A REFINEMENT OF THE DISCRETE JENSEN'S INEQUALITY

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Abstract. We give a refinement of the discrete Jensen's inequality in the convex and mid-convex cases. For mid-convex functions our result is a common generalization of known inequalities. We illustrate the scope of the results by applying them to some special situations.

1. Introduction and the main results

The following inequalities are known by the collective title "discrete Jensen's inequalities":

THEOREM A. (see [3]) *Let C be a convex subset of a real vector space X, let $x_i \in C$, and let $p_i \geqslant 0$ $(i = 1,\dots,n)$ with $\sum_{i=1}^{n} p_i = 1$.*

(a) If $f : C \to \mathbb{R}$ is a convex function, then

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leqslant \sum_{i=1}^{n} p_i f(x_i). \tag{1}$$

(b) If $f : C \to \mathbb{R}$ is a mid-convex function, and p_i is rational $(i = 1,\dots,n)$, then (1) also holds.

In the previous setting, the function $f : C \to \mathbb{R}$ is convex if

$$f(\beta x + (1-\beta)y) \leqslant \beta f(x) + (1-\beta)f(y), \quad x,y \in C, \quad 0 \leqslant \beta \leqslant 1,$$

and mid-convex if

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{1}{2}f(x) + \frac{1}{2}f(y), \quad x,y \in C.$$

Various attempts were made in the last years to refine these inequalities (see [1], [2], [4]–[10]). The following two improvements of (1) for mid-convex functions involve similar ideas. The first result was published in [9].

THEOREM B. *Let C be a convex subset of a real vector space X, and let $f : C \to \mathbb{R}$ be a mid-convex function. If $x_i \in C$ $(i = 1, \ldots, n)$, and*

$$B_{k,n} := \frac{1}{\binom{n}{k}} \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n} f\left(\frac{x_{i_1} + \ldots + x_{i_k}}{k}\right), \quad 1 \leqslant k \leqslant n,$$

then

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \leqslant B_{n,n} \leqslant \ldots \leqslant B_{k,n} \leqslant \ldots \leqslant B_{1,n} = \frac{1}{n}\sum_{i=1}^{n} f(x_i). \tag{2}$$

The second result was obtained in [7].

THEOREM C. *Let C be a convex subset of a real vector space X, and let $f : C \to \mathbb{R}$ be a mid-convex function. If $x_i \in C$ $(i = 1, \ldots, n)$, and*

$$\bar{B}_{k,n} := \frac{1}{\binom{n+k-1}{k}} \sum_{1 \leqslant i_1 \leqslant \ldots \leqslant i_k \leqslant n} f\left(\frac{x_{i_1} + \ldots + x_{i_k}}{k}\right), \quad 1 \leqslant k,$$

then

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \leqslant \ldots \leqslant \bar{B}_{k+1,n} \leqslant \bar{B}_{k,n} \leqslant \ldots \leqslant \bar{B}_{1,n} = \frac{1}{n}\sum_{i=1}^{n} f(x_i). \tag{3}$$

Of course, inequalities (2) and (3) remain true if f is a convex function, since every convex function is mid-convex. On the one hand our goal is to clarify the connection between Theorem B and Theorem C by giving such a refinement of (1) for mid-convex functions which contains (2) and (3) as a special case. On the other hand we would like to refine (1) for convex functions on the basis of (2) and (3). The main results will be applied to some special situations in the second part of the paper.

We begin with some notations.

Let X be a set. The power set of X is denoted by $P(X)$. $|X|$ means the number of elements in X.

The usual symbol $\mathbb{N}$ is used for the set of natural numbers (including 0).

Let $u \geqslant 1$ and $v \geqslant 2$ be fixed integers. Define the functions

$$S_{v,w} : \{1, \ldots, u\}^v \to \{1, \ldots, u\}^{v-1}, \quad 1 \leqslant w \leqslant v,$$

$$S_v : \{1, \ldots, u\}^v \to P\left(\{1, \ldots, u\}^{v-1}\right),$$

and

$$T_v : P(\{1, \ldots, u\}^v) \to P\left(\{1, \ldots, u\}^{v-1}\right)$$

by

$$S_{v,w}(i_1, \ldots, i_v) := (i_1, i_2, \ldots, i_{w-1}, i_{w+1}, \ldots, i_v), \quad 1 \leqslant w \leqslant v,$$

$$S_v(i_1, \ldots, i_v) := \bigcup_{w=1}^{v} \{S_{v,w}(i_1, \ldots, i_v)\},$$

and

$$T_v(I) := \begin{cases} \varnothing, & \text{if} \quad I = \varnothing \\ \bigcup_{(i_1,\dots,i_v)\in I} S_v(i_1,\dots,i_v), & \text{if} \quad I \neq \varnothing \end{cases}.$$

Next, let the function

$$\alpha_{v,i} : \{1,\dots,u\}^v \to \mathbb{N}, \quad 1 \leqslant i \leqslant u,$$

be given by: $\alpha_{v,i}(i_1,\dots,i_v)$ means the number of occurrences of i in the sequence $(i_1,\dots,i_v)$.

For each $I \in P(\{1,\dots,u\}^v)$ let

$$\alpha_{I,i} := \sum_{(i_1,\dots,i_v)\in I} \alpha_{v,i}(i_1,\dots,i_v), \quad 1 \leqslant i \leqslant u.$$

It is easy to see that the dependence of the functions $S_{v,w}$, S_v, T_v and $\alpha_{v,i}$ on u does not play an important role, so we can use simplified notations.

