Mathematical
Inequalities
& Applications
Volume 14, Number 4 (2011), 793–810

COMPACTNESS OF EMBEDDING BETWEEN SPACES WITH MULTIWEIGHTED DERIVATIVES – THE CASE $p \leqslant q$

ZAMIRA ABDIKALIKOVA, ASKAR BAIARYSTANOV AND RYSKUL OINAROV

(*Communicated by V. Burenkov*)

Abstract. This paper deal with a new Sobolev type function space called the space with multiweighted derivatives. This space is a generalization of the usual one dimensional Sobolev space. As basis for this space serves some differential operators containing weight functions.

We establish necessary and sufficient conditions for the boundedness and compactness of the embedding between the spaces with multiweighted derivatives with different weights and different metrics.

1. Introduction

Let R be the set of real numbers, m and n be natural numbers, $1 \leqslant p,q < \infty$, $\bar{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_n)$, $\alpha_i \in R$, $i = 0,1,\ldots,n$, $|\bar{\alpha}| = \sum_{i=0}^{n} \alpha_i$, $I = (0,1)$ or $I = (1,+\infty)$ and $\frac{1}{p} + \frac{1}{p'} = 1$ (if $p = 1$, then $p' = \infty$).

For functions $f\colon I \to R$ we define the differential operations $D^i_{\bar{\alpha}} f$ as follows:

$$D^0_{\bar{\alpha}} f(t) = t^{\alpha_0} f(t),$$

$$D^i_{\bar{\alpha}} f(t) = t^{\alpha_i} \frac{d}{dt} t^{\alpha_{i-1}} \frac{d}{dt} \ldots t^{\alpha_1} \frac{d}{dt} t^{\alpha_0} f(t),\ i = 1,2,\ldots,n,$$

where each derivative is understood as a weak derivative (see e.g. [6]).

DEFINITION 1.1. The operation $D^i_{\bar{\alpha}} f$ is called α - multiweighted derivative of the function f of order i, $i = 0,1,\ldots,n$.

With help of the operator $D^i_{\bar{\alpha}}$, $i = 0,1,\ldots,n$, we define a space $W^n_{p,\bar{\alpha}} = W^n_{p,\bar{\alpha}}(I)$, $1 \leqslant p < \infty$, $I = (0,1)$ or $I = (1,+\infty)$, of functions $f : I \to R$ which have α - multiweighted n:th order derivatives and for which the following norm is finite:

$$\|f\|_{W^n_{p,\bar{\alpha}}} = \|D^n_{\bar{\alpha}} f\|_p + \sum_{i=0}^{n-1} |D^i_{\bar{\alpha}} f(1)|,$$

Mathematics subject classification (2010): 46E35, 46E30.
Keywords and phrases: Weighted function space, multiweighted derivative, embedding theorems, compactness.

Paper MIA-14-65

where $\|\cdot\|_p$ denotes the usual norm of the space $L_p(I)$, $1 \leqslant p < \infty$.

When $\alpha_i = 0$, $i = 0,1,\dots,n-1$, and $\alpha_n = \gamma$ the space $W^n_{p,\bar{\alpha}}$ coincides with the usual Kudryavtsev space $L^n_{p,\gamma} = L^n_{p,\gamma}(I)$ (see [10]) with the finite norm

$$\|f\|_{L^n_{p,\gamma}} = \|t^\gamma f^{(n)}\|_p + \sum_{i=0}^{n-1} |f^{(i)}(1)|.$$

Spaces of $W^n_{p,\bar{\alpha}}$ type appeared first in [5] with the purpose to characterize the behavior of a solution of singular differential equations in a neighbourhood of the singular point $t = 0$. Detailed investigation of the properties of the spaces $W^n_{p,\bar{\alpha}}$ and their applications in the theory of differential equations can be found in the papers by Baidel'dinov (e.g. [4]), by Kalybay (e.g. [8]) and others.

Moreover, for $\bar{\beta} = (\beta_0, \beta_1, \dots, \beta_m)$, $\beta_i \in R$, $i = 0,1,\dots,m$, we define the space $W^m_{q,\bar{\beta}} = W^m_{q,\bar{\beta}}(I)$.

The main aim of this paper is to investigate boundedness and compactness of the embedding

$$W^n_{p,\bar{\alpha}}(I) \hookrightarrow W^m_{q,\bar{\beta}}(I) \tag{1.1}$$

when $1 < p \leqslant q < \infty$, $0 \leqslant m < n$.

Boundedness of the embedding (1.1) has been considered in [1] when $1 \leqslant p \leqslant q < \infty$ and in [7] when $1 < q < p < \infty$ in different assumptions concerning α_i, $i = 0,1,\dots,n$. In [3] necessary and sufficient conditions for boundedness and compactness of the embedding (1.1) have been established when $1 \leqslant q < p < \infty$.

In this paper we derive necessary and sufficient conditions for the boundedness and compactness of the embedding (1.1) when $1 < p \leqslant q < \infty$, without dependence on powers of degeneration of the space $W^n_{p,\bar{\alpha}}(I)$, i.e. for which relations between the parameters $\bar{\alpha}$, $\bar{\beta}$, p and q the bounded and compact embedding (1.1) holds.

The main results are given in Sections 3 and 4. In Section 2 we present some notations, facts and statements, which are necessary for the proofs of the main results. In Section 3 the embedding (1.1) is considered for $I = (0,1)$ and in Section 4 for $I = (1,+\infty)$.

In this paper we use the following *conventions*: If $i > j$, then the sum $\sum_{k=i}^{j}$ is considered to be equal to zero; and the notation $A \ll B$ means that $A \leqslant cB$, where the constant $c > 0$ may depend on unessential parameters.

2. Preliminaries

For $i, j = 0,1,\dots,n-1$ we define the following set of functions:

$$K_{i+1,j}(t,x) \equiv K_{i+1,j}(t,x,\bar{\alpha}) = \int_t^x t_{i+1}^{-\alpha_{i+1}} \int_{t_{i+1}}^x t_{i+2}^{-\alpha_{i+2}} \dots \int_{t_{j-1}}^x t_j^{-\alpha_j} dt_j dt_{j-1} \dots dt_{i+1} \text{ when } i < j,$$

$$K_{i+1,j}(t,x) \equiv K_{i+1,j}(t,x,\bar{\alpha}) \equiv 1 \text{ when } i = j,$$

$$K_{i+1,j}(t,x) \equiv K_{i+1,j}(t,x,\bar{\alpha}) \equiv 0 \text{ when } i > j \text{ for } 0 < t \leqslant x.$$

By changing variables, when $i < j$ the following property of uniformity of the functions $K_{i+1,j}$ can be established (see [3]):

$$K_{i+1,j}(zt,zx) = z^{\sum\limits_{k=i+1}^{j}(1-\alpha_k)} K_{i+1,j}(t,x).$$

In particular, when $x = 1$ and $t = 1$, we have that

$$\begin{aligned} K_{i+1,j}(zt,z) &= z^{\sum\limits_{k=i+1}^{j}(1-\alpha_k)} K_{i+1,j}(t,1), \\ K_{i+1,j}(z,zx) &= z^{\sum\limits_{k=i+1}^{j}(1-\alpha_k)} K_{i+1,j}(1,x), \end{aligned} \tag{2.1}$$

respectively.

