Mathematical
Inequalities
& Applications
Volume 14, Number 4 (2011), 811–817

DIFFERENTIAL INEQUALITIES FOR IMPLICIT PERTURBATIONS OF FIRST ORDER DIFFERENTIAL EQUATIONS WITH APPLICATIONS

BAPURAO C. DHAGE

(*Communicated by Y. Zhou*)

Abstract. In this paper, some fundamental differential inequalities for an implicit perturbation of nonlinear first order ordinary differential equations have been established. Our results of this paper generalize some well-known results on ordinary differential inequalities for unperturbed first order nonlinear differential equations.

1. Introduction

Differential inequalities are very crucial in the qualitative study of nonlinear differential equations and an extensive literature of differential inequalities along with some nice applications may be found in the research monograph of Lakshmikantham and Leela [2]. It is known that the differential inequalities play a significant role in the study of extremal solutions for nonlinear differential equations via the method of upper and lower solutions. The differential inequalities for nonlinear initial and boundary value problems of ordinary differential equations of different orders have already discussed in the literature, however, to the best of our knowledge, the differential inequalities for implicit perturbations of second type for such type of differential equations have not been so far studied in the literature. In the present paper, we establish some fundamental differential inequalities for nonlinear initial value problems of nonlinear implicit first order ordinary differential equations. The details of different types of perturbations for a nonlinear differential and integral equations are given in Dhage [1]. In the following section, we give the statement of the differential problem under study and derive the related differential inequalities under some suitable natural conditions.

2. Implicit Differential Equation

Let $\mathbb{R}$ be the real line and $J = [t_0, t_0 + a)$ be a bounded interval in $\mathbb{R}$ for some $t_0, a \in \mathbb{R}$ with $a > 0$. Let $C(J \times \mathbb{R}, \mathbb{R})$ denote the class of continuous functions $f : J \times \mathbb{R} \to \mathbb{R}$ and let $\mathscr{C}(J \times \mathbb{R}, \mathbb{R})$ denote the class of functions $g : J \times \mathbb{R} \to \mathbb{R}$ such that

(i) the map $t \mapsto g(t,x)$ is measurable for each $x \in \mathbb{R}$, and

Mathematics subject classification (2010): 34K10.
Keywords and phrases: Differential inequalities, implicit perturbations.

Paper MIA-14-66

(ii) the map $x \mapsto g(t,x)$ is continuous for each $t \in J$.

The class $\mathscr{C}(J \times \mathbb{R}, \mathbb{R})$ is called the Carathéodory class of functions on $J \times \mathbb{R}$ which are Lebesgue integrable when bounded by a Lebesgue integrable function on J.

The differential equation in question is the following first order implicit differential equation (in short IDE),

$$\left.\begin{aligned} \frac{d}{dt}[f(t,x(t))] &= g(t,x(t)) \text{ a.e. } t \in J \\ x(t_0) &= x_0 \in \mathbb{R} \end{aligned}\right\} \tag{2.1}$$

where, $f \in C(J \times \mathbb{R}, \mathbb{R})$ and $g \in \mathscr{C}(J \times \mathbb{R}, \mathbb{R})$.

By a solution of the IDE (2.1) we mean a function $x \in C(J, \mathbb{R})$ such that

(i) the function $t \mapsto f(t,x)$ is absolutely continuous for each $x \in \mathbb{R}$, and

(i) x satisfies the equations in (2.1),

where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on J.

The IDE (2.1) can be discussed via hybrid fixed point theory for existence results only, but to the knowledge of author, there is no any result available in this direction. It is clear that the IDE (2.1) is a implicit perturbation of second type of the well-known initial value problems of nonlinear first order ordinary differential equations (DE),

$$\left.\begin{aligned} x'(t) &= g(t,x(t)) \text{ a.e. } t \in J \\ x(t_0) &= x_0 \in \mathbb{R}. \end{aligned}\right\} \tag{2.2}$$

The details of different types of perturbations of the DE (2.2) appear in Dhage [1]. The DE (2.2) has been extensively studied in the literature for different aspects of the solutions. The strict and nonstrict differential inequalities related to the DE (2.2) are available in the literature (see Lakshmikantham and Leela [2] and the references therein). It is known that differential inequalities are useful for proving the existence of extremal solutions of the DE (2.2) defined on J. In the following section, we prove some fundamental differential inequalities concerning the IDE (2.1).

3. Implicit Differential Inequalities

We need the following definition in the subsequent development of the paper.

DEFINITION 3.1. A function $f : \mathbb{R} \to \mathbb{R}$ is called nondecreasing if for all $x, y \in \mathbb{R}$, $x \leqslant y$ implies that $f(x) \leqslant f(y)$. Again, f is called increasing if $x < y$ implies $f(x) < f(y)$ for all $x, y \in \mathbb{R}$. Similarly, the nonincreasing and decreasing functions on $\mathbb{R}$ into itself are defined.

We also consider the following hypothesis in what follows.

