEOUS SOBOLEV SPACES ON $\mathbf{R}^n$

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Abstract. We prove optimal embeddings in the subcritical case of the inhomogeneous Sobolev spaces built-up over function spaces in $\mathbf{R}^n$ with $K-$ monotone and rearrangement invariant norm into another rearrangement invariant function spaces. The investigation is based on pointwise and integral estimates of the rearrangement or the oscillation of the rearrangement of f in terms of the rearrangement of the derivatives of f.

1. Introduction

Let L_{loc} be the space of all locally integrable functions f on $\mathbf{R}^n$, $n \geqslant 2$ with the Lebesgue measure, finite almost everywhere. Denote by L^p, $1 \leqslant p < \infty$, the usual Lebesgue space with a norm $\|f\|_p = (\int_{\mathbf{R}^n} |f(x)|^p dx)^{1/p}$. The classical inhomogeneous Sobolev space W_p^k is defined as a Banach space consisting of all $f \in L^p$, such that all generalized derivatives up to order k are in L^p, and having a norm $\|f\|_{W_p^k} = \left\| |D^k f| \right\|_p + \|f\|_p$, where $|D^k f| := \sum_{|\alpha|=k} |D^\alpha f|$. The following continuous embedding is well known:

$$W_p^k \hookrightarrow L^{r,p},\ 1/r = 1/p - k/n,\ k/n < 1/p < 1, \tag{1.1}$$

where $L^{r,p}$ stands for the Lorentz space with a norm

$$\|f\|_{L^{r,p}} = \left(\int_0^\infty t^{p/r} [f^{**}(t)]^p dt/t \right)^{1/p}.$$

Here $f^{**}(t) := \frac{1}{t}\int_0^t f^*(s)ds$ and f^* is the decreasing rearrangement of f, given by $f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \leqslant t\}$, $t > 0$, where μ_f is the distribution function of f, defined by $\mu_f(\lambda) = |\{x \in \mathbf{R}^n : |f(x)| > \lambda\}|_n$, and $|\cdot|_n$ denotes Lebesgue's $n-$ measure. We use the notations $a_1 \lesssim a_2$ or $a_2 \gtrsim a_1$ for nonnegative functions or functionals to mean that the quotient a_1/a_2 is bounded; also, $a_1 \approx a_2$ means that $a_1 \lesssim a_2$ and $a_1 \gtrsim a_2$. We say that a_1 is equivalent to a_2 if $a_1 \approx a_2$.

Note that the embedding (1.1) is not optimal in the sense that the target space $L^{r,p}$ can be replaced by a smaller one, namely the intersection $L^{r,p} \cap L^p$. The norm in the

Mathematics subject classification (2010): 46E35.

Keywords and phrases: Sobolev inhomogeneous spaces, optimal embeddings, rearrangement invariant spaces.

Research partially supported by the Abdus Salam School of Mathematical Sciences, GC University Lahore, and by a grant from HEC, Pakistan.

Paper MIA-14-62

intersection is the maximum of the two corresponding norms. What will be important in the sequel is the following relation

$$\|f\|_{L^{r,p}\cap L^p} \approx \left(\int_0^1 t^{p/r}[f^*(t)]^p dt/t\right)^{1/p} + \left(\int_1^\infty [f^*(t)]^p dt\right)^{1/p}, \ r > p. \tag{1.2}$$

The main goal of this paper is to generalize the embedding (1.1), considering the generalized inhomogeneous Sobolev spaces W^kE, built-up over rearrangement invariant spaces E, with a norm $\|f\|_{W^kE} := \||D^kf|\|_E + \|f\|_E$. More precisely, we suppose that $\|f\|_E := \rho_E(f^*)$, where ρ_E is a norm defined on $\mathscr{M}^+$, the space of all non-negative measurable functions on $(0,\infty)$ with respect to the Lebesgue measure, finite almost everywhere and $f^*(\infty) = 0$. We suppose that ρ_E is rearrangement invariant and $K-$monotone in the sense (cf. [1], Definition 1.16, p. 305):

$$\int_0^t g_1^*(s)ds \leqslant \int_0^t g_2^*(s)ds \text{ implies } \rho_E(g_1^*) \leqslant \rho_E(g_2^*), \ g_1, \ g_2 \in \mathscr{M}^+. \tag{1.3}$$