For $0 \leqslant i \leqslant j \leqslant n-1$ we define:

$$M_{i,j} = \max_{i \leqslant s \leqslant j} (j - s + 1 - \sum_{k=s+1}^{j+1} \alpha_k)$$

and

$$k_{i,j} = \min\{k :\ i \leqslant k \leqslant j,\ \sum_{s=i+1}^{k} \alpha_s - k = \max_{i \leqslant \xi \leqslant j} (\sum_{s=i+1}^{\xi} \alpha_s - \xi)\}.$$

For convenience we denote $M_i = M_{i,n-1}$ and $k_i \equiv k_{i,n-1}$. Note that $M_i \geqslant M_{i+1}$ and $M_0 = \max\limits_{0 \leqslant i \leqslant n-1} M_i$.

Furthermore, we need upper and lower estimates for the functions $K_{i+1,j}(t,1)$ when $0 < t \leqslant 1$ and $K_{i+1,n-1}(1,t)$ when $1 \leqslant t < \infty$, $0 \leqslant i \leqslant j \leqslant n-1$. In [2] it was obtained upper and lower estimates for the functions $u_i(t) = t^{\alpha_0} K_{1,i}(t,1,-\bar{\alpha})$, $i = 0,1,\ldots,n-1$. Below we give three statements about estimates for the functions $K_{i+1,j}(t,1)$ and $K_{i+1,j}(1,t)$, which follow from these results. Moreover, for convenience we use the following equalities:

$$\begin{aligned} \min_{i \leqslant s \leqslant j} \left(\alpha_0 + \sum_{k=i+1}^{s} (1-\alpha_k) \right) &= \min_{i \leqslant s \leqslant j} \left[\alpha_0 + j - i + 1 - \sum_{k=i+1}^{j+1} \alpha_k - (j - s + 1 - \sum_{k=s+1}^{j+1} \alpha_k) \right] \\ &= \alpha_0 + j - i + 1 - \sum_{k=i+1}^{j+1} \alpha_k - M_{i,j}. \end{aligned}$$

LEMMA 2.1. *Let* $0 \leqslant i \leqslant j \leqslant n-1$. *Then*

$$K_{i+1,j}(t,1) \ll t^{\,j-i+1-\sum\limits_{k=i+1}^{j+1}\alpha_k - M_{i,j}} |\ln t|^{l_{i,j}},\ t \in (0,1],$$

where $l_{i,j}$ *is the number of* k, $k_{i,j}+1 \leqslant k \leqslant j$, *such that* $\sum\limits_{s=k_{i,j}+1}^{k} (\alpha_s - 1) = 0$ *if* $k_{i,j} < j$, *and* $l_{i,j} = 0$ *if* $k_{i,j} = j$.

LEMMA 2.2. *Let* $0 \leqslant i \leqslant n-1$*. Then there exists* δ*,* $0<\delta<1$*, such that for any* $t \in (0,\delta]$ *the following estimate*

$$K_{i+1,n-1}(t,1) \gg t^{n-i-\sum\limits_{k=i+1}^{n}\alpha_k - M_i}$$

holds.

LEMMA 2.3. *Let* $0 \leqslant i \leqslant n-1$*. Then*

$$x^{-\alpha_n}K_{i+1,n-1}(1,x) \ll x^{M_i-1}|\ln x|^{l_i},\ x \geqslant 1,$$

where l_i *is the number of* k*,* $i+1 \leqslant k \leqslant k_i-1$*, such that* $\sum\limits_{s=k}^{k_i-1}(\alpha_s-1)=0$ *when* $k_i>i+1$*, and* $l_i=0$ *when* $k_i=i+1$*.*

For the proof of our main results we also need the following statements from [3], [7] and [9], respectively:

LEMMA 2.4. *The functions* $f_s(t)=t^{-\alpha_0}K_{1,s}(t,1,\bar{\alpha})$*,* $0 \leqslant m \leqslant s \leqslant n$*, are not solutions of the equation*

$$D^m_{\bar{\beta}}f(t)=0,\ \forall t \in (0,1].$$

LEMMA 2.5. *For all* $f \in W^n_{p,\bar{\alpha}}$*,* $0 \leqslant m < n$*, we have that*

$$D^k_{\bar{\beta}}f(t)=\sum_{i=0}^{k}c_{k,i}t^{\mu_{k,i}}D^i_{\bar{\alpha}}f(t),\ k=0,1,\ldots,m, \tag{2.2}$$

where $\mu_{k,i}=\sum\limits_{j=0}^{k}\beta_j-\sum\limits_{j=0}^{i}\alpha_j+i-k$*,* $i=0,1,\ldots,k$*,* $k=0,1,\ldots,m$*; and the coefficients* $c_{k,i}$*,* $i=0,1,\ldots,k-1$*,* $k=0,1,\ldots,m$*, are defined by the recurrent formula:*

$$c_{k,k}=1, \qquad c_{k,0}=c_{k-1,0}\left(\sum_{j=0}^{k-1}\beta_j-\alpha_0-k+1\right),$$

$$c_{k,i}=c_{k-1,i-1}+c_{k-1,i}\left(\sum_{j=0}^{k-1}\beta_j-\sum_{j=0}^{i}\alpha_j+i-k+1\right),\ i=1,2,\ldots,k-1,$$

and

$$D^i_{\bar{\alpha}}f(t)=\sum_{j=i}^{n-1}(-1)^{j-i}K_{i+1,j}(t,1)D^j_{\bar{\alpha}}f(1)+\int_t^1 x^{-\alpha_n}K_{i+1,n-1}(t,x)D^n_{\bar{\alpha}}f(x)dx, \tag{2.3}$$

$i=0,1,\ldots,n-1$.

3. Embedding theorems for the space $W^n_{p,\overline{\alpha}}(0,1)$

Denote $i_0=\min\{i:\ 0\leqslant i\leqslant m,\, c_{m,i}\neq 0\}$, where $c_{m,i}$, $i=0,1,\ldots,m$, are defined as in Lemma 2.5.

Our main result in this Section reads:

THEOREM 3.1. *Let $I=(0,1)$, $1<p\leqslant q<\infty$, $0\leqslant m<n$ and $M_{i_0}\geqslant\frac{1}{p}$. Then the following conditions are equivalent:*
i) The embedding (1.1) is bounded;
ii) The embedding (1.1) is compact;
iii)

$$|\bar{\beta}|-|\bar{\alpha}|+n-m+\frac{1}{q}>M_{i_0}. \tag{3.1}$$

Proof. Let us first prove that $i)\Rightarrow iii)$. Assume that the embedding (1.1) is bounded. Then

$$\|D^m_{\bar{\beta}}f\|_q\leqslant c\|f\|_{W^n_{p,\bar{\alpha}}},\ \forall f\in W^n_{p,\bar{\alpha}}. \tag{3.2}$$

Let us take the function $f_0(t)=t^{-\alpha_0}K_{1,n-1}(t,1)$. Since $D^n_{\bar{\alpha}}f_0(t)=0$, $\forall t\in(0,1)$, and $D^i_{\bar{\alpha}}f_0(1)=0$, $i=0,1,\ldots,n-2$, $|D^{n-1}_{\bar{\alpha}}f_0(1)|=1$, then $f_0\in W^n_{p,\bar{\alpha}}$ and $\|f_0\|_{W^n_{p,\bar{\alpha}}}=1$. Hence, from (3.2) we have that

$$\|D^m_{\bar{\beta}}f_0\|_q\leqslant c. \tag{3.3}$$

Moreover, due to Lemma 2.4 we get that $\|D^m_{\bar{\beta}}f_0\|_q>0$. From (2.2) and (3.3) it follows that

$$\int_0^1\Big|\sum_{i=i_0}^{m}(-1)^i c_{m,i}t^{\mu_{m,i}}K_{i+1,n-1}(t,1)\Big|^q dt<\infty. \tag{3.4}$$