The following hypotheses will give the basic context of our results.

(H_1) Let $n \geqslant 1$ and $k \geqslant 2$ be fixed integers, and let I_k be a subset of $\{1,\dots,n\}^k$ such that

$$\alpha_{I_k,i} \geqslant 1, \quad 1 \leqslant i \leqslant n. \tag{4}$$

(H_2) Let V be a real vector space, let C be a convex subset of V, and let $x_1,\dots,x_n \in C$.

(H_3) Let $p_1,\dots,p_n$ be positive numbers such that $\sum_{j=1}^{n} p_j = 1$.

(H_4) Let the function $f : C \to \mathbb{R}$ be convex.

(H_5) Let the function $f : C \to \mathbb{R}$ be mid-convex, and let $p_1,\dots,p_n$ be rational.

We need some further preparations.

Starting from I_k, we introduce the sets $I_l \subset \{1,\dots,n\}^l$ $(k-1 \geqslant l \geqslant 1)$ inductively by

$$I_{l-1} := T_l(I_l), \quad k \geqslant l \geqslant 2.$$

Obviously, $I_1 = \{1,\dots,n\}$, by (4), and this insures that $\alpha_{I_1,i} = 1$ $(1 \leqslant i \leqslant n)$. From (4) again, we have that $\alpha_{I_l,i} \geqslant 1$ $(k-1 \geqslant l \geqslant 1, 1 \leqslant i \leqslant n)$. It is evident that

$$\alpha_{1,i}(j) = \begin{cases} 1, & \text{if} \quad j = i \\ 0, & \text{if} \quad j \neq i \end{cases}, \quad 1 \leqslant i \leqslant n. \tag{5}$$

For any $k \geqslant l \geqslant 2$ and for any $(j_1,\dots,j_{l-1}) \in I_{l-1}$ let

$$H_{I_l}(j_1,\dots,j_{l-1})$$
$$:= \{((i_1,\dots,i_l),m) \in I_l \times \{1,\dots,l\} \mid S_{l,m}(i_1,\dots,i_l) = (j_1,\dots,j_{l-1})\}.$$

Using these sets we define the functions $t_{I_k,l} : I_l \to \mathbb{N}$ $(k \geqslant l \geqslant 1)$ inductively by

$$t_{I_k,k}(i_1,\ldots,i_k) := 1, \quad (i_1,\ldots,i_k) \in I_k; \tag{6}$$

$$t_{I_k,l-1}(j_1,\ldots,j_{l-1}) := \sum_{((i_1,\ldots,i_l),m) \in H_{I_l}(j_1,\ldots,j_{l-1})} t_{I_k,l}(i_1,\ldots,i_l). \tag{7}$$

The main results of this paper involve some special expressions, which we now describe. For any $k \geqslant l \geqslant 1$ let

$$A_{l,l} = A_{l,l}(I_k, x_1, \ldots, x_n, p_1, \ldots, p_n)$$

$$:= \sum_{(i_1,\ldots,i_l) \in I_l} \left(\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_l,i_s}} \right) f \left(\frac{\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_l,i_s}} x_{i_s}}{\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_l,i_s}}} \right),$$

and associate to each $k - 1 \geqslant l \geqslant 1$ the number

$$A_{k,l} = A_{k,l}(I_k, x_1, \ldots, x_n, p_1, \ldots, p_n)$$

$$:= \frac{1}{(k-1)\ldots l} \sum_{(i_1,\ldots,i_l) \in I_l} t_{I_k,l}(i_1,\ldots,i_l) \left(\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}} \right) f \left(\frac{\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}} x_{i_s}}{\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}}} \right).$$

With these preparations out of the way we come to

THEOREM 1. *Assume that either* $(H_1)-(H_4)$ *or* $(H_1)-(H_3)$ *and* (H_5) *are satisfied. Then*
(a)

$$f \left(\sum_{r=1}^{n} p_r x_r \right) \leqslant A_{k,k} \leqslant A_{k,k-1} \leqslant \ldots \leqslant A_{k,2} \leqslant A_{k,1} = \sum_{r=1}^{n} p_r f(x_r). \tag{8}$$

(b) Suppose $\left| H_{I_l}(j_1,\ldots,j_{l-1}) \right| = \beta_{l-1}$ *for any* $(j_1,\ldots,j_{l-1}) \in I_{l-1}$ $(k \geqslant l \geqslant 2)$. *Then*

$$A_{k,l} = A_{l,l} = \frac{n}{l|I_l|} \sum_{(i_1,\ldots,i_l) \in I_l} \left(\sum_{s=1}^{l} p_{i_s} \right) f \left(\frac{\sum_{s=1}^{l} p_{i_s} x_{i_s}}{\sum_{s=1}^{l} p_{i_s}} \right), \quad (k \geqslant l \geqslant 1),$$

and thus

$$f \left(\sum_{r=1}^{n} p_r x_r \right) \leqslant A_{k,k} \leqslant A_{k-1,k-1} \leqslant \ldots \leqslant A_{2,2} \leqslant A_{1,1} = \sum_{r=1}^{n} p_r f(x_r).$$

2. Discussion, and applications

Throughout Examples (2–7) the conditions (H_2), (H_3) and either (H_4) or (H_5) will be assumed.