Then $\|f\|_E$ satisfies the triangle inequality, since $(f+g)^{**} \leqslant (f^*+g^*)^{**}$.

We also require that the norm ρ_E satisfies the Minkovski inequality:

$$\rho_E\left(\int_0^\infty \varphi(u)g(tu)du\right) \leqslant \int_0^\infty \varphi(u)\rho_E(g(tu))du, \ \varphi, g \in \mathscr{M}^+. \tag{1.4}$$

For example, if E is a rearrangement invariant Banach function space as in [1], then by the Luxemburg representation theorem $\|f\|_E = \rho_E(f^*)$ for some norm ρ_E satisfying (1.3) and (1.4). More general example is given by the Riesz-Fischer monotone spaces as in [1], p. 305.

Note that we have the equivalence that can be established as in [1] p. 337,

$$\|f\|_{W^kE} \approx \sum_{j=0}^k \||D^jf|\|_E. \tag{1.5}$$

Recall the definition of the lower and upper Boyd indices α_E and β_E. Let

$$h_E(s) = \sup\left\{\frac{\rho_E(g_s^*)}{\rho_E(g^*)} : g \in \mathscr{M}^+\right\}, \ g_s(t) := g(t/s)$$

be the dilation function generated by ρ_E. Then

$$\alpha_E := \sup_{0<t<1} \frac{\log h_E(t)}{\log t} \text{ and } \beta_E := \inf_{1<t<\infty} \frac{\log h_E(t)}{\log t}.$$

If ρ_E is monotone, then the function h_E is submultiplicative, increasing, $h_E(1) = 1$, $1 \leqslant h_E(s)h_E(1/s)$, therefore $0 \leqslant \alpha_E \leqslant \beta_E$. If ρ_E is $K-$monotone, then by interpolation, (analogously to [1], p. 148) we see that $h_E(s) \leqslant \max(1,s)$. Hence $0 \leqslant \alpha_E \leqslant \beta_E \leqslant 1$.

We suppose that $\alpha_E > k/n$, i.e. in this paper we consider the subcritical case. If $E = L^p$, then $\alpha_E = 1/p$ and the condition $k/n < 1/p$ appears in (1.1).

If $\beta_E < 1$, we have analogously to [1] p. 150,

$$\rho_E(f^*) \approx \rho_E(f^{**}). \tag{1.6}$$

The condition $\beta_E < 1$ is equivalent to (see [1] p. 147),

$$\int_0^1 h_E(1/s)ds < \infty. \tag{1.7}$$

For example, consider the classical Lorentz spaces $\Lambda^q(w),\ 1 \leqslant q \leqslant \infty,\ w$ - positive weight, i.e. a positive function from $\mathscr{M}^+$; $f \in \Lambda^q(w)$ if $\|f\|_{\Lambda^q_w} := \rho_{w,q}(f^*) < \infty$, $\rho_{w,q}(g) := (\int_0^\infty [g(t)w(t)]^q dt/t)^{1/q}$. In general, the functional $f \mapsto \|f\|_{\Lambda^q_w}$ is not a norm. But in many cases we can find an equivalent norm. Consider the so-called Γ spaces, $\Gamma^q(w)$ with a norm $\|f\|_{\Gamma^q(w)} := \rho_{w,q,\Gamma}(f^*)$, where $\rho_{w,q,\Gamma}(g) := (\int_0^\infty [g^{**}(t)w(t)]^q dt/t)^{1/q}$. The following condition should be satisfied (otherwise the space will be trivial)