(A_0) The function $x \mapsto f(t,x)$ is increasing in $\mathbb{R}$ for each $t \in J$.

There do exist functions f satisfying the hypothesis (A_0). In fact the function $f(t,x) = t+x$, $x \in \mathbb{R}$ satisfies the hypothesis (A_0).

Our first differential inequality related to the IDE (2.1) is

THEOREM 3.1. *Assume that the hypothesis (A_0) holds. Suppose that there exist functions $y,z \in C(J,\mathbb{R})$ such that*

$$\frac{d}{dt}[f(t,y(t))] < g(t,y(t)) \text{ a.e. } t \in J \tag{3.1}$$

and

$$\frac{d}{dt}[f(t,z(t))] \geqslant g(t,z(t)) \text{ a.e. } t \in J \tag{3.2}$$

Then,

$$y(t_0) < z(t_0) \tag{3.3}$$

implies

$$y(t) < z(t) \tag{3.4}$$

for all $t \in J$.

Proof. Suppose that the inequality (3.4) is false. Then the set Z^* defined by

$$Z^* = \{t \in J \mid y(t) \geqslant z(t), \quad t \in J \setminus Z\} \tag{3.5}$$

is non-empty, where Z is a set of measure zero. Denote $t_1 = \inf Z^*$. Without loss of generality, we may assume that $y(t_1) = z(t_1)$ and $y(t) < z(t)$ for all $t < t_1$.

Define the function Y and Z on J by

$$Y(t) = f(t,y(t)) \quad \text{and} \quad Z(t) = f(t,z(t))$$

for all $t \in J$. Then we have

$$Y(t_1) = Z(t_1) \tag{3.6}$$

and by virtue of hypothesis (A_0), we get

$$Y(t) < Z(t) \tag{3.7}$$

for all $t < t_1$.

From (3.7) it follows that

$$\frac{Y(t_1+h) - Y(t_1)}{h} > \frac{Z(t_1+h) - Z(t_1)}{h}$$

for small $h < 0$. The above inequality implies that

$$Y'(t_1) \geqslant Z'(t_1)$$

or

$$g(t_1,y(t_1)) > g(t_1,z(t_1)).$$

This is a contradiction and hence the the set Z^* is empty. As a result, the inequality (3.4) holds for all $t \in J$. This completes the proof.

Similarly, we can prove

THEOREM 3.2. *Assume that the hypothesis (A_0) holds. Suppose that there exist functions $y,z \in C(J,\mathbb{R})$ such that*

$$\frac{d}{dt}[f(t,y(t))] \leqslant g(t,y(t)) \text{ a.e. } t \in J \tag{3.8}$$

and

$$\frac{d}{dt}[f(t,z(t))] > g(t,z(t)) \text{ a.e. } t \in J \tag{3.9}$$

Then, the inequality (3.3) *implies the inequality* (3.4) *on* J.

THEOREM 3.3. *Assume that the hypothesis (A_0) holds. Let the function $u \in C(J,\mathbb{R})$ satisfies* (3.1) *with y replaced by u and let the function $v \in C(J,\mathbb{R})$ satisfies* (3.8) *with z replaced by v on J. If w is any solution of the IDE* (2.1) *existing on J with*

$$u(t_0) < w(t_0) < v(t_0), \tag{3.10}$$

then

$$u(t) < w(t) < v(t) \tag{3.11}$$

for all $t \in J$.

COROLLARY 3.1. *Assume that the hypothesis (A_0) holds. Let $g_1, g_2 \in \mathscr{C}(J \times \mathbb{R},\mathbb{R})$ and*

$$g_1(t,x) < g_2(t,x) \tag{3.12}$$

for all $t \in J$ and $x \in \mathbb{R}$. If u_1 and u_2 are any two solutions respectively of the IDEs

$$\frac{d}{dt}[f(t,u(t))] = g_1(t,u(t)) \text{ a.e. } t \in J \tag{3.13}$$

and

$$\frac{d}{dt}[f(t,u(t))] = g_2(t,u(t)) \text{ a.e. } t \in J, \tag{3.14}$$

satisfying

$$u_1(t_0) < u_2(t_0). \tag{3.15}$$

Then,

$$u_1(t) < u_2(t) \tag{3.16}$$

for all $t \in J$.

Our next result is concerned with the nonstrict differential inequality related to IDE (2.1) on J.