Theorem 1 contains Theorem B, as the first example shows.

EXAMPLE 2. *Let*

$$I_k := \left\{ (i_1,\ldots,i_k) \in \{1,\ldots,n\}^k \mid i_1 < \ldots < i_k \right\}, \quad 1 \leqslant k \leqslant n.$$

Then $\alpha_{I_n,i} = 1$ $(i = 1,\ldots,n)$ ensuring (H_1) with $k = n$. It is easy to check that $T_k(I_k) = I_{k-1}$ $(k = 2,\ldots,n)$, $|I_k| = \binom{n}{k}$ $(k = 1,\ldots,n)$, and for every $k = 2,\ldots,n$

$$\left| H_{I_k}(j_1,\ldots,j_{k-1}) \right| = n - (k-1), \quad (j_1,\ldots,j_{k-1}) \in I_{k-1},$$

and therefore, thanks to Theorem 1 (b),

$$A_{k,k} = \frac{1}{\binom{n-1}{k-1}} \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \left(\sum_{s=1}^{k} p_{i_s} \right) f \left(\frac{\sum\limits_{s=1}^{k} p_{i_s} x_{i_s}}{\sum\limits_{s=1}^{k} p_{i_s}} \right), \quad k = 1,\ldots,n.$$

and

$$f \left(\sum_{r=1}^{n} p_r x_r \right) \leqslant A_{k,k} \leqslant A_{k-1,k-1} \leqslant \ldots \leqslant A_{2,2} \leqslant A_{1,1} = \sum_{r=1}^{n} p_r f(x_r). \tag{9}$$

If $p_1 = \ldots = p_n = \frac{1}{n}$, then by Lemma 10 (f)

$$A_{k,k} = \frac{1}{\binom{n}{k}} \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n} f \left(\frac{x_{i_1} + \ldots x_{i_k}}{k} \right), \quad k = 1,\ldots,n,$$

and thus (9) gives Theorem B.

The next example illustrates that Theorem C is a special case of Theorem 1.

EXAMPLE 3. *Let*

$$I_k := \left\{ (i_1,\ldots,i_k) \in \{1,\ldots,n\}^k \mid i_1 \leqslant \ldots \leqslant i_k \right\}, \quad k \geqslant 1.$$

Obviously, $\alpha_{I_k,i} \geqslant 1$ $(i = 1,\ldots,n)$, and therefore (H_1) is satisfied. It is not hard to see that $T_k(I_k) = I_{k-1}$ $(k = 2,\ldots)$, $|I_k| = \binom{n+k-1}{k}$ $(k = 1,\ldots)$, and for each $l = 2,\ldots,k$

$$\left| H_{I_l}(j_1,\ldots,j_{l-1}) \right| = n, \quad (j_1,\ldots,j_{l-1}) \in I_{l-1}.$$

Consequently, by applying Theorem 1 (b), we deduce that

$$A_{k,k} = \frac{1}{\binom{n+k-1}{k-1}} \sum_{1 \leqslant i_1 \leqslant \ldots \leqslant i_k \leqslant n} \left(\sum_{s=1}^{k} p_{i_s} \right) f \left(\frac{\sum_{s=1}^{k} p_{i_s} x_{i_s}}{\sum_{s=1}^{k} p_{i_s}} \right), \quad k \geqslant 1,$$

and

$$f \left(\sum_{r=1}^{n} p_r x_r \right) \leqslant \ldots \leqslant A_{k,k} \leqslant \ldots \leqslant A_{k,1} = \sum_{r=1}^{n} p_r f(x_r). \tag{10}$$

By taking $p_1 = \ldots = p_n = \frac{1}{n}$ we obtain from Lemma 10 (f) that

$$A_{k,k} = \frac{1}{\binom{n+k-1}{k}} \sum_{1 \leqslant i_1 \leqslant \ldots \leqslant i_k \leqslant n} f \left(\frac{x_{i_1} + \ldots x_{i_k}}{k} \right), \quad k \geqslant 1,$$

and thus (10) gives Theorem C.

The following two examples are particular cases of Theorem 1 (b).

EXAMPLE 4. *Let*
$$I_k := \{1, \ldots, n\}^k, \quad k \geqslant 1.$$

Trivially, $\alpha_{I_k,i} \geqslant 1 \ (i = 1, \ldots, n)$, hence (H_1) holds. It is evident that $T_k(I_k) = I_{k-1}$ $(k = 2, \ldots)$, $|I_k| = n^k \ (k = 1, \ldots)$, and for every $l = 2, \ldots, k$

$$\left| H_{I_l}(j_1, \ldots, j_{l-1}) \right| = n^l, \quad (j_1, \ldots, j_{l-1}) \in I_{l-1},$$

and so Theorem 1 (b) leads to

$$A_{k,k} = \frac{1}{kn^{k-1}} \sum_{(i_1, \ldots, i_k) \in I_k} \left(\sum_{s=1}^{k} p_{i_s} \right) f \left(\frac{\sum_{s=1}^{k} p_{i_s} x_{i_s}}{\sum_{s=1}^{k} p_{i_s}} \right), \quad k \geqslant 1,$$

and

$$f \left(\sum_{r=1}^{n} p_r x_r \right) \leqslant \ldots \leqslant A_{k,k} \leqslant \ldots \leqslant A_{1,1} = \sum_{r=1}^{n} p_r f(x_r), \quad k \geqslant 1.$$