$$\left(\int_0^\infty \min(1,t^{-q})w^q(t)dt/t\right)^{1/q} < \infty.$$

Then this space is continuously embedded in the sum $L^1 + L^\infty$. Using this embedding the completeness of the space can be established in a standard way. The space $E = \Gamma^q(w)$ with $\rho_E = \rho_{w,q,\Gamma}$ satisfies the conditions (1.3), (1.4). When $w(t) = t^{1/r}$, $1 \leqslant r < \infty$, we use the standard notation $L^{r,q}$ for $\Lambda^q(t^{1/r})$.

In some cases the Lorentz space $E = \Lambda^q(w),\ 1 \leqslant q < \infty$, also satisfies the conditions (1.3), (1.4). For example, if $w^q(t)/t$ is not increasing, then (see [1], p. 72), the functional $\rho_{w,q}$ is a $K-$monotone norm.

Note also that if $E = \Gamma^q(w),\ \rho_E = \rho_{w,q,\Gamma}$, then (1.6) is equivalent to $\beta_E < 1$ (see [1], p. 150).

We are going to prove optimal embeddings of W^kE into rearrangement invariant function spaces G with a norm $\|f\|_G \approx \rho_G(f^*)$, where ρ_G is a monotone norm. In order to define the classes of the domain norms ρ_E and target norms ρ_G, where optimality of the embedding $W^kE \hookrightarrow G$ is investigated, we observe, in addition to (1.2), that we have two limiting embeddings: $W_1^k \hookrightarrow \Lambda^1(t^{1-k/n})$ and $W^k\Lambda^1(t^{k/n}) \hookrightarrow L^\infty$. For these reasons, we define the class N_d of domain norms ρ_E with the following properties: ρ_E is rearrangement invariant, $K-$monotone, satisfying Minkovski's inequality, $\alpha_E > k/n$ and $E \hookrightarrow L^1 + \Lambda^1(t^{k/n})$. The class N_t of target norms ρ_G has the properties: ρ_G is monotone and $G \hookrightarrow \Lambda^1(t^{1-k/n}) + L^\infty$. Finally, the class N of couples $\rho_E \in N_d$, $\rho_G \in N_t$ is defined by the condition

$$\rho_G(\chi_{(1,\infty)}g^*) \approx \rho_E(\chi_{(1,\infty)}g^*),\ g \in \mathscr{M}^+, \tag{1.8}$$

where $\chi_{(a,b)}$ is the characteristic function of the interval (a,b).

DEFINITION 1.1. (admissible couple) We say that the couple $\rho_E,\ \rho_G$ in N is admissible if the continuous embedding is valid:

$$W^kE \hookrightarrow G. \tag{1.9}$$

Moreover, ρ_E (E) is called domain norm (domain space), and ρ_G (G) is called target norm (target space). In this way we shall reserve the letter E for the domain space and ρ_E for the domain norm, while the letter G is reserved for the target space and ρ_G for the target norm.

Now we recall the definition of optimal norms (see for example [6]).

DEFINITION 1.2. (optimal target norm) Given the domain norm $\rho_E \in N_d$, the optimal target norm, denoted by $\rho_{G(E)}$, satisfies (1.8) and is the strongest target norm in N_t, i.e.

$$\rho_G(g^*) \lesssim \rho_{G(E)}(g^*),\ g \in \mathscr{M}^+, \tag{1.10}$$

for any target norm $\rho_G \in N_t$ such that the couple ρ_E, ρ_G is admissible.

DEFINITION 1.3. (optimal domain norm) Given the target norm $\rho_G \in N_t$, the optimal domain norm, denoted by $\rho_{E(G)}$, satisfies (1.8) and is the weakest domain norm in N_d, i.e.

$$\rho_{E(G)}(g^*) \lesssim \rho_E(g^*),\ g \in \mathscr{M}^+, \tag{1.11}$$

for any domain norm $\rho_E \in N_d$ such that the couple ρ_E, ρ_G is admissible.

