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ON QUERMASSINTEGRAL AND WILLS CONJECTURE

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Abstract. In this paper we investigate the relative quermassintegrals of convex bodies. We obtain an lower bound of the relative quermassintegral $W_i(K;E)$. Specially, we give another stronger Wills inequality.

1. Introduction

A geometric inequality describes the relations between the invariants of geometric objects. Perhaps the best and the most remarkable one is the classical isoperimetric inequality that relates volume to area of a plane domain: Among domains with fixed areas, the disc has the shortest circumlength. That is, the domain D with area A and perimeter L satisfies

$$L^2 - 4\pi A \geqslant 0, \tag{1.1}$$

with equality if and only if D is a disc.

A strengthening inequality of (1.1) was given by T. Bonnesen, who demonstrated in 1929, that is if D has circumradius R and inradius r, then

$$L^2 - 4\pi A \geqslant \pi^2 (R-r)^2, \tag{1.2}$$

with equality when and only when D is a disc. Inequality (1.2) can be derived from the inequality

$$0 \geqslant A - \lambda L + \lambda^2 \pi, \quad r \leqslant \lambda \leqslant R. \tag{1.3}$$

Inequality (1.3) with $\lambda = r$ is known as the Bonnesen's inradius inequality.

The inequality (1.3) have been extended by J. Zhou and F. Chen (see [10]) onto a plane of constant curvature (the hyperbolic plane and the projective plane). They obtain the Bonnesen-type inequalities for the domains in a plane of constant curvature. On higher dimensional space, J. M. Wills [9] conjectured in 1970 that

$$0 \geqslant V - rS + (n-1) r^n \omega_n, \tag{1.4}$$

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where V and S denote the n-dimensional volume of a convex body K and the n-dimensional surface area of K in R^n, respectively, r denotes the inradius of K, and ω_n is the volume of the n-dimensional unit ball. Note that,

$$\omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})},$$

where $\Gamma(\cdot)$ is the gamma function. Inequality (1.4) would be an extension of Bonnesen's inradius inequality to higher dimensions.

The inequaity (1.4) was proved simultaneously in 1973 by J. Bokowski [1] and V. I. Diskant [4]. In 1979 R. Osserman [8] showed that

$$0 \geqslant V - rS + (n-1)r^n\omega_n \left(\frac{S}{nr^{n-1}\omega_n}\right)^{\frac{n-2}{n-1}}. \tag{1.5}$$

In 1988, J. S. Yanger [5] derived an stronger result,

$$0 \geqslant V - rS + (n-1)r^2W_2 + (n-1)(n-2)\int_0^r (r-s)V(K_{-s},\dots,K_{-s},K_{-r},B,B)ds,$$

where W_2 is the second quermassintegral of K, K_{-s} and K_{-r} are the inner parallel bodies of K, and $V(K_{-s},\dots,K_{-s},K_{-r},B,B)$ is a mixed volume.

In 1997 N. S. Brannen [3] derived another inequality

$$0 \geqslant V - rS + n\sum_{i=0}^{n-2}\int_0^r tV(\underbrace{K_{-t},\dots,K_{-t}}_{i}K,\dots,K,B,B)dt,$$

and the strengthening of the Wills conjecture is

$$0 \geqslant V - rS + \left[n\sum_{i=0}^{n-2}\frac{i}{i+1}\binom{n-1}{i}r^{i+1}W_{i+1}(K_{-r})\right] + (n-1)r^n\omega_n, \tag{1.6}$$

where W_{i+1} is the $(i+1)$-st quermassintegral.

In this paper we give another lower bound of the relative quermassintegral $W_i(K;E)$, which has a corollary strengthening the Wills conjecture. We obtain

$$W_i(K;E) \geqslant \sum_{j=0}^{n-i}\binom{n-i}{j}\lambda^j W_{i+j}(K_{-\lambda};E). \tag{1.7}$$

A strengthening of the Wills conjecture is the following

$$0 \geqslant V(K) - rnW_1(K;E) + \sum_{j=1}^{n-1}\binom{n-1}{j}\int_0^r \lambda^j W_{j+1}(K_{-\lambda};E)d\lambda. \tag{1.8}$$

2. Quermassintegral

Let K and L be convex bodies in n-dimensional Euclidean space E^n. Their Minkowski sum is defined by $K+L := \{k+l : k \in K, l \in L\}$, if λ is a scalar then $\lambda K := \{\lambda k : k \in K\}$. Let $K \in R^n$ be a Minkowski linear combination of m convex bodies, i.e.,

$$K = \lambda_1 K_1 + \lambda_2 K_2 + \ldots + \lambda_m K_m, \quad \lambda_1, \ldots, \lambda_m \geqslant 0,$$

then the volume of K can be expressed as the n-th degree homogeneous polynomial in the λ_i as follows:

$$V(K) = \sum_{1 \leqslant p_1, \ldots, p_n \leqslant m} V(K_{p_1}, K_{p_2}, \ldots, K_{p_n}) \lambda_{p_1} \lambda_{p_2} \ldots \lambda_{p_n}.$$

Here the summation is extended over all p_i independently as i varies from 1 to n (see [7]). The coefficient $V(K_{p_1}, K_{p_2}, \ldots, K_{p_n})$ is called **mixed volume**. Mixed volume has several properties, we present here, which we will use it later (see [1]).