According to Lemma 2.2 for small enough $t>0$ we obtain that

$$K_{i+1,n-1}(t,1)\gg t^{\,n-i-\sum\limits_{k=i+1}^{n}\alpha_k-M_i},\ i=0,1,\ldots,n-1.$$

Therefore,

$$t^{\mu_{m,i}}K_{i+1,n-1}(t,1)\gg t^{|\bar{\beta}|-|\bar{\alpha}|+n-m-M_i},\ i=i_0,i_0+1,\ldots,m,$$

in a neighbourhood of $t=0$. Since $c_{m,i_0}\neq 0$ and $M_{i_0}\geqslant M_i$, $i_0\leqslant i\leqslant m$, then for $M_{i_0}>\frac{1}{p}$ the order of the underintegral function in (3.4) in a neighbourhood of $t=0$

is not less than $t^{|\bar{\beta}|-|\bar{\alpha}|+n-m-M_{i_0}}$. Consequently, the function $t^{(|\bar{\beta}|-|\bar{\alpha}|+n-m-M_{i_0})q}$ is integrable in a neighbourhood of $t=0$ and this is equivalent to the condition

$$|\bar{\beta}|-|\bar{\alpha}|+n-m+\frac{1}{q}>M_{i_0}.$$

Hence, the implication $i) \Rightarrow iii)$ is proved.

Obviously, it is sufficient to show that $iii) \Rightarrow ii)$. Assume that $iii)$ holds and let $f \in W^n_{p,\bar{\alpha}}$. According to (2.2) and (2.3) we have that

$$D^m_{\bar{\beta}} f(t) = \sum_{i=i_0}^{m} c_{m,i} t^{\mu_{m,i}} \sum_{j=i}^{n-1} (-1)^{j-i} K_{i+1,j}(t,1) D^j_{\bar{\alpha}} f(1)$$
$$+ \sum_{i=i_0}^{m} c_{m,i} t^{\mu_{m,i}} \int_t^1 x^{-\alpha_n} K_{i+1,n-1}(t,x) D^n_{\bar{\alpha}} f(x)dx. \tag{3.5}$$

Moreover, from (2.2) it follows that

$$\sum_{k=0}^{m-1} |D^k_{\bar{\beta}} f(1)| \ll \sum_{k=i_0}^{n-1} |D^k_{\bar{\alpha}} f(1)|,$$

and, therefore, for boundedness of the embedding (1.1) it is sufficient that the conditions

$$\int_0^1 |t^{\mu_{m,i}} K_{i+1,j}(t,1)|^q dt < \infty, \;\; i = i_0, i_0+1, \ldots, m, \;\; j = i, i+1, \ldots, n-1, \tag{3.6}$$

hold and that the following integral operators

$$K_i F(t) = t^{\mu_{m,i}} \int_t^1 x^{-\alpha_n} K_{i+1,n-1}(t,x) F(x)dx, \;\; i = i_0, i_0+1, \ldots, m, \tag{3.7}$$

are bounded from $L_p(0,1)$ to $L_q(0,1)$. Moreover, for the compactness of the embedding (1.1), due to finiteness of the first sum on the right hand side in (3.5), it is sufficient to prove that the operators (3.7) from $L_p(0,1)$ to $L_q(0,1)$ are compact.

First we prove that (3.6) holds. For $i_0 \leqslant i \leqslant m$, according to Lemma 2.1, we have that

$$\int_0^1 |t^{\mu_{m,i}} K_{i+1,j}(t,1)|^q dt \ll \int_0^1 t^{q(\mu_{m,i}+j-i+1-\sum\limits_{k=i+1}^{j+1} \alpha_k - M_{i,j})} |\ln t|^{q l_{i,j}} dt.$$

The last integral converges if and only if

$$\mu_{m,i} + j - i + 1 - \sum_{k=i+1}^{j+1} \alpha_k - M_{i,j} + \frac{1}{q} > 0, \tag{3.8}$$

$i = i_0, i_0+1, \dots, m,\ j = i, i+1, \dots, n-1$, i.e.,

$$|\bar{\beta}| - |\bar{\alpha}| + n - m + \frac{1}{q} > \max_{i \leqslant s \leqslant j} \left(j - s + 1 - \sum_{k=s+1}^{j+1} \alpha_k \right) - j - \sum_{k=j+2}^{n} \alpha_k + n$$
$$= \max_{i \leqslant s \leqslant j} \left(n - s + 1 - \sum_{k=s+1}^{n} \alpha_k \right).$$

But by the definition $M_{i_0} \geqslant \max\limits_{i \leqslant s \leqslant j} \left(n - s + 1 - \sum\limits_{k=s+1}^{n} \alpha_k \right)$ when $i_0 \leqslant i \leqslant m$ and $i \leqslant j \leqslant n-1$, then from (3.1) it follows that (3.8) holds and, hence, that (3.6) holds.

According to the results in [11] the integral operators (3.7) are compact from $L_p(0,1)$ to $L_q(0,1)$ when $1 < p \leqslant q < \infty$ if and only if

$$\max_{i \leqslant j \leqslant n-1} \sup_{0<z<1} A_{i,j}(z) < \infty,\ i = i_0, i_0+1, \dots, m, \tag{3.9}$$

and

$$\lim_{z \to 0} A_{i,j}(z) = \lim_{z \to 1} A_{i,j}(z) = 0, \tag{3.10}$$

$i = i_0, i_0+1, \dots, m,\ j = i, i+1, \dots, n-1$; where

$$A_{i,j}(z) = \left(\int_0^z |t^{\mu_{i,m}} K_{i+1,j}(t,z)|^q dt \right)^{\frac{1}{q}} \left(\int_z^1 |x^{-\alpha_n} K_{j+1,n-1}(z,x)|^{p'} dx \right)^{\frac{1}{p'}}. \tag{3.11}$$

Due to (3.6) the first integral in (3.11) converges for all $0 \leqslant z \leqslant 1$, and the underintegral function of the second integral is continuous on $(0,1]$, we find that the function $A_{i,j}(z)$ is continuous on $(0,1]$ and $\lim\limits_{z \to 1} A_{i,j}(z) = 0$ for all $i = i_0, i_0+1, \dots, m,\ j = i, i+1, \dots, n-1$. Therefore, the fulfilment of (3.9) and (3.10) depends on the behavior of the function $A_{i,j}(z)$ when $z \to 0$.