DEFINITION 1.4. (optimal couple) The admissible couple $\rho_E,\ \rho_G$ in N is said to be optimal if $\rho_E = \rho_{E(G)}$ and $\rho_G = \rho_{G(E)}$.

The optimal norms are uniquely determined up to equivalence, while the corresponding optimal Banach spaces are unique. Our main result is Theorem 3.3, where for a given domain norm $\rho_E \in N_d$ we construct an optimal target norm and prove that the couple is optimal. For example, the couple $E = \Gamma^q(w),\ G = \Gamma^q(t^{-k/n}w) \cap E$, $1 \leqslant q \leqslant \infty,\ \alpha_E > k/n$, is optimal. In particular, if $E = L^p,\ 1 \leqslant p < \infty,\ k/n < 1/p$, then the couple $L^p,\ L^{r,p} \cap L^p,\ 1/r = 1/p - k/n$ is optimal.

The problem of optimal embeddings for inhomogeneous Sobolev spaces in bounded domains is treated in several papers by somewhat different methods [7], [6], [8], [12], [11], [4], [3]. The case of inhomogeneous Sobolev spaces W^1E in $\mathbf{R}^n$ is investigated in [13] in the class of rearrangement invariant Banach function spaces as in [1]. Our domain spaces are more general. In particular, we do not use the Fatou property and duality arguments.

2. Pointwise estimates for the rearrangement

LEMMA 2.1. ([5]) *For $k = 1$ and $k = 2$*

$$f^{**}(t) - f^{**}(2t) \lesssim t^{k/n} \left| D^k f \right|^{**}(t),\ f \in C_0^\infty, \tag{2.1}$$

where C_0^∞ is the class of C^∞ functions in $\mathbf{R}^n$ with compact support.

When $n = 1$, $k = 1$ the estimate (2.1) is equivalent to one given in [9], Lemma 5. For $k = 1$ it was proved in [2] using another method.

LEMMA 2.2. *If* $f \in W^k E$, *then*

$$f^{**}(t) \lesssim \int_t^1 u^{k/n} |D^k f|^{**}(u) \frac{du}{u} + \sum_{j=0}^{k-1} |D^j f|^{**}(1),\ 0 < t < 1. \tag{2.2}$$

Proof. We prove (2.2) by induction. First we note that (see [5])

$$f^{**}(t) = \int_t^1 \delta f^{**}(u) \frac{du}{u} + f^{**}(1), \tag{2.3}$$

where $\delta f^{**}(t) := f^{**}(t) - f^*(t) \lesssim f^{**}(t) - f^{**}(2t)$. Using (2.1) and (2.3) we can write

$$f^{**}(t) \lesssim \int_t^1 u^{1/n} |D^1 f|^{**}(u) \frac{du}{u} + f^{**}(1),\ 0 < t < 1,$$

i.e. (2.2) for $k = 1$. By induction and (2.3) for $|D^m f|$, we have for $0 < t < 1$,

$$f^{**}(t) \lesssim \int_t^1 u^{m/n} \left(\int_u^1 \delta |D^m f|^{**}(s) \frac{ds}{s} + |D^m f|^{**}(1) \right) \frac{du}{u} + \sum_{j=0}^{m-1} |D^j f|^{**}(1).$$

Using again (2.1), we get

$$f^{**}(t) \lesssim \int_t^1 u^{m/n} \left(\int_u^1 s^{1/n} |D^{m+1} f|^{**}(s) \frac{ds}{s} \right) \frac{du}{u} + \sum_{j=0}^{m} |D^j f|^{**}(1),\ 0 < t < 1.$$

It remains to apply Fubini's theorem. □

3. Optimal Sobolev embeddings

3.1. Admissible couples

Here we give a characterization of all admissible couples ρ_E, ρ_G in N.