Especially, for $p_1 = \ldots p_n = \frac{1}{n}$ we find from Lemma 10 (f) that

$$A_{k,k} = \frac{1}{n^k} \sum_{(i_1, \ldots, i_k) \in I_k} f \left(\frac{x_{i_1} + \ldots x_{i_k}}{k} \right), \quad k = 1, \ldots, n.$$

EXAMPLE 5. *For $1 \leqslant k \leqslant n$ let I_k consist of all sequences $(i_1,\ldots,i_k)$ of k distinct numbers from $\{1,\ldots,n\}$. Then $\alpha_{I_n,i} \geqslant 1$ $(i=1,\ldots,n)$, hence (H_1) is valid. It is immediate that $T_k(I_k) = I_{k-1}$ $(k=2,\ldots,n)$, $|I_k| = n(n-1)\ldots(n-k+1)$ $(k=1,\ldots,n)$, and for each $k = 2,\ldots,n$*

$$\left|H_{I_k}(j_1,\ldots,j_{k-1})\right| = (n-(k-1))k, \quad (j_1,\ldots,j_{k-1}) \in I_{k-1}.$$

and from them, on account of Theorem 1 (b), follows

$$A_{k,k} = \frac{n}{kn(n-1)\ldots(n-k+1)} \sum_{(i_1,\ldots,i_k)\in I_k} \left(\sum_{s=1}^{k} p_{i_s}\right) f\left(\frac{\sum_{s=1}^{k} p_{i_s}x_{i_s}}{\sum_{s=1}^{k} p_{i_s}}\right), \quad k=1,\ldots,n$$

and

$$f\left(\sum_{r=1}^{n} p_r x_r\right) \leqslant A_{n,n} \leqslant \ldots \leqslant A_{k,k} \leqslant \ldots \leqslant A_{1,1} = \sum_{r=1}^{n} p_r f(x_r).$$

If we set $p_1 = \ldots = p_n = \frac{1}{n}$, then

$$A_{k,k} = \frac{1}{n(n-1)\ldots(n-k+1)} \sum_{(i_1,\ldots,i_k)\in I_k} f\left(\frac{x_{i_1}+\ldots x_{i_k}}{k}\right), \quad k=1,\ldots,n.$$

In the sequel two interesting consequences of Theorem 1 (a) are given.

EXAMPLE 6. *Let $c_i \geqslant 1$ be an integer $(i=1,\ldots,n)$, let $k := \sum_{i=1}^{n} c_i$, and let $I_k = P^{c_1,\ldots,c_n}$ consist of all sequences $(i_1,\ldots,i_k)$ in which the number of occurrences of $i \in \{1,\ldots,n\}$ is c_i $(i=1,\ldots,n)$. Evidently, (H_1) is satisfied. A simple calculation shows that*

$$I_{k-1} = \bigcup_{i=1}^{n} P^{c_1,\ldots,c_{i-1},c_i-1,c_{i+1},\ldots,c_n}, \quad \alpha_{I_k,i} = \frac{k!}{c_1!\ldots c_n!}c_i, \quad i=1,\ldots,n,$$

and

$$t_{I_k,k-1}(i_1,\ldots,i_{k-1}) = k,$$
$$\text{if} \quad (i_1,\ldots,i_{k-1}) \in P^{c_1,\ldots,c_{i-1},c_i-1,c_{i+1},\ldots,c_n}, \quad i=1,\ldots,n,$$

and

$$f\left(\sum_{r=1}^{n} p_r x_r\right) = A_{k,k}$$

$$= \frac{c_1!\ldots c_n!}{k!} \sum_{(i_1,\ldots,i_k)\in I_k} \left(\sum_{s=1}^{k} \frac{p_{i_s}}{c_{i_s}}\right) f\left(\frac{\sum_{s=1}^{k} \frac{p_{i_s}}{c_{i_s}}x_{i_s}}{\sum_{s=1}^{k} \frac{p_{i_s}}{c_{i_s}}}\right).$$

According to Theorem 1 *(a)*

$$f\left(\sum_{r=1}^{n}p_r x_r\right) \leqslant A_{k,k-1} \leqslant \sum_{r=1}^{n}p_r f(x_r),$$

where

$$A_{k,k-1} = \frac{1}{k-1}\sum_{i=1}^{n}(c_i - p_i)f\left(\frac{\sum_{r=1}^{n}p_r x_r - \frac{p_i}{c_i}x_i}{1 - \frac{p_i}{c_i}}\right).$$