In the integrals (3.11), by changing variables $t \to tz,\ x \to xz$, respectively, and, using the property of uniformity (2.1), we find that

$$\left(\int_0^z |t^{\mu_{m,i}} K_{i+1,j}(t,z)|^q dt \right)^{\frac{1}{q}} = z^{\mu_{m,i} + \sum\limits_{k=i+1}^{j} (1-\alpha_k) + \frac{1}{q}} \left(\int_0^1 |t^{\mu_{m,i}} K_{i+1,j}(t,1)|^q dt \right)^{\frac{1}{q}}$$
$$= c_{i,j} z^{|\bar{\beta}| - \sum\limits_{k=0}^{j} \alpha_k + j - m + \frac{1}{q}}, \tag{3.12}$$

where, due to (3.6), $c_{i,j} = \left(\int\limits_0^1 |t^{\mu_{m,i}} K_{i+1,j}(t,1)|^q dt \right)^{\frac{1}{q}} < \infty$ when $j = i, i+1, \dots, n-1$, $i = i_0, i_0+1, \dots, m$, and

$$\left(\int_z^1 |x^{-\alpha_n} K_{j+1,n-1}(z,x)|^{p'} dx\right)^{\frac{1}{p'}}$$

$$= z^{-\alpha_n+\frac{1}{p'}+\sum\limits_{k=j+1}^{n-1}(1-\alpha_k)} \left(\int_1^{\frac{1}{z}} |x^{-\alpha_n} K_{j+1,n-1}(1,x)|^{p'} dx\right)^{\frac{1}{p'}}$$

[according to Lemma 2.3]

$$\ll z^{-\frac{1}{p}-\sum\limits_{k=j+1}^{n}\alpha_k+n-j} \left(\int_1^{\frac{1}{z}} |x^{p'(M_j-1)}| \ln x|^{p' l_j} dx\right)^{\frac{1}{p'}}. \tag{3.13}$$

Since

$$\int_1^{\infty} x^{p'(M_j-1)} |\ln x|^{p' l_j} dx < \infty \text{ when } M_j < \frac{1}{p}, \; j = 0, 1, \ldots, n-1,$$

then for small enough $z > 0$ we have that

$$\int_1^{\frac{1}{z}} x^{p'(M_j-1)} |\ln x|^{p' l_j} dx \ll \begin{cases} z^{-p'(M_j-1)-1} |\ln z|^{p' l_j} & \text{when } M_j > \frac{1}{p}, \\ 1 & \text{when } M_j < \frac{1}{p}, \\ |\ln z|^{p' l_j + 1} & \text{when } M_j = \frac{1}{p}. \end{cases} \tag{3.14}$$

From (3.11)–(3.14) for small $z > 0$ we get that

$$A_{i,j}(z) \ll z^{|\bar{\beta}|-|\bar{\alpha}|+n-m+\frac{1}{q}-M_j} |\ln z|^{l_j} \text{ when } M_j > \frac{1}{p}, \tag{3.15}$$

$$A_{i,j}(z) \ll z^{|\bar{\beta}|-|\bar{\alpha}|+n-m+\frac{1}{q}-\frac{1}{p}} \text{ when } M_j < \frac{1}{p}, \tag{3.16}$$

$$A_{i,j}(z) \ll z^{|\bar{\beta}|-|\bar{\alpha}|+n-m+\frac{1}{q}-\frac{1}{p}} |\ln z|^{l_j+\frac{1}{p'}} \text{ when } M_j = \frac{1}{p}. \tag{3.17}$$

Moreover, by the assumptions of Theorem 3.1, it yields that $M_{i_0} \geqslant \frac{1}{p}$, and by the definition we have that $M_{i_0} \geqslant M_j$ when $i_0 \leqslant j \leqslant n-1$. Therefore, from *iii*) it follows that

$$|\bar{\beta}| - |\bar{\alpha}| + n - m + \frac{1}{q} - \max\left\{M_j, \frac{1}{p}\right\} > 0$$

for all $j = i_0, i_0+1, \ldots, n-1$. Hence, (3.15)–(3.17) imply that $\lim_{z\to 0} A_{i,j}(z) = 0$ for all $i = i_0, i_0+1, \ldots, m$, $j = i, i+1, \ldots, n-1$, i.e. (3.9) and (3.10) hold.

Thus *iii)* implies (3.6) and the compactness of the integral operators (3.7). Consequently, also the implication *iii)* $\Rightarrow$ *ii)* is proved. The proof is complete. □

THEOREM 3.2. *Let* $I = (0,1)$, $1 < p \leqslant q < \infty$, $0 \leqslant m < n$ *and* $M_{i_0} < \dfrac{1}{p}$.

a) The embedding (1.1) *is bounded if and only if*

$$|\bar{\beta}| - |\bar{\alpha}| + n - m + \frac{1}{q} \geqslant \frac{1}{p}. \tag{3.18}$$

b) The embedding (1.1) *is compact if and only if*

$$|\bar{\beta}| - |\bar{\alpha}| + n - m + \frac{1}{q} > \frac{1}{p}. \tag{3.19}$$

Proof. Let us first prove a). Let the embedding (1.1) be bounded. Consider the function $f_0(t) = t^{n-|\bar{\alpha}|-\frac{1}{p}+\varepsilon}$, where $\varepsilon > 0$. Then

$$D^n_{\bar{\alpha}} f_0(t) = \prod_{j=0}^{n-1} \left(n - j - \sum_{k=j+1}^{n} \alpha_k - \frac{1}{p} + \varepsilon \right) t^{-\frac{1}{p}+\varepsilon},$$

and, consequently, $f_0 \in W^n_{p,\bar{\alpha}}$. A direct calculation implies that

$$D^m_{\bar{\beta}} f_0(t) = \prod_{i=0}^{m-1} \left(\sum_{k=0}^{i} \beta_k - |\bar{\alpha}| + n - i - \tfrac{1}{p} + \varepsilon \right) t^{|\bar{\beta}|-|\bar{\alpha}|+n-m-\frac{1}{p}+\varepsilon}.$$

Since we have only finite many factors in the product, then there exists $\varepsilon_0 > 0$ such that, for each $\varepsilon \in (0, \varepsilon_0)$:

$$\prod_{i=0}^{m-1} \left(\sum_{k=0}^{i} \beta_k - |\bar{\alpha}| + n - i - \frac{1}{p} + \varepsilon \right) \neq 0.$$

Due to the boundedness of the embedding (1.1) it must hold that $D^m_{\bar{\beta}} f_0 \in L_q(0,1)$, but this is possible if and only if

$$|\bar{\beta}| - |\bar{\alpha}| + n - m - \frac{1}{p} + \varepsilon + \frac{1}{q} > 0 \text{ when } \varepsilon \in (0, \varepsilon_0).$$

Hence, by letting $\varepsilon \to 0$, we have (3.18).

On the contrary, assume that (3.18) holds. In Theorem 3.1 it was shown that the embedding (1.1) is bounded if (3.6) holds and the integral operators (3.7) are bounded from $L_p(0,1)$ to $L_q(0,1)$ and this is equivalent to the condition (3.9). By the assumptions of Theorem 3.2 it yields that $M_{i_0} < \dfrac{1}{p}$ and, therefore, from (3.18) it follows that (3.1) holds, which, in its turn, implies (3.6), as it was in Theorem 3.1. Since $\frac{1}{p} > M_{i_0} \geqslant M_j$, $i_0 \leqslant j \leqslant n-1$, then from (3.16) and (3.18) it follows that (3.9) holds. Thus a) is proved.

Let us now prove b). Assume that the embedding (1.1) is compact. Then (3.18) holds. We suppose that in (3.18) it will be equality, i.e. that

$$|\bar{\beta}| - |\bar{\alpha}| + n - m + \frac{1}{q} = \frac{1}{p}. \tag{3.20}$$

We consider the following set of functions:

$$f_\varepsilon(t) = c_\varepsilon t^{-\alpha_0} \int_t^1 K_{1,n-1}(t,x) x^{-\alpha_n} \chi_{(0,\varepsilon)}(x) x^{-\frac{\varepsilon}{p}} dx,\ 0 < \varepsilon < 1,$$

where c_ε is a constant and $\chi_{(0,\varepsilon)}(\cdot)$ denotes the characteristic function of the interval $(0,\varepsilon)$.