THEOREM 3.1. (Case $\beta_E < 1$) *Let* $\alpha_E > 0,\ \beta_E < 1$. *The couple* ρ_E, ρ_G *in* N *is admissible if and only if*

$$\rho_G(\chi_{(0,1)} T g) \lesssim \rho_E(\chi_{(0,1)} g),\ g \in \mathscr{M}^+, \tag{3.1}$$

where

$$Tg(t) := \int_t^1 s^{k/n} g(s) ds/s,\ 0 < t < 1. \tag{3.2}$$

Proof. From (2.2) it follows

$$f^*(t) \lesssim T\left(|D^k f|^{**} \right)(t) + \sum_{j=0}^{k-1} |D^j f|^{**}(1),\ 0 < t < 1.$$

Using (3.1), $\beta_E < 1$ and $E \hookrightarrow L^1 + L^\infty$, we obtain

$$\rho_G(\chi_{(0,1)} f^*) \lesssim \rho_E\left(|D^k f|^*\right) + \sum_{j=0}^{k-1} \rho_E\left(|D^j f|^*\right).$$

Taking into account also (1.8) and (1.5), we get the embedding (1.9).

Now we prove that (1.9) implies (3.1). The proof given below is valid without the restriction $\beta_E < 1$. To this end we choose the test function in the form

$$f(x) = \int_0^1 u^k g(u^n) \psi(|x/u|) \frac{du}{u}, \; g \in \mathscr{M}^+, \tag{3.3}$$

where $\psi \in C_0^\infty(-d,d)$, $0 \leqslant \psi \leqslant 1$, $\psi(u) = 1$ if $|u| \leqslant c$ and the constant $c < d$ is such that the ball with radius c has volume one. We can suppose that $\rho_E(g) < \infty$. In particular, $g \in L^1 + L^\infty$. We have for $0 < t < 1$,

$$f^*(t) \geqslant \int_{t^{1/n}}^1 u^k g(u^n) \frac{du}{u} = \frac{1}{n} \int_t^1 u^{k/n} g(u) \frac{du}{u} = \frac{1}{n} Tg(t). \tag{3.4}$$

On the other hand,

$$f^*(t) \leqslant \int_{C_1 t^{1/n}}^1 u^k g(u^n) \frac{du}{u} \lesssim \int_{Ct}^\infty \chi_{(0,1)}(u) g(u) \frac{du}{u}.$$

Applying Minkovski's inequality and using $\alpha_E > 0$, we obtain

$$\rho_E(f^*) \lesssim \rho_E(\chi_{(0,1)} g). \tag{3.5}$$

Analogously, we can prove that all derivatives $D^\alpha f$ up to order k exist as generalized ones and they are locally integrable. In particular,

$$|D^k f|^*(t) \lesssim \int_{Ct}^\infty \chi_{(0,1)}(u) g(u) \frac{du}{u},$$

and applying again Minkovski's inequality, we get

$$\rho_E(|D^k f|^*) \lesssim \rho_E(\chi_{(0,1)} g).$$

Together with (3.5) this proves that $\|f\|_{W^k E} \lesssim \rho_E(\chi_{(0,1)} g)$. Hence (3.1) follows from (1.9) and (3.4). $\square$

THEOREM 3.2. (case $\beta_E = 1$) *Let $\alpha_E > 0$, $\beta_E = 1$. The couple ρ_E, ρ_G in N is admissible if and only if the condition (3.1) is satisfied for all $g \in \mathscr{M}^+$.*

Proof. We need to prove only sufficiency. We start with the following estimate, proved in [10] for $k = 1$

$$\int_0^t s^{-k/n} \delta f^{**}(s) ds \lesssim \int_0^t |D^k f|^*(s) ds, \; f \in C_0^\infty. \tag{3.6}$$