EXAMPLE 7. *Let*

$$I_2 := \left\{(i_1,i_2) \in \{1,\ldots,n\}^2 \mid i_1|i_2\right\}.$$

The notation $i_1|i_2$ means that i_1 divides i_2. Since $i|i$ $(i=1,\ldots,n)$, (H_1) holds. In this case

$$\alpha_{I_2,i} = \left[\frac{n}{i}\right] + d(i), \quad i = 1,\ldots,n,$$

where $\left[\frac{n}{i}\right]$ is the largest natural number that does not exceed $\frac{n}{i}$, and $d(i)$ denotes the number of positive divisors of i. By Theorem 1 (a), we have

$$f\left(\sum_{r=1}^{n}p_r x_r\right) \leqslant \sum_{(i_1,i_2)\in I_2}\left(\frac{p_{i_1}}{\left[\frac{n}{i_1}\right]+d(i_1)} + \frac{p_{i_2}}{\left[\frac{n}{i_2}\right]+d(i_2)}\right)f\left(\frac{\frac{p_{i_1}}{\left[\frac{n}{i_1}\right]+d(i_1)}x_{i_1} + \frac{p_{i_2}}{\left[\frac{n}{i_2}\right]+d(i_2)}x_{i_2}}{\frac{p_{i_1}}{\left[\frac{n}{i_1}\right]+d(i_1)} + \frac{p_{i_2}}{\left[\frac{n}{i_2}\right]+d(i_2)}}\right)$$

$$\leqslant \sum_{r=1}^{n}p_r f(x_r).$$

The final example is connected to the arithmetic-geometric mean inequality.

EXAMPLE 8. *Assume (H_1) and (H_3). Let $V := \mathbb{R}$, let $C :=]0,\infty[$, and let $f := -\ln$. Then (H_1)–(H_4) hold, therefore recalling Theorem 1 (a),*

$$\prod_{i=1}^{n}x_i^{p_i} \leqslant \prod_{(i_1,\ldots,i_k)\in I_k}\left(\frac{\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}x_{i_s}}{\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}}\right)^{\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}} \leqslant \sum_{i=1}^{n}p_i x_i.$$

This inequality is a refinement of the arithmetic-geometric mean inequality.

Suppose $p_1 = \ldots = p_n = \frac{1}{n}$, and $\alpha_{I_k,1} = \ldots = \alpha_{I_k,n} = \frac{|I_k|k}{n}$. *Then the previous inequality has the form*

$$\sqrt[n]{x_1\ldots x_n} \leqslant \prod_{(i_1,\ldots,i_k)\in I_k}\left(\frac{x_{i_1}+\ldots x_{i_k}}{k}\right)^{\frac{1}{|I_k|}} \leqslant \frac{x_1+\ldots x_n}{n}.$$

3. Preliminary results and the proofs of the main results

We begin with a deeper property of the function $t_{I_k,1}$.

LEMMA 9. *If* (H_1) *is satisfied, then*

$$t_{I_k,1}(i) = \alpha_{I_k,i}(k-1)!, \quad 1 \leqslant i \leqslant n. \tag{11}$$

Proof. For a fixed $1 \leqslant i \leqslant n$ we first prove by induction on l that

$$\sum_{(i_1,\ldots,i_l) \in I_l} \alpha_{l,i}(i_1,\ldots,i_l) t_{I_k,l}(i_1,\ldots,i_l) \tag{12}$$

$$= \begin{cases} \alpha_{I_k,i}, & \text{if} \quad l = k \\ \alpha_{I_k,i}(k-1)(k-2)\ldots l, & \text{if} \quad k-1 \geqslant l \geqslant 1 \end{cases}.$$

If $l = k$, then (4) and (6) give (12). Suppose then that l $(k \geqslant l \geqslant 2)$ is an integer for which (12) holds. By (7)

$$\sum_{(j_1,\ldots,j_{l-1}) \in I_{l-1}} \alpha_{l-1,i}(j_1,\ldots,j_{l-1}) t_{I_k,l-1}(j_1,\ldots,j_{l-1})$$

$$= \sum_{(j_1,\ldots,j_{l-1}) \in I_{l-1}} \alpha_{l-1,i}(j_1,\ldots,j_{l-1}) \left(\sum_{((i_1,\ldots,i_l),m) \in H_{I_l}(j_1,\ldots,j_{l-1})} t_{I_k,l}(i_1,\ldots,i_l) \right).$$

From this and the definition of $S_{l,m}$ $(1 \leqslant m \leqslant l)$ it follows

$$\sum_{(j_1,\ldots,j_{l-1}) \in I_{l-1}} \alpha_{l-1,i}(j_1,\ldots,j_{l-1}) t_{I_k,l-1}(j_1,\ldots,j_{l-1})$$

$$= \sum_{(j_1,\ldots,j_{l-1}) \in I_{l-1}} \left(\sum_{\{((i_1,\ldots,i_l),m) \in H_{I_l}(j_1,\ldots,j_{l-1}) | i_m \neq i\}} \alpha_{l,i}(i_1,\ldots,i_l) t_{I_k,l}(i_1,\ldots,i_l) \right.$$

$$\left. + \sum_{\{((i_1,\ldots,i_l),m) \in H_{I_l}(j_1,\ldots,j_{l-1}) | i_m = i\}} \left(\alpha_{l,i}(i_1,\ldots,i_l) - 1 \right) t_{I_k,l}(i_1,\ldots,i_l) \right)$$

$$= \sum_{(i_1,\ldots,i_l) \in I_l} \left((l - \alpha_{l,i}(i_1,\ldots,i_l)) \alpha_{l,i}(i_1,\ldots,i_l) t_{I_k,l}(i_1,\ldots,i_l) \right.$$

$$\left. + \alpha_{l,i}(i_1,\ldots,i_l) \left(\alpha_{l,i}(i_1,\ldots,i_l) - 1 \right) t_{I_k,l}(i_1,\ldots,i_l) \right)$$

$$= (l-1) \sum_{(i_1,\ldots,i_l) \in I_l} \alpha_{l,i}(i_1,\ldots,i_l) t_{I_k,l}(i_1,\ldots,i_l),$$

and therefore the induction hypothesis shows that

$$\sum_{(j_1,\ldots,j_{l-1}) \in I_{l-1}} \alpha_{l-1,i}(j_1,\ldots,j_{l-1}) t_{I_k,l-1}(j_1,\ldots,j_{l-1})$$

$$= \alpha_{I_k,i}(k-1)\ldots l(l-1).$$

(12) for $l = 1$, taking into consideration (5), implies (11). The proof is complete. $\square$

In Theorem 1 (b) our arguments depend on the following lemma.