Since $D^n_{\bar{\alpha}} f_\varepsilon(t) = c_\varepsilon (-1)^n \chi_{(0,\varepsilon)}(t) t^{-\frac{\varepsilon}{p}}$, then $f_\varepsilon \in W^n_{p,\bar{\alpha}}$ for all $\varepsilon \in (0,1)$.

We now choose the constant c_ε such that $\|f_\varepsilon\|_{W^n_{p,\bar{\alpha}}} = \|D^n_{\bar{\alpha}} f_\varepsilon\|_p = 1$, i.e.

$$c_\varepsilon = (1-\varepsilon)^{\frac{1}{p}} \varepsilon^{\frac{\varepsilon - 1}{p}}.$$

Let us show that the set of functions f_ε, $0 < \varepsilon < 1$, converges weakly to zero when $\varepsilon \to 0$. By definition of the space $W^n_{p,\bar{\alpha}}$ it follows that it is isometric to the space $L_p(I) \times R^n$. Therefore, $(W^n_{p,\bar{\alpha}})^* = (L_p(I) \times R^n)^* = L_{p'}(I) \times R^n$. Since $D^i_{\bar{\alpha}} f_\varepsilon(1) = 0$, $i = 0, 1, \ldots, n-1$, then, according to Hölder's inequality, for each $G \equiv (g,a) \in L_{p'}(I) \times R^n$ we have that

$$\begin{aligned} |\langle f_\varepsilon, G\rangle| &= \left| \int_0^1 D^n_{\bar{\alpha}} f_\varepsilon(t) g(t) dt \right| = c_\varepsilon \left| \int_0^\varepsilon t^{-\frac{\varepsilon}{p}} g(t) dt \right| \\ &\leqslant c_\varepsilon \left(\int_0^\varepsilon t^{-\varepsilon} dt \right)^{\frac{1}{p}} \left(\int_0^\varepsilon |g(t)|^{p'} dt \right)^{\frac{1}{p'}} = \left(\int_0^\varepsilon |g(t)|^{p'} dt \right)^{\frac{1}{p'}}. \end{aligned}$$

It yields that $\langle f_\varepsilon, G\rangle \to 0$ when $\varepsilon \to 0$ for all $G \in \left(W^n_{p,\bar{\alpha}}\right)^*$. Then, according to the compactness of embedding (1.1), the set of functions f_ε, $0 < \varepsilon < 1$, when $\varepsilon \to 0$ converges strongly to zero in $W^m_{q,\bar{\beta}}$. By using (2.2), (2.3) and (3.5) we find that

$$\begin{aligned} D^m_{\bar{\beta}} f_\varepsilon(t) &= \sum_{i=i_0}^m c_{m,i} t^{\mu_{m,i}} D^i_{\bar{\alpha}} f_\varepsilon(t) \\ &= c_\varepsilon \sum_{i=i_0}^m (-1)^i c_{m,i} t^{\mu_{m,i}} \int_t^1 K_{i+1,n-1}(t,x) x^{-\alpha_n} \chi_{(0,\varepsilon)}(x) x^{-\frac{\varepsilon}{p}} dx. \end{aligned} \tag{3.21}$$

Now we show that for $i = i_0, i_0+1, \ldots, m$ it holds that

$$\int_0^1 |t^{\mu_{m,i}} \int_t^1 K_{i+1,n-1}(t,x) x^{-\alpha_n} \chi_{(0,\varepsilon)}(x) x^{-\frac{\varepsilon}{p}} dx|^q dt < \infty, \tag{3.22}$$

for all $\varepsilon \in (0,1)$.

By changing variables, due to Lemma 2.3, we get that

$$\begin{aligned} &\int_0^1 \left| t^{\mu_{m,i}} \int_t^1 K_{i+1,n-1}(t,x) x^{-\alpha_n - \frac{\varepsilon}{p}} dx \right|^q dt \\ &\ll \int_0^1 \left| t^{\mu_{m,i} - \alpha_n - \frac{\varepsilon}{p} + 1 + \sum\limits_{k=i+1}^{n-1} (1-\alpha_k)} \int_1^{\frac{1}{t}} z^{M_i - 1 - \frac{\varepsilon}{p}} |\ln z|^{l_i} dz \right|^q dt. \end{aligned} \tag{3.23}$$

Since $M_{i_0} < \dfrac{1}{p}$ and $M_i \leqslant M_{i_0}$ for $i = i_0, i_0+1, \ldots, n$, then, for all $\varepsilon \in (0,1)$, it yields that $M_i - 1 - \dfrac{\varepsilon}{p} < 0$, $i = i_0, i_0+1, \ldots, n$. Consequently,

$$\int_1^{\frac{1}{t}} z^{M_i - 1 - \frac{\varepsilon}{p}} |\ln z|^{l_i} dz \leqslant \int_1^{\frac{1}{t}} |\ln z|^{l_i} dz \leqslant \frac{1}{t} |\ln t|^{l_i}.$$

Therefore (3.23) implies that

$$\begin{aligned} &\int_0^1 |t^{\mu_{m,i}} \int_t^1 K_{i+1,n-1}(t,x) x^{-\alpha_n - \frac{\varepsilon}{p}} dx|^q dt \\ &\ll \int_0^1 t^{(\mu_{m,i} - \alpha_n - \frac{\varepsilon}{p} + \sum\limits_{k=i+1}^{n-1} (1-\alpha_k)) q} |\ln t|^{q l_i} dt. \end{aligned} \tag{3.24}$$

From (3.20) we obtain that

$$\mu_{m,i} - \alpha_n - \frac{\varepsilon}{p} + \sum_{k=i+1}^{n-1} (1-\alpha_k) > -\frac{1}{q}$$

for all $\varepsilon \in (0,1)$ and, consequently, the last integral in (3.24) converges, which means that (3.22) holds.

We take the q - norm in both sides of (3.21) and find that

$$\|D^m_{\bar{\beta}} f_\varepsilon\|_q = c_\varepsilon \left(\int_0^1 |\sum_{i=i_0}^m (-1)^i c_{m,i} t^{\mu_{m,i}} \int_t^1 K_{i+1,n-1}(t,x) x^{-\alpha_n - \frac{\varepsilon}{p}} \chi_{(0,\varepsilon)}(x) dx|^q dt \right)^{\frac{1}{q}}$$
$$= c_\varepsilon \left(\int_0^\varepsilon |\sum_{i=i_0}^m (-1)^i c_{m,i} t^{\mu_{m,i}} \int_t^\varepsilon K_{i+1,n-1}(t,x) x^{-\alpha_n - \frac{\varepsilon}{p}} dx|^q dt \right)^{\frac{1}{q}}. \quad (3.25)$$

In (3.25) we first change variables $t \to \varepsilon t$ in the outer integral and next we change variables $x \to \varepsilon x$ in the inner integral. Then, by taking into account the relation (3.20), we get that

$$\|D^m_{\bar{\beta}} f_\varepsilon\|_q = \varepsilon^{|\bar{\beta}| - |\bar{\alpha}| + n - m + \frac{1}{q} - \frac{1}{p}} T_\varepsilon = T_\varepsilon,$$

where

$$T_\varepsilon = (1-\varepsilon)^{\frac{1}{p}} \left(\int_0^1 |\sum_{i=i_0}^m (-1)^i c_{m,i} t^{\mu_{m,i}} \int_t^1 K_{i+1,n-1}(t,x) x^{-\alpha_n - \frac{\varepsilon}{p}} dx|^q dt \right)^{\frac{1}{q}}.$$