If $k=2$ and $n>2$ this estimate is also valid. It follows from

$$\int_0^t (s^{1-2/n}(-f^*(s))')^*(u)du \lesssim \int_0^t |D^2 f|^*(s)ds,\ n>2, \tag{3.7}$$

which is proved in [3]. Indeed, let $g(t) := t^{-2/n}\int_0^t u(-f^*(u))'du$. Since $g(t) = t^{1-2/n}\delta f^{**}(t)$ we get using also (2.1), that $g(0)=0$. Now we can integrate by parts:

$$\int_0^t s^{-2/n}\delta f^{**}(s)ds = \int_0^t g(s)ds/s = -ng(t)/2 + n\int_0^t s^{1-2/n}(-f^*(s))'ds/2,$$

thus (3.6) for $k=2$ follows.

Since $\alpha_E > 0$, inequalities (3.6) for $k=1$ or $k=2$ imply

$$\rho_E(t^{-k/n}\delta f^{**}(t)) \lesssim \rho_E(|D^k f|^*). \tag{3.8}$$

Indeed, the argument is similar to [10] in proving Lemma 2. Introduce the Hardy operators

$$Pg(t) := \frac{1}{t}\int_0^t g(s)ds,\ Qg(t) := \int_t^\infty g(s)ds/s,$$

which commute. Then (3.6) gives

$$\int_0^t Q(s^{-k/n}\delta f^{**}(s))ds \lesssim \int_0^t Q(|D^k f|^*(s))ds, \tag{3.9}$$

whence by $K-$monotonicity of ρ_E,

$$\rho_E(Q(t^{-k/n}\delta f^{**}(t))) \lesssim \rho_E(Q(|D^k f|^*)). \tag{3.10}$$

We need the estimate

$$\rho_E(t^{-a}Qg(t)) \lesssim \rho_E(t^{-a}g(t)) \text{ if } \alpha_E > a,\ 0 \leqslant a < 1,\ g \in \mathscr{M}^+. \tag{3.11}$$

The proof is standard, we just have to use that ρ_E satisfies (1.4) and that $\alpha_E > a$ is equivalent to $\int_0^1 s^{-a}h_E(s)ds/s < \infty$. (cf. [1] p. 147)

From (3.10) we get (3.8) since $\alpha_E > 0$ implies the boundedness of Q , while the monotonicity of $t\delta f^{**}(t)$, Q and ρ_E give the needed estimate from below.

By induction, we can prove (3.8) for all $k>2$, provided $\alpha_E > (k-2)/n$. Let $h_k(t) = t^{-k/n}\delta f^{**}(t)$. If (3.8) is true for $j>2$, $\alpha_E > (j-2)/n$, then by (2.1) and using $f^{**}(t) = Q(\delta f^{**})$, which is valid for $f \in E$ (note that $f^*(\infty)=0$ due to the embedding $E \hookrightarrow L^1 + \Lambda^1(t^{k/n})$), we can write

$$\rho_E(h_{j+1}) \lesssim \rho_E(t^{-(j-1)/n}|D^2 f|^{**}(t)) = \rho_E(t^{-(j-1)/n}Q(\delta|D^2 f|^{**}(t))),$$

and if $(j-1)/n < \alpha_E$ then by (3.11),

$$\rho_E(t^{-(j-1)/n}Q(\delta|D^2 f|^{**}(t))) \lesssim \rho_E(t^{-(j-1)/n}\delta|D^2 f|^{**}(t)) \lesssim \rho_E(|D^{j+1} f|^*).$$