LEMMA 10. *Assume* (H_1). *If*

$$\left|H_{I_l}(j_1,\ldots,j_{l-1})\right| = \beta_{l-1}, \text{ for all } (j_1,\ldots,j_{l-1}) \in I_{l-1}, \quad k \geqslant l \geqslant 2, \qquad (13)$$

then

(a) $\beta_{l-1} = l\frac{|I_l|}{|I_{l-1}|}$ $(k \geqslant l \geqslant 2)$.

(b) $t_{I_k,l}(j_1,\ldots,j_l) = \beta_{k-1}\ldots\beta_l = k\ldots(l+1)\frac{|I_k|}{|I_l|}$ $((j_1,\ldots,j_l) \in I_l,\ k-1 \geqslant l \geqslant 1)$.

(c) $\alpha_l := \alpha_{I_l,n} = \ldots = \alpha_{I_l,1}$ $(k \geqslant l \geqslant 1)$.

(d) $\alpha_l = \frac{l|I_l|}{n}$ $(k \geqslant l \geqslant 1)$.

(e) $A_{k,l} = A_{l,l} = \dfrac{n}{l\,|I_l|} \displaystyle\sum_{(i_1,\ldots,i_l)\in I_l} \left(\sum_{s=1}^{l} p_{i_s}\right) f\left(\dfrac{\displaystyle\sum_{s=1}^{l} p_{i_s} x_s}{\displaystyle\sum_{s=1}^{l} p_{i_s}}\right), \quad k \geqslant l \geqslant 1.$

(f) If $p_1 = \ldots = p_n = \frac{1}{n}$, then

$$A_{k,l} = A_{l,l} = \frac{1}{|I_l|} \sum_{(i_1,\ldots,i_l)\in I_l} f\left(\frac{x_{i_1} + \ldots + x_{i_l}}{l}\right), \quad k \geqslant l \geqslant 1.$$

Proof. (a) By the definition of $H_{I_l}(j_1,\ldots,j_{l-1})$

$$\sum_{(j_1,\ldots,j_{l-1})\in I_{l-1}} \left|H_{I_l}(j_1,\ldots,j_{l-1})\right| = l\,|I_l|, \quad k \geqslant l \geqslant 2.$$

Consequently, (13) yields (a).

(b) We prove this by induction on l, the case $l = k$ being

$$t_{I_k,k-1}(j_1,\ldots,j_{k-1}) := \sum_{((i_1,\ldots,i_k),m)\in H_{I_k}(j_1,\ldots,j_{k-1})} t_{I_k,k}(i_1,\ldots,i_k)$$
$$= \beta_{k-1}, \quad (j_1,\ldots,j_{k-1}) \in I_{k-1}.$$

Let l $(k-1 \geqslant l \geqslant 2)$ be an integer such that the result holds. Then

$$t_{I_k,l-1}(j_1,\ldots,j_{l-1}) := \sum_{((i_1,\ldots,i_l),m)\in H_{I_l}(j_1,\ldots,j_{l-1})} t_{I_k,l}(i_1,\ldots,i_l)$$
$$= \sum_{((i_1,\ldots,i_l),m)\in H_{I_l}(j_1,\ldots,j_{l-1})} \beta_{k-1}\ldots\beta_l$$
$$= \left|H_{I_l}(j_1,\ldots,j_{l-1})\right|\beta_{k-1}\ldots\beta_l$$
$$= \beta_{k-1}\ldots\beta_l\beta_{l-1}.$$

The second equality in (b) comes from (a).

(c) Part (b) and (11) show that

$$t_{I_k,1}(i) = \beta_{k-1}\ldots\beta_1 = \alpha_{I_k,i}(k-1)!, \quad 1 \leqslant i \leqslant n,$$

and thus $\alpha_{I_k,n} = \ldots = \alpha_{I_k,1}$. It follows from (b) and (12) that

$$\sum_{(i_1,\ldots,i_l)\in I_l} \alpha_{l,i}(i_1,\ldots,i_l)t_{I_k,l}(i_1,\ldots,i_l)$$

$$= \beta_{k-1}\ldots\beta_l \sum_{(i_1,\ldots,i_l)\in I_l} \alpha_{l,i}(i_1,\ldots,i_l)$$

$$= \beta_{k-1}\ldots\beta_l\alpha_{I_l,i} = \alpha_{I_k,i}(k-1)(k-2)\ldots l, \quad k-1 \geqslant l \geqslant 1, \quad 1 \leqslant i \leqslant n,$$

giving

$$\alpha_{I_l,i} = \frac{\alpha_k(k-1)(k-2)\ldots l}{\beta_{k-1}\ldots\beta_l}, \quad k-1 \geqslant l \geqslant 1, \quad 1 \leqslant i \leqslant n,$$

and this implies the result for $k-1 \geqslant l \geqslant 1$.