Due to (3.22) we have that $T_\varepsilon < \infty$ for all $\varepsilon \in (0,1)$. Moreover,

$$T_0 = \lim_{\varepsilon \to 0} T_\varepsilon$$
$$= \lim_{\varepsilon \to 0} (1-\varepsilon)^{\frac{1}{p}} \left(\int_0^1 |\sum_{i=i_0}^m (-1)^i c_{m,i} t^{\mu_{m,i}} \int_t^1 K_{i+1,n-1}(t,x) x^{-\alpha_n - \frac{\varepsilon}{p}} dx|^q dt \right)^{\frac{1}{q}}$$
$$= \left(\int_0^1 |\sum_{i=i_0}^m (-1)^i c_{m,i} t^{\mu_{m,i}} \int_t^1 K_{i+1,n-1}(t,x) x^{-\alpha_n} dx|^q dt \right)^{\frac{1}{q}}$$
$$= \left(\int_0^1 |D^m_{\bar{\beta}}(t^{-\alpha_0} K_{1,n}(t,1))|^q dt \right)^{\frac{1}{q}} \neq 0,$$

since, according to Lemma 2.4, $D^m_{\bar{\beta}}(t^{-\alpha_0} K_{1,n}(t,1)) \neq 0$ for all $t \in (0,1]$. Consequently, $\|D^m_{\bar{\beta}} f_\varepsilon\|_q \not\to 0$ when $\varepsilon \to 0$, that is f_ε does not converge to zero in $W^m_{q,\bar{\beta}}$ when $\varepsilon \to 0$. This contradiction shows that in (3.18) it will be strict inequality, i.e. that (3.19) holds.

Conversely, assume that (3.19) holds. Then (3.16) and (3.18) yield that $\lim_{z \to 0} A_{i,j}(z) = 0$ for all $i = i_0, i_0+1, \ldots, m,\ j = i, i+1, \ldots, n-1$, i.e. the integral operators (3.7) are compact from $L_p(0,1)$ to $L_q(0,1)$ and, thus, the embedding (1.1) is compact.

The proof is complete. □

Now we consider the following embedding

$$W^n_{p,\bar{\alpha}}(I) \hookrightarrow W^m_{q,\bar{\alpha}}(I), \quad 0 \leqslant m < n. \tag{3.26}$$

In this case $i_0 = m$. In particular, Theorem 3.1 and Theorem 3.2 imply the following:

COROLLARY 3.3. *Let $I = (0,1)$, $0 \leqslant m < n$ and $1 < p \leqslant q < \infty$.*
a) If $M_m \geqslant \frac{1}{p}$, then the following conditions are equivalent:
i) The embedding (3.26) is bounded;
ii) The embedding (3.26) is compact;
iii) $\frac{1}{q} > M_m - \left(n - m - \sum_{k=m+1}^{n} \alpha_k\right)$.

b) If $M_m < \frac{1}{p}$, then the embedding (3.26) is bounded if and only if $\frac{1}{q} \geqslant \frac{1}{p} - \left(n - m - \sum_{k=m+1}^{n} \alpha_k\right)$ and the embedding (3.26) is compact if and only if $\frac{1}{q} > \frac{1}{p} - \left(n - m - \sum_{k=m+1}^{n} \alpha_k\right)$.

In particular, from Corollary 3.3 it follows that the estimate of the intermediate derivatives

$$\|D^m_{\bar{\alpha}} f\|_p \leqslant c\left(\|D^n_{\bar{\alpha}} f\|_p + \sum_{i=0}^{n-1} |D^i_{\bar{\alpha}} f(1)|\right), \; 0 \leqslant m < n,$$

holds for functions $f \in W^n_{p,\bar{\alpha}}$ if and only if

$$n - m - \sum_{k=m+1}^{n} \alpha_k > 0 \text{ when } M_m \geqslant \frac{1}{p},$$

and

$$n - m - \sum_{k=m+1}^{n} \alpha_k \geqslant 0 \text{ when } M_m < \frac{1}{p}.$$

On the interval $I = (0,1)$ when $\alpha_k = 0$, $k = 0,1,\ldots,n-1$, $\alpha_n = \gamma$, $\beta_i = 0$, $i = 0,1,\ldots,m-1$, and $\beta_m = \upsilon$ we consider Kudryavtsev spaces $L^n_{p,\gamma}$ and $L^m_{q,\upsilon}$, respectively. Then $M_{i_0} = \max_{i_0 \leqslant s \leqslant n-1} (n - s - \gamma) = n - \gamma - i_0$. Theorem 3.1 and Theorem 3.2 yields the following:

COROLLARY 3.4. *Let $I = (0,1)$, $0 \leqslant m < n$ and $1 < p \leqslant q < \infty$.*
a) If $n - i_0 - \gamma \geqslant \frac{1}{p}$, then the following conditions are equivalent:
i) The embedding $L^n_{p,\gamma}(I) \hookrightarrow W^m_{q,\bar{\beta}}(I)$ is bounded;
ii) The embedding $L^n_{p,\gamma}(I) \hookrightarrow W^m_{q,\bar{\beta}}(I)$ is compact;

iii) $|\bar{\beta}| - m + i_0 + \frac{1}{q} > 0.$

b) If $n - i_0 - \gamma < \frac{1}{p}$*, then the embedding* $L^n_{p,\gamma}(I) \hookrightarrow W^m_{q,\bar{\beta}}(I)$ *is bounded if and only if* $|\bar{\beta}| - \gamma + n - m + \frac{1}{q} \geqslant \frac{1}{p}$ *and the embedding* $L^n_{p,\gamma}(I) \hookrightarrow W^m_{q,\bar{\beta}}(I)$ *is compact if and only if* $|\bar{\beta}| - \gamma + n - m + \frac{1}{q} > \frac{1}{p}$.

COROLLARY 3.5. *Let* $I = (0,1)$, $0 \leqslant m < n$ *and* $1 < p \leqslant q < \infty$.

a) If $M_{i_0} \geqslant \frac{1}{p}$*, then the following conditions are equivalent:*

i) The embedding $W^n_{p,\bar{\alpha}}(I) \hookrightarrow L^m_{q,\upsilon}(I)$ *is bounded;*

ii) The embedding $W^n_{p,\bar{\alpha}}(I) \hookrightarrow L^m_{q,\upsilon}(I)$ *is compact;*

iii) $\upsilon - |\bar{\alpha}| + n - m + \frac{1}{q} > M_{i_0}$.

b) If $M_{i_0} < \frac{1}{p}$*, then the embedding* $W^n_{p,\bar{\alpha}}(I) \hookrightarrow L^m_{q,\upsilon}(I)$ *is bounded if and only if* $\upsilon - |\bar{\alpha}| + n - m + \frac{1}{q} \geqslant \frac{1}{p}$ *and the embedding* $W^n_{p,\bar{\alpha}}(I) \hookrightarrow L^m_{q,\upsilon}(I)$ *is compact if and only if* $\upsilon - |\bar{\alpha}| + n - m + \frac{1}{q} > \frac{1}{p}$.