LEMMA 1. *Let $V(K_1, K_2, \ldots, K_n)$ denote the mixed volume, then*

- $V(K_1, K_2, \ldots, K_n)$ *is symmetric in its arguments.*
- $V(K_1, K_2, \ldots, K_n)$ *is a Minkowski multi-linear functional.*
- $V(K_1, K_2, \ldots, K_n)$ *is monotone, such that if $K_1 \subset K_1'$, then*

$$V(K_1, K_2, \ldots, K_n) \leqslant V(K_1', K_2, \ldots, K_n).$$

The mixed volume $V(\underbrace{K, \ldots, K}_{n-i} \underbrace{B, \ldots, B}_{i})$ with i copies of the unit ball B in R^n will be denoted by $W_i(K)$ and is called the i-th quermassintegral of K. It can be shown that $W_0(K) = V(K)$, $nW_1(K) = S(K)$ and $W_n(K) = \omega_n$.

The generally relative quermassintegrals of convex bodies are defined by

$$W_i(K;E) = V(\underbrace{K, \ldots, K}_{n-i}, \underbrace{E, \ldots, E}_{i}). \tag{2.1}$$

The i-th relative quermassintegral of $sK + tL$ $(s, t \geqslant 0)$ satisfies the following (see [7] for more details).

PROPOSITION 1. *Let K and L be convex bodies in R^n, E be a fixed convex body in R^n, s, $t \geqslant 0$ and $W_i(\,\cdot\,;E)$ be the i-th relative quermassintegral, then*

$$W_i(sK+tL;E) = \sum_{j=0}^{n-i} \binom{n-i}{j} s^{n-i-j} t^j V(\underbrace{K, \ldots, K}_{n-i-j}, \underbrace{L, \ldots, L}_{j}, \underbrace{E, \ldots, E}_{i}). \tag{2.2}$$

This formula is an extension of the Steiner's formula in three dimensional space. The relative inner parallel body of K with respect to the fixed convex body E at a distance λ $(0 \leqslant \lambda \leqslant r(K;\, E))$ is defined by

$$K_{-\lambda(E)} = \{x:\; x+\lambda E \subseteq K\},$$

we simply it as $K_{-\lambda}$. Where the relative inradius $r(K;\, E)$ is defined by

$$r(K;E) = sup\{r:\; some\, translate\, of\; rE \subseteq K\}.$$

LEMMA 2. *Let E be a fixed convex body and K be any convex body in R^n, r denotes the relative inradius of K, for each λ $(0 \leqslant \lambda \leqslant r)$, then*

$$K_{-\lambda} + \lambda E \subseteq K. \tag{2.3}$$

Now we have the following theorem.

THEOREM 1. *Let E be a fixed convex body, K be any convex body in R^n with i-th relative quermassintegral $W_i(K;E)$, and relative inradius r. For all $0 \leqslant \lambda \leqslant r$, we have*

$$W_i(K;E) \geqslant \sum_{j=0}^{n-i} \binom{n-i}{j} \lambda^j W_{i+j}(K_{-\lambda};E). \tag{2.4}$$

Proof. By formula (2.3) and combining with the third property of Lemma 1, we have

$$\begin{aligned} W_i(K;E) &= V(\underbrace{K,\ldots,K}_{n-i},\underbrace{E,\ldots,E}_{i}) \\ &\geqslant V(\underbrace{K_{-\lambda}+\lambda E,\ldots,K_{-\lambda}+\lambda E}_{n-i},\underbrace{E,\ldots,E}_{i}) = W_i(K_{-\lambda}+\lambda E;E). \end{aligned}$$

By formula (2.2), it says

$$W_i(K;E) \geqslant W_i(K_{-\lambda}+\lambda E;E) = \sum_{j=0}^{n-i} \binom{n-i}{j} \lambda^j W_{i+j}(K_{-\lambda};E). \tag{2.5}$$

We complete the proof of (2.4). □

Note that $nW_1(K_\lambda;E) = V'(K_\lambda)$, where $V'(K_\lambda)$ denotes the derivative of $V(K_\lambda)$, then we obtain the following two corollaries by replacing with the $i=0$ or $i=1$.