(d) It is an easy consequence of (c).

(e) Using the definition of $A_{k,l}$ $(k-1 \geqslant l \geqslant 1)$, then (b), (c) and (d), we get

$$A_{k,l} := \frac{1}{(k-1)\ldots l} \sum_{(i_1,\ldots,i_l)\in I_l} t_{I_k,l}(i_1,\ldots,i_l)\left(\sum_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right) f\left(\frac{\sum_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_k,i_s}}x_s}{\sum_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_k,i_s}}}\right)$$

$$= \frac{1}{\alpha_k(k-1)\ldots l}k\ldots(l+1)\frac{|I_k|}{|I_l|}\sum_{(i_1,\ldots,i_l)\in I_l}\left(\sum_{s=1}^{l}p_{i_s}\right)f\left(\frac{\sum_{s=1}^{l}p_{i_s}x_s}{\sum_{s=1}^{l}p_{i_s}}\right)$$

$$= \frac{k}{l\alpha_k}\frac{|I_k|}{|I_l|}\sum_{(i_1,\ldots,i_l)\in I_l}\left(\sum_{s=1}^{l}p_{i_s}\right)f\left(\frac{\sum_{s=1}^{l}p_{i_s}x_s}{\sum_{s=1}^{l}p_{i_s}}\right)$$

$$= \frac{n}{l\,|I_l|}\sum_{(i_1,\ldots,i_l)\in I_l}\left(\sum_{s=1}^{l}p_{i_s}\right)f\left(\frac{\sum_{s=1}^{l}p_{i_s}x_s}{\sum_{s=1}^{l}p_{i_s}}\right), \quad (k-1 \geqslant l \geqslant 1).$$

Similarly, the definition of $A_{l,l}$ $(k \geqslant l \geqslant 1)$, (c) and (d) insure that

$$
A_{l,l} := \sum_{(i_1,\ldots,i_l) \in I_l} \left(\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_l,i_s}} \right) f \left(\frac{\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_l,i_s}} x_s}{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_l,i_s}}} \right)
$$

$$
= \frac{1}{\alpha_l} \sum_{(i_1,\ldots,i_l) \in I_l} \left(\sum_{s=1}^{l} p_{i_s} \right) f \left(\frac{\sum_{s=1}^{l} p_{i_s} x_s}{\sum_{s=1}^{l} p_{i_s}} \right)
$$

$$
= \frac{n}{l\,|I_l|} \sum_{(i_1,\ldots,i_l) \in I_l} \left(\sum_{s=1}^{l} p_{i_s} \right) f \left(\frac{\sum_{s=1}^{l} p_{i_s} x_s}{\sum_{s=1}^{l} p_{i_s}} \right), \quad (k \geqslant l \geqslant 1).
$$

(f) This is a special case of (e).
The proof is now complete. $\square$

REMARK 11. *Assume* (H_1). *Lemma* 10 *shows that* (13) *implies* $\alpha_l := \alpha_{I_l,n} = \ldots = \alpha_{I_l,1}$ $(k \geqslant l \geqslant 1)$. *The converse of this is not true in general, as it is seen by easy examples.*

The following lemma will be fundamental.

LEMMA 12. *Assume that either* $(H_1)-(H_4)$ *or* $(H_1)-(H_3)$ *and* (H_5) *are satisfied. Then*

$$
A_{k,l} \leqslant A_{k,l-1}, \quad k \geqslant l \geqslant 2.
$$

Proof. Assume $(H_1)-(H_4)$. We prove first that $A_{k,k} \leqslant A_{k,k-1}$. Since

$$
A_{k,k} = \sum_{(i_1,\ldots,i_k) \in I_k} \left(\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} \right) f \left(\frac{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} x_{i_s}}{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}}} \right)
$$

$$
= \sum_{(i_1,\ldots,i_k) \in I_k} \left(\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} \right) f \left(\sum_{m=1}^{k} \left(\frac{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} - \frac{p_{i_m}}{\alpha_{I_k,i_m}}}{(k-1) \sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}}} \cdot \frac{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} x_{i_s} - \frac{p_{i_m}}{\alpha_{I_k,i_m}} x_{i_m}}{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} - \frac{p_{i_m}}{\alpha_{I_k,i_m}}} \right) \right),
$$

and

$$\frac{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} - \frac{p_{i_m}}{\alpha_{I_k,i_m}}}{(k-1)\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}}} \geqslant 0, \quad 1 \leqslant m \leqslant k,$$

and

$$\sum_{m=1}^{k} \left(\frac{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} - \frac{p_{i_m}}{\alpha_{I_k,i_m}}}{(k-1)\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}}} \right) = 1, \quad (i_1,\ldots,i_k) \in I_k,$$

the Jensen's inequality (1) for convex functions implies

$$A_{k,k} \leqslant \sum_{(i_1,\ldots,i_k)\in I_k} \left(\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} \right) \sum_{m=1}^{k} \left(\frac{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_j}} - \frac{p_{i_m}}{\alpha_{I_k,i_m}}}{(k-1)\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}}} f \left(\frac{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} x_{i_s} - \frac{p_{i_m}}{\alpha_{I_k,i_m}} x_{i_m}}{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} - \frac{p_{i_m}}{\alpha_{I_k,i_m}}} \right) \right)$$

$$\tag{14}$$

$$= \frac{1}{k-1} \sum_{(i_1,\ldots,i_k)\in I_k} \left(\sum_{m=1}^{k} \left(\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} - \frac{p_{i_m}}{\alpha_{I_k,i_m}} \right) f \left(\frac{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} x_{i_s} - \frac{p_{i_m}}{\alpha_{I_k,i_m}} x_{i_m}}{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} - \frac{p_{i_m}}{\alpha_{I_k,i_m}}} \right) \right).$$