THEOREM 3.4. *Assume that the hypothesis* (A_0) *holds and there exists bounded function* $h \in L^1(J,\mathbb{R}_+)$ *such that the function* $x \mapsto g(t,x)+h(t)f(t,x)$ *is nonincreasing in* $\mathbb{R}$ *for all* $t \in J$. *Suppose that there exist functions* $y,z \in C(J,\mathbb{R})$ *such that*

$$\left.\begin{aligned} \frac{d}{dt}[f(t,y(t))] &\leqslant g(t,y(t)) \text{ a.e. } t \in J \\ y(t_0) &\leqslant x_0, \end{aligned}\right\} \tag{3.17}$$

and

$$\left.\begin{aligned} \frac{d}{dt}[f(t,z(t))] &\geqslant g(t,z(t)) \text{ a.e. } t \in J \\ y(t_0) &\geqslant x_0. \end{aligned}\right\} \tag{3.18}$$

Then,

$$y(t) \leqslant z(t) \tag{3.19}$$

for all $t \in J \setminus Z$, *where* Z *is a set of measure zero.*

Proof. Suppose that (3.19) is not true. Then there exist $t_1,t_2 \in J$, $t_1 < t_2$ such that $y(t_1) = y(t_2)$ and $y(t) > y(t)$ for all $t \in (t_1,t_2)$.

As in the proof of Theorem 3.1, we set

$$Y(t) = f(t,y(t)) \ , \ Z(t) = f(t,z(t))$$

and

$$m(t) = Y(t) - Z(t)$$

for all $t \in J$. Then, $m(t_1) = 0$ and $m(t) = Y(t) - Z(t) > 0$ for all $t \in (t_1,t_2)$. From the monotonicity of g it follows that

$$\begin{aligned} m'(t) &= Y'(t) - Z'(t) \\ &\leqslant -h(t)[Y(t) - Z(t)] \\ &= -h(t)m(t). \end{aligned}$$

Therefore,

$$m(t) \leqslant m(t_1)e^{-H(t)} = 0$$

for all $t \in (t_1,t_2)$, where $H(t) = \int_{t_0}^{t} h(t)\,dt$. Hence, $Y(t) \leqslant Z(t)$, or, $y(t) \leqslant z(t)$ for all $t \in (t_1,t_2)$ This is a contradiction to our assumption. The proof is complete.

A weaker version of the above nonstrict differential inequality with a different method may be proved as follows. Here, unlike Theorem 3.4, we do not assume any monotonic hypothesis of the functions involved in the QDE (2.1).

THEOREM 3.5. *Assume that the hypothesis* (A_0) *holds and there exists a real number* $L>0$ *such that*

$$g(t,x_1)-g(t,x_2)\leqslant L\big[f(t,x_1)-f(t,x_2)\big] \tag{3.20}$$

for all $t\in J$ *and for all* $x_1,x_2\in\mathbb{R}$ *with* $x_1\geqslant x_2$. *Suppose that there exist functions* $y,z\in C(J,\mathbb{R})$ *such that inequalities* (3.17) *and* (3.18) *are satisfied. Then the nonstrict inequality* (3.19) *holds on* J.

Proof. Let $\varepsilon>0$ and let a real number $L>0$ be given. Set

$$f(t,z_\varepsilon)=f(t,z)+\varepsilon e^{2Lt} \tag{3.21}$$

so that

$$f(t,z_\varepsilon)>f(t,z)\Longrightarrow z_\varepsilon>z.$$

Let $Z_\varepsilon=f(t,z_\varepsilon)$ so that $Z=f(t,z)$ for $t\in J$. Then, by inequality (3.18),

$$Z_\varepsilon'=Z'+2L\varepsilon e^{2Lt}\geqslant g(t,z)+2L\varepsilon e^{2Lt}. \tag{3.22}$$

Since

$$g(t,z_\varepsilon)-g(t,z)\leqslant L\left(f(t,z_\varepsilon)-f(t,z)\right)$$

for all $t\in J$, one has

$$Z_\varepsilon'(t)\geqslant g(t,z_\varepsilon(t))-L\varepsilon e^{2Lt}+2L\varepsilon e^{2Lt}>g(t,z_\varepsilon(t)),$$

or

$$\frac{d}{dt}\left[f(t,z_\varepsilon(t))\right]>g(t,z_\varepsilon(t)) \tag{3.23}$$

for all $t\in J$. Also, we have

$$z_\varepsilon(t_0)>z(t_0)>y(t_0).$$

Hence by an application of Theorem 3.1 with $z=z_\varepsilon$ yields that

$$y(t)<z_\varepsilon(t) \tag{3.24}$$

for all $t\in J$. Taking the limit as $\varepsilon\to 0$, we get $y(t)\leqslant z(t)$ for all $t\in J$.

REMARK 3.1. The conclusion of Theorems 3.1, 3.2 and 3.3 also remains true if we replace the derivatives in the inequalities (3.1)–(3.2) and (3.7)–(3.8) by Dini-derivative D_- of the function $f(t,x(t))$ on the bounded interval J.

REFERENCES

[1] B. C. DHAGE, *Quadratic perturbations of periodic boundary value problems of second order ordinary differential equations*, Diff. Equ. Appl., **2** (2010), 465–486.
[2] V. LAKSHMIKANTHAM AND S. LEELA, *Differential and Integral Inequalities*, Academic Press, New York, 1969.

(Received July 17, 2009)

Bapurao C. Dhage
Kasubai, Gurukul Colony
Ahmedpur-413 515, Dist: Latur Maharashtra
India
e-mail: bcdhage@yahoo.co.in