Hence

$$\rho_E(h_{j+1}) \lesssim \rho_E(|D^{j+1} f|^*),\ \alpha_E > (j-1)/n.$$

Thus (3.8) is proved. Finally, since $f^{**} = T(h_k) + f^{**}(1)$, $h_k(t) = t^{-k/n}\delta f^{**}(t)$, we get from (3.1) and (3.8), the estimate

$$\rho_G(\chi_{(0,1)}f^*) \lesssim \rho_E(|D^k f|^*) + \rho_E(f^*).$$

Together with (1.8) this gives (1.9). □

3.2. Optimal norms in the subcritical case $k/n < \alpha_E$

THEOREM 3.3. *Let $\rho_E \in N_d$. Then the target norm $\rho_{G(E)}$, defined by*

$$\rho_{G(E)}(g) := \rho_E(t^{-k/n}\chi_{(0,1)}(t)g(t)) + \rho_E(\chi_{(1,\infty)}g),\ g \in \mathscr{M}^+ \tag{3.12}$$

is optimal. Moreover, the couple $\rho_E, \rho_{G(E)}$ is optimal.

Proof. Evidently, $\rho_{G(E)}$ is a monotone norm, satisfying (1.8). To prove that the couple $\rho_E, \rho_{G(E)}$ is admissible, it is enough to check

$$\rho_E\left(t^{-k/n}\chi_{(0,1)}(t)\int_t^1 s^{k/n}\chi_{(0,1)}(s)g(s)\frac{ds}{s}\right) \lesssim \rho_E(\chi_{(0,1)}g),\ g \in \mathscr{M}^+.$$

But this follows from (3.11) since $k/n < \alpha_E$. Further, let ρ_E, ρ_G be an admissible couple in N. Then by (2.3) and (3.1),

$$\rho_G(\chi_{(0,1)}g^*) \leqslant \rho_G(\chi_{(0,1)}T(t^{-k/n}\delta g^{**}(t))) + g^{**}(1) \lesssim \rho_E(t^{-k/n}\chi_{(0,1)}(t)g^{**}(t)) + g^{**}(1).$$

Using also (1.8), we obtain

$$\rho_G(g^*) \lesssim \rho_E(t^{-k/n}\chi_{(0,1)}(t)g^{**}(t)) + g^{**}(1) + \rho_E(\chi_{(1,\infty)}g^*)$$

and since $\rho_E(t^{-k/n}\chi_{(0,1)}(t)g^{**}(t)) \geqslant g^{**}(1)\rho_E(\chi_{(0,1)})$, it follows

$$\rho_G(g^*) \lesssim \rho_E(t^{-k/n}\chi_{(0,1)}(t)g^{**}(t)) + \rho_E(\chi_{(1,\infty)}g^*).$$

Using Minkovski's inequality, we obtain

$$\rho_E(t^{-k/n}\chi_{(0,1)}(t)g^{**}(t)) \lesssim \rho_E(t^{-k/n}\chi_{(0,1)}(t)g^*(t)).$$

Therefore $\rho_G(g^*) \lesssim \rho_{G(E)}(g^*)$. This means that the target norm defined by (3.12) is optimal.

It remains to prove that the domain norm ρ_E is also optimal. Suppose that the couple ρ, $\rho_{G(E)}$ in N is admissible. Then

$$\rho_{G(E)}(\chi_{(0,1)}Tg) \lesssim \rho(\chi_{(0,1)}g),\ \rho_{G(E)}(\chi_{(1,\infty)}g) \lesssim \rho(\chi_{(1,\infty)}g),\ g \in \mathscr{M}^+.$$

Therefore

$$\rho_E(\chi_{(0,1)}g^*) \lesssim \rho_{G(E)}(\chi_{(0,1)}(t)t^{k/n}g^*(t)) \lesssim \rho(\chi_{(0,1)}g^*)$$

and

$$\rho_E(\chi_{(1,\infty)}g^*) = \rho_{G(E)}(\chi_{(1,\infty)}g^*) \lesssim \rho(\chi_{(1,\infty)}g^*).$$

This implies $\rho_E(g^*) \lesssim \rho(g^*)$, which means that the domain norm ρ_E is optimal. □

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(Received March 5, 2011)

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