4. Embedding theorems for the space $W^n_{p,\overline{\alpha}}(1,+\infty)$

The connection between the spaces $W^n_{p,\bar{\alpha}}(0,1)$ and $W^n_{p,\bar{\alpha}}(1,\infty)$ can be seen by making the variable transformation $x = \frac{1}{t}$. In this way every function $f \in W^n_{p,\bar{\alpha}}(1,\infty)$ can be transformed to a function $\tilde{f}(x) = f(\frac{1}{x})$ from the space $W^n_{p,\bar{\tilde{\alpha}}}(0,1)$, where $\bar{\tilde{\alpha}} = (\tilde{\alpha}_0, \tilde{\alpha}_1, \dots, \tilde{\alpha}_n)$, $\tilde{\alpha}_n = -\alpha_n + 2 - \frac{2}{p}$, $\tilde{\alpha}_i = -\alpha_i + 2$, $i = 1,2,\dots,n-1$, $\tilde{\alpha}_0 = -\alpha_0$. Moreover,

$$\|D^n_{\bar{\alpha}} f\|_{p,(1,+\infty)} = \left(\int_1^{+\infty} |D^n_{\bar{\alpha}} f(t)|^p dt \right)^{\frac{1}{p}}$$

$$= \left(\int_1^{+\infty} |t^{\alpha_n} \frac{d}{dt} t^{\alpha_{n-1}} \frac{d}{dt} \dots t^{\alpha_1} \frac{d}{dt} t^{\alpha_0} f(t)|^p dt \right)^{\frac{1}{p}}$$

$$= \left(\int_0^1 |x^{-\alpha_n} \frac{d}{x^{-2}dx} x^{-\alpha_{n-1}} \frac{d}{x^{-2}dx} \dots x^{-\alpha_1} \frac{d}{x^{-2}dx} x^{-\alpha_0} f\left(\frac{1}{x}\right) |^p \frac{dx}{x^2} \right)^{\frac{1}{p}}$$

$$= \left(\int_0^1 |x^{-\alpha_n+2-\frac{2}{p}} \frac{d}{dx} x^{-\alpha_{n-1}+2} \frac{d}{dx} \dots x^{-\alpha_1+2} \frac{d}{dx} x^{-\alpha_0} f\left(\frac{1}{x}\right)|^p dx \right)^{\frac{1}{p}}$$

$$= \left(\int_0^1 |x^{\tilde{\alpha}_n} \frac{d}{dx} x^{\tilde{\alpha}_{n-1}} \frac{d}{dx} \dots x^{\tilde{\alpha}_1} \frac{d}{dx} x^{\tilde{\alpha}_0} \tilde{f}(x)|^p dx \right)^{\frac{1}{p}} = \|D^n_{\bar{\tilde{\alpha}}} \tilde{f}\|_{p,(0,1)},$$

and $D^i_{\bar{\tilde{\alpha}}} f(1) = D^i_{\bar{\alpha}} f(1)$, $i = 0, 1, \dots, n-1$.

Analogously, from the space $W^m_{q,\bar{\beta}}(1,+\infty)$ we can pass to the space $W^m_{q,\bar{\tilde{\beta}}}(0,1)$.

Then the embedding (1.1) is equivalent to the embedding:

$$W^n_{p,\bar{\tilde{\alpha}}}(0,1) \hookrightarrow W^m_{q,\bar{\tilde{\beta}}}(0,1),$$

and all notions and statements for the space $W^n_{p,\bar{\tilde{\alpha}}}(0,1)$ can be rewritten for the space $W^n_{p,\bar{\alpha}}(1,+\infty)$.

Therefore,

$$\begin{aligned} \tilde{M}_i &= \max_{i \leqslant s \leqslant n-1} (n - s - \sum_{k=s+1}^{n} \tilde{\alpha}_k) \\ &= \max_{i \leqslant s \leqslant n-1} \left(n - s - \sum_{k=s+1}^{n-1} (-\alpha_k + 2) + \alpha_n - 2 + \frac{2}{p} \right) \\ &= \max_{i \leqslant s \leqslant n-1} \left(-(n - s - \sum_{k=s+1}^{n} \alpha_k) + \frac{2}{p} \right) = -\mathscr{M}_i + \frac{2}{p}, \end{aligned}$$

where $\mathscr{M}_i = \min\limits_{i \leqslant s \leqslant n-1} (n - s - \sum\limits_{k=s+1}^{n} \alpha_k)$, $i = 0, 1, \dots, n-1$.

Since $|\bar{\tilde{\beta}}| = \sum\limits_{i=1}^{m-1} (-\beta_i + 2) - \beta_0 - \beta_m + 2 - \dfrac{2}{q} = -|\bar{\beta}| + 2m - \dfrac{2}{q}$ and $|\bar{\tilde{\alpha}}| = -|\bar{\alpha}| + 2n - \dfrac{2}{p}$, then from (3.1) and (3.18), respectively, we have that

$$\begin{aligned} |\bar{\tilde{\beta}}| - |\bar{\tilde{\alpha}}| + n - m + \frac{1}{q} &= |\bar{\alpha}| - |\bar{\beta}| + 2m - 2n + n - m + \frac{1}{q} - \frac{2}{q} + \frac{2}{p} \\ &= |\bar{\alpha}| - |\bar{\beta}| + m - n - \frac{1}{q} + \frac{2}{p} > \tilde{M}_{i_0} \end{aligned} \tag{4.1}$$

and

$$|\bar{\tilde{\beta}}| - |\bar{\tilde{\alpha}}| + n - m + \frac{1}{q} = |\bar{\alpha}| - |\bar{\beta}| + m - n - \frac{1}{q} + \frac{2}{p} \geqslant \frac{1}{p}. \tag{4.2}$$

When $\tilde{M}_{i_0} = -\mathscr{M}_{i_0} + \dfrac{2}{p} \geqslant \dfrac{1}{p}$, it is equivalent to $\mathscr{M}_{i_0} \leqslant \dfrac{1}{p}$ and from (4.1) it follows that

$$|\bar{\alpha}| - |\bar{\beta}| + m - n - \frac{1}{q} + \frac{2}{p} > -\mathscr{M}_{i_0} + \frac{2}{p},$$

that is

$$|\bar{\beta}|-|\bar{\alpha}|+n-m+\frac{1}{q}<\mathscr{M}_{i_0} \text{ when } \mathscr{M}_{i_0}\leqslant\frac{1}{p}.$$

When $\tilde{M}_{i_0}<\frac{1}{p}$, that is $\mathscr{M}_{i_0}>\frac{1}{p}$, from (4.2) we get that

$$|\bar{\alpha}|-|\bar{\beta}|+m-n-\frac{1}{q}+\frac{2}{p}\geqslant\frac{1}{p},$$

i.e.

$$|\bar{\beta}|-|\bar{\alpha}|+n-m+\frac{1}{q}\leqslant\frac{1}{p} \text{ when } \mathscr{M}_{i_0}>\frac{1}{p}.$$

Then Theorem 3.1 and Theorem 3.2, and Corollaries 3.3–3.5, respectively, imply the following results:

THEOREM 4.1. *Let* $I=(1,+\infty)$, $1<p\leqslant q<\infty$, $0\leqslant m<n$ *and* $\mathscr{M}_{i_0}\leqslant\frac{1}{p}$. *Then the following conditions are equivalent:*

i) The embedding (1.1) *is bounded;*

ii) The embedding (1.1) *is compact;*

iii)

$$|\bar{\beta}|-|\bar{\alpha}|+n-m+\frac{1}{q}<\mathscr{M}_{i_0}.$$

THEOREM 4.2. *Let* $I=(1,+\infty)$, $1<p\leqslant q<\infty$, $0\leqslant m<n$ *and* $\mathscr{M}_{i_0}>\frac{1}{p}$.