COROLLARY 1. *When $i=0$, Theorem 1 becomes*

$$V(K_{-\lambda}) \leqslant V(K) - \lambda V'(K_{-\lambda}) - \sum_{j=2}^{n} \binom{n}{j} \lambda^j W_j(K_{-\lambda};E), \tag{2.6}$$

which gives us an upper bound for the volume of $K_{-\lambda}$.

COROLLARY 2. *When $i=1$ in Theorem 1, we obtain*

$$V(K_{-\lambda}) \geqslant V(K) - n\lambda W_1(K;E) + n\sum_{j=1}^{n-1}\binom{n-1}{j}\int_0^{\lambda} t^j W_{j+1}(K_{-t};E)dt. \tag{2.7}$$

Proof. When $i=1$, Theorem 1 becomes

$$W_1(K;E) \geqslant \sum_{j=0}^{n-1}\binom{n-1}{j}\lambda^j W_{j+1}(K_{-\lambda};E). \tag{2.8}$$

Replace with $W_1(K_{-\lambda};E)$ by $\frac{1}{n}V'(K_{-\lambda})$, it says

$$V'(K_{-\lambda}) \leqslant nW_1(K;E) - n\sum_{j=1}^{n-1}\binom{n-1}{j}\lambda^j W_{j+1}(K_{-\lambda};E), \tag{2.9}$$

take t instead of λ in (2.9), and integrate both sides from 0 to λ with respect to t, we obtain

$$V(K) - V(K_{-\lambda}) \leqslant n\lambda W_1(K;E) - n\sum_{j=1}^{n-1}\binom{n-1}{j}\int_0^{\lambda} t^j W_{j+1}(K_{-t};E)dt.$$

So we obtain

$$V(K_{-\lambda}) \geqslant V(K) - n\lambda W_1(K;E) + n\sum_{j=1}^{n-1}\binom{n-1}{j}\int_0^{\lambda} t^j W_{j+1}(K_{-t};E)dt. \tag{2.10}$$

We complete the proof of Corollary 2. □

By Corollary 1 and Corollary 2 we obtain

COROLLARY 3. *Let K be a convex body in E^n, $K_{-\lambda}$ be the relative inner parallel body of K $(0 \leqslant \lambda \leqslant r)$, $W_i(K;E)$ denotes the i-th relative quermassintegral, then we have*

$$\begin{aligned} W_1(K;E) - W_1(K_{-\lambda};E) \geqslant \frac{1}{\lambda}\sum_{j=1}^{n-1}\binom{n-1}{j}\int_0^{\lambda} t^j W_{j+1}(K_{-t};E)dt \\ + \frac{1}{n}\sum_{j=2}^{n}\binom{n}{j}\lambda^{j-1}W_{j+1}(K_{-\lambda};E). \end{aligned}$$

The following lemma is useful in the next computation.

LEMMA 3. *Let n, λ and r be real numbers, then the following equality holds*

$$n\binom{n-1}{i}\int_0^r t^{n-1-i}(r-t)^i dt = r^n. \tag{2.11}$$

Let $E = B$ (a unit ball in R^n) and $0 \leqslant \lambda \leqslant r$, since the mixed volume are monotone, and $(r-\lambda)B \subseteq K_{-\lambda}$, we obtain

$$\begin{aligned} W_i(K_{-\lambda}) &= V(K_{-\lambda},\dots,K_{-\lambda},\underbrace{B,\dots,B}_{i}) \\ &\geqslant V((r-\lambda)B,\dots,(r-\lambda)B,\underbrace{B,\dots,B}_{i}) = (r-\lambda)^{n-i}\omega_n. \end{aligned} \tag{2.12}$$

So we can say that the (2.7) is a strengthening of the Wills conjecture.

In fact if we let $E = B$ and $\lambda = r$ in (2.7), then

$$\begin{aligned} 0 &\geqslant V(K) - rS(K) + n\sum_{j=1}^{n-1}\binom{n-1}{j}\int_0^r t^j W_{j+1}(K_{-t})dt \\ &\geqslant V(K) - rS(K) + n\sum_{j=1}^{n-1}\binom{n-1}{j}\int_0^r t^j W_{j+1}((r-t)B)dt \\ &= V(K) - rS(K) + n\sum_{j=1}^{n-1}\binom{n-1}{j}\int_0^r t^j (r-t)^{n-1-j}\omega_n dt \\ &= V(K) - rS(K) + (n-1)r^n\omega_n. \end{aligned} \tag{2.13}$$

So we proved the following

THEOREM 2. *Let K be a convex body in R^n, $r = r(K;E)$ be the relative inradius of K, and $W_i(K;E)$ denote the i-th relative quermassintegral of K. Then the inequality* (2.7) *is stronger than the Wills inequality.*

REMARK 1. *When $n = 2$,* (2.13) *becomes the Bonnesen inequality on plane.*

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