In light of the meaning of $t_{I_k,k-1}$, this yields

$$A_{k,k} \leqslant \frac{1}{k-1} \sum_{(j_1,\ldots,j_{k-1})\in I_{k-1}} t_{I_k,k-1}(j_1,\ldots,j_{k-1}) \left(\sum_{s=1}^{k-1} \frac{p_{j_s}}{\alpha_{I_k,j_s}} \right) f \left(\frac{\sum_{s=1}^{k-1} \frac{p_{j_s}}{\alpha_{I_k,j_s}} x_{j_s}}{\sum_{s=1}^{k-1} \frac{p_{j_s}}{\alpha_{I_k,j_s}}} \right)$$

$$= A_{k,k-1}.$$

Suppose now that $k-1 \geqslant l \geqslant 2$. By an argument analogous to that employed in the first part we have that

$$A_{k,l} \leqslant \frac{1}{(k-1)\ldots l(l-1)} \sum_{(i_1,\ldots,i_l)\in I_l} \left(t_{I_k,l}(i_1,\ldots i_l) \right.$$

$$\cdot \sum_{m=1}^{l} \left(\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}} - \frac{p_{i_m}}{\alpha_{I_k,i_m}} \right) f \left(\frac{\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}} x_{i_s} - \frac{p_{i_m}}{\alpha_{I_k,i_m}} x_{i_m}}{\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}} - \frac{p_{i_m}}{\alpha_{I_k,i_m}}} \right) \right),$$

and therefore the definitions of the set $H_{I_l}(j_1, \ldots, j_{l-1})$ and the function $t_{I_k, l-1}$ give

$$
\begin{aligned}
A_{k,l} \leqslant &\frac{1}{(k-1)\ldots l\,(l-1)} \\
&\cdot \sum_{(j_1,\ldots,j_{l-1})\in I_{l-1}} \left(\left(\sum_{((i_1,\ldots,i_l),m)\in H_{I_l}(j_1,\ldots,j_{l-1})} t_{I_k,l}(i_1,\ldots,i_l) \right) \right. \\
&\left. \cdot \left(\sum_{s=1}^{l-1} \frac{p_{j_s}}{\alpha_{I_k,j_s}} \right) f \left(\frac{\sum_{s=1}^{l-1} \frac{p_{j_s}}{\alpha_{I_k,j_s}} x_{j_s}}{\sum_{s=1}^{l-1} \frac{p_{j_s}}{\alpha_{I_k,j_s}}} \right) \right) = A_{k,l-1},
\end{aligned}
$$

and this completes the proof in the considered case.

    We turn now to the other case: assume (H_1)–(H_3) and (H_5). Since the numbers $\alpha_{I_k,i}$ $(1 \leqslant i \leqslant n)$ are integers the proof is entirely similar as above (the Jensen's inequality (1) for mid-convex functions can be applied in (14)).

    The proof is now complete. $\square$

After these preliminaries we arrive to the proof of Theorem 1.

Proof. First, assume (H_1)–(H_4).
(a) Since

$$
\sum_{r=1}^{n} p_r x_r = \sum_{(i_1,\ldots,i_k)\in I_k} \left(\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} \frac{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} x_{i_s}}{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}}} \right),
$$

and

$$
\sum_{(i_1,\ldots,i_k)\in I_k} \left(\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} \right) = 1,
$$

it follows from the Jensen's inequality (1) for convex functions that

$$
f\left(\sum_{r=1}^{n} p_r x_r \right) \leqslant \sum_{(i_1,\ldots,i_k)\in I_k} \left(\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} \right) f\left(\frac{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,ij}} x_{i_s}}{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}}} \right) = A_{k,k}, \tag{15}
$$

which proves the first inequality in (8).

    The inequalities

$$
A_{k,k} \leqslant A_{k,k-1} \leqslant \ldots \leqslant A_{k,2} \leqslant A_{k,1}
$$

can be obtained from Lemma 12.

It remains only to show that

$$A_{k,1} = \sum_{r=1}^{n} p_r f(x_r) \tag{16}$$

By the definition of $A_{k,1}$

$$A_{k,1} = \frac{1}{(k-1)!} \sum_{s=1}^{n} t_{I_k,1}(s) \frac{p_s}{\alpha_{I_k,s}} f(x_s),$$

and therefore Lemma 9 insures (16).

(b) It follows from (a) by applying Lemma 10 (e).

If $(H_1)-(H_3)$ and (H_5) are satisfied, then we can prove as before, since the numbers $\alpha_{I_k,i}$ $(1 \leqslant i \leqslant n)$ are integers, and since the Jensen's inequality (1) for mid-convex functions can be used in (15).

The proof of the theorem is complete. $\quad\square$

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