a) The embedding (1.1) *is bounded if and only if*

$$|\bar{\beta}|-|\bar{\alpha}|+n-m+\frac{1}{q}\leqslant\frac{1}{p}.$$

b) The embedding (1.1) *is compact if and only if*

$$|\bar{\beta}|-|\bar{\alpha}|+n-m+\frac{1}{q}<\frac{1}{p}.$$

COROLLARY 4.3. *Let* $I=(1,+\infty)$, $0\leqslant m<n$ *and* $1<p\leqslant q<\infty$.

a) If $\mathscr{M}_m\leqslant\frac{1}{p}$, *then the following conditions are equivalent:*

i) The embedding (3.26) *is bounded;*

ii) The embedding (3.26) *is compact;*

iii) $\mathscr{M}_m-\left(n-m-\sum\limits_{k=m+1}^{n}\alpha_k\right)+\frac{1}{q}>0.$

b) If $\mathscr{M}_m>\frac{1}{p}$, *then the embedding* (3.26) *is bounded if and only if* $\frac{1}{p}+\frac{1}{q}-\left(n-m-\sum\limits_{k=m+1}^{n}\alpha_k\right)\geqslant 0$ *and the embedding* (3.26) *is compact if and only if* $\frac{1}{p}+\frac{1}{q}-\left(n-m-\sum\limits_{k=m+1}^{n}\alpha_k\right)>0$.

Corollary 4.3 yields that for the functions $f \in W^n_{p,\bar{\alpha}}(I)$ the following estimate of intermediate derivatives holds

$$\|D^m_{\bar{\alpha}} f\|_p \leqslant c \left(\|D^n_{\bar{\alpha}} f\|_p + \sum_{i=0}^{n-1} |D^i_{\bar{\alpha}} f(1)| \right), \ 0 \leqslant m < n,$$

if and only if

$$n - m - \sum_{k=m+1}^{n} \alpha_k < 0 \text{ when } \mathscr{M}_m \leqslant \frac{1}{p},$$

and

$$n - m - \sum_{k=m+1}^{n} \alpha_k \leqslant 0 \text{ when } \mathscr{M}_m > \frac{1}{p}.$$

In the space $L^n_{p,\gamma}(1,+\infty)$ we have that $M_{i_0} = 1-\gamma$. Hence, we have the following

COROLLARY 4.4. *Let $I = (1,+\infty)$, $0 \leqslant m < n$ and $1 < p \leqslant q < \infty$.*

a) If $1-\gamma \leqslant \frac{1}{p}$, then the following conditions are equivalent:

i) The embedding $L^n_{p,\gamma}(I) \hookrightarrow W^m_{q,\bar{\beta}}(I)$ is bounded;

ii) The embedding $L^n_{p,\gamma}(I) \hookrightarrow W^m_{q,\bar{\beta}}(I)$ is compact;

iii) $|\bar{\beta}| + n - m + \frac{1}{q} - 1 < 0$.

b) If $1-\gamma > \frac{1}{p}$, then the embedding $L^n_{p,\gamma}(I) \hookrightarrow W^m_{q,\bar{\beta}}(I)$ is bounded if and only if $|\bar{\beta}| - \gamma + n - m + \frac{1}{q} \leqslant \frac{1}{p}$ and the embedding $L^n_{p,\gamma}(I) \hookrightarrow W^m_{q,\bar{\beta}}(I)$ is compact if and only if $|\bar{\beta}| - \gamma + n - m + \frac{1}{q} < \frac{1}{p}$.

COROLLARY 4.5. *Let $I = (1,+\infty)$, $0 \leqslant m < n$ and $1 < p \leqslant q < \infty$.*

a) If $\mathscr{M}_{i_0} \leqslant \frac{1}{p}$, then the following conditions are equivalent:

i) The embedding $W^n_{p,\bar{\alpha}}(I) \hookrightarrow L^m_{q,\upsilon}(I)$ is bounded;

ii) The embedding $W^n_{p,\bar{\alpha}}(I) \hookrightarrow L^m_{q,\upsilon}(I)$ is compact;

iii) $\upsilon - |\bar{\alpha}| + n - m + \frac{1}{q} < \mathscr{M}_{i_0}$.

b) If $\mathscr{M}_{i_0} > \frac{1}{p}$, then the embedding $W^n_{p,\bar{\alpha}}(I) \hookrightarrow L^m_{q,\upsilon}(I)$ is bounded if and only if $\upsilon - |\bar{\alpha}| + n - m + \frac{1}{q} \leqslant \frac{1}{p}$ and the embedding $W^n_{p,\bar{\alpha}}(I) \hookrightarrow L^m_{q,\upsilon}(I)$ is compact if and only if $\upsilon - |\bar{\alpha}| + n - m + \frac{1}{q} < \frac{1}{p}$.

Acknowledgement. The authors would like to thank Professor Lars–Erik Persson for generous advices, which have improved the final version of this paper.

REFERENCES

[1] Z. T. ABDIKALIKOVA, *Embedding Theorems for Spaces with multiweighted derivatives,* Math. Journal., **2**, 4(6) (2002), 11–18 (in Russian).

[2] Z. T. ABDIKALIKOVA AND A. A. KALYBAY, *Summability of a Tchebysheff system of functions,* J. Funct. Spaces Appl., **8** (2010), 87–102.

[3] Z. T. ABDIKALIKOVA, R. OINAROV AND L.-E. PERSSON, *Boundedness and Compactness of the Embedding between Spaces with Multiweighted Derivatives when $1 \leqslant q < p < \infty$,* to appear in Czech. Math. Journal (No.256/09), 20 pp.

[4] B. L. BAIDEL'DINOV, *Near – boundary properties of multiweighted spaces,* Dokl. Akad. Nauk, **367**, 5 (1999), 583–585 (in Russian).

[5] B. L. BAIDEL'DINOV AND R. OINAROV, *Properties of one weighted space,* In: Proc. Conf. “Function Spaces, Theory of Approximations and Non-Linear Analysis”, Moscow 1995, 10–11 (in Russian).

[6] V. I. BURENKOV, *Sobolev Spaces on Domains,* Teubner Texts in Mathematics, Teubner Verlag, Stuttgart, **137** (1998), 312 pp.

[7] A. A. KALYBAY, *Interrelation of spaces with multiweighted derivatives,* Vestnik of Karaganda State University, Kazakhstan 1999, 13–22 (in Russian).

[8] A. A. KALYBAY, *A new development of Nikol'skii-Lizorkin and Hardy type inequalities with applications,* PhD Thesis, Luleå University of Technology, 2006.

[9] A. A. KALYBAY AND R. OINAROV, *Some properties of spaces with multiweighted derivatives,* In: Progress in Analysis, Vol. I, II (Berlin, 2001), World Sci. Publ., River Edge, NJ 2003, 1–13.

[10] L. D. KUDRYAVTSEV, *Equivalent norms in weighted spaces,* Proc. Steklov Inst. Math., **170**, 1 (1987), 185–218.

[11] R. OINAROV, *Boundedness and compactness of superposition of fractional integration operators and their applications,* In: Function Spaces, Differential Operators and Nonlinear Analysis 2004, Math. Institute, Acad. Sci., Czech Republic, 2005, 213–235.

(Received May 31, 2009)

Zamira Abdikalikova
International IT University
Zhandosov st., 8A
050040 Almaty, Kazakhstan
e-mail: zamir-a-t@yandex.ru

Askar Baiarystanov
Kazakh National Agrarian University
Abay st., 8
050010 Almaty, Kazakstan

Ryskul Oinarov
L. N. Gumilyev Eurasian National University
Munaytpasov st., 5
010008 Astana, Kazakhstan
e-mail: o_ryskul@mail.ru