Mathematical
Inequalities
& Applications
Volume 14, Number 4 (2011), 833–837

A POWER MEAN INEQUALITY FOR THE GRÖTZSCH RING FUNCTION

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(*Communicated by J. Pečarić*)

Abstract. The Grötzsch ring function has numerous applications in geometric function theory and its properties have been investigated by many authors. Here we extend an earlier functional inequality involving the Grötzsch ring function and the geometric mean, due to Anderson, Vamanamurthy and Vuorinen, to the case of power mean.

1. Introduction

Throughout this paper we let $r' = \sqrt{1-r^2}$ for $0 < r < 1$. The complete elliptic integrals of the first and second kinds [8, 10] are defined by

$$\begin{cases} \mathscr{K}(r) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-r^2\sin^2\theta}}, \\ \mathscr{K}'(r) = \mathscr{K}(r'), \\ \mathscr{K}(0) = \frac{\pi}{2}, \qquad \mathscr{K}(1) = \infty, \end{cases}$$

and

$$\begin{cases} \mathscr{E}(r) = \int_0^{\pi/2} \sqrt{1-r^2\sin^2\theta}\, d\theta, \\ \mathscr{E}'(r) = \mathscr{E}(r'), \\ \mathscr{E}(0) = \frac{\pi}{2}, \qquad \mathscr{E}(1) = 1. \end{cases}$$

In the sequel, we will use the symbols $\mathscr{K}$ and $\mathscr{E}$ for $\mathscr{K}(r)$ and $\mathscr{E}(r)$, respectively. For $r \in (0,1)$, the Grötzsch ring function [13] is defined as

$$\mu(r) = \frac{\pi}{2}\frac{\mathscr{K}'}{\mathscr{K}}.$$

The Grötzsch ring function $\mu(r)$ is a strictly decreasing homeomorphism of the interval $(0,1)$ onto $(0,\infty)$, with limit values $\mu(0+) = \infty$, $\mu(1-) = 0$. This function represents the modulus of the plane Grötzsch ring $B^2 \setminus [0,r]$, where B^2 is the open unit disk in the complex plane, and it occurs frequently in the study of conformal invariants, quasiconformal mapings and Ramanujan's classical modular equations [1, 3, 6, 12, 13, 14, 16, 18, 19, 17].

Mathematics subject classification (2010): 30C62, 33E05.
Keywords and phrases: Grötzsch ring function, power mean, inequality.

Paper MIA-14-69

Recently, many interesting inequalities and elementary estimates of $\mu(r)$ were obtained, see [3, 4, 5, 6, 13, 14, 15, 17]. By [6, Theorem 5.12 and p. 369], it is easy to see that the function $\mu(r)$ satisfies the following geometric mean inequality:

$$\sqrt{\mu(s)\mu(t)} \leqslant \mu(\sqrt{st}), \tag{1}$$

here $s,t \in (0,1)$, equality holds if and only if $s = t$.

The geometric mean belongs to the family of power means, which is defined for $x,y > 0$ and real parameters λ by

$$M^{\lambda}(x,y) = \left(\frac{x^{\lambda}+y^{\lambda}}{2}\right)^{1/\lambda} \quad \text{for} \quad \lambda \neq 0, \quad M^{0}(x,y) = \sqrt{xy}.$$

Many interesting properties of power means are given in [9] and [11].

Using this notation, inequality (1) can be written as

$$M^{0}(\mu(s),\mu(t)) \leqslant \mu(M^{0}(s,t)). \tag{2}$$

It is natural to look for the extension of (2) to other power means. More precisely, we ask: for which real number λ is the inequality $M^{\lambda}(\mu(s),\mu(t)) \leqslant \mu(M^{\lambda}(s,t))$ valid for all $0 < s,t < 1$? It is the aim of this paper to answer this question.

Our main result is the following theorem:

THEOREM. *Let λ be a real number. The inequality*

$$M^{\lambda}(\mu(x),\mu(y)) \leqslant \mu(M^{\lambda}(x,y)) \tag{3}$$

holds for all $x,y \in (0,1)$ if and only if $\lambda \leqslant 0$. For $\lambda \leqslant 0$, the sign of equality is valid in (3) if and only if $x = y$. There is no value of λ for which the reverse inequality holds for all $x,y \in (0,1)$.

Note that inequality (3) applies e.g. when $\lambda = -1$ and $M^{-1}(x,y)$ equals the harmonic mean $H(x,y) = 2xy/(x+y)$. Another generalization of (1) can be easily obtained from a result of R. Balasubramanian, S. Ponnusamy, M. Vuorinen in [7, Theorem 1.5].

2. Lemmas

In order to prove our main result we need some lemmas, which we present in this section. We establish some properties of certain functions, which are defined in terms of the complete elliptic integrals of the first and second kinds, $\mathscr{K}$ and $\mathscr{E}$, respectively.

For $0 < r < 1$, now we list some derivative formulas(cf. [6, Appendix E, pp. 474–475]:

$$\frac{d\mathscr{K}}{dr} = \frac{\mathscr{E} - r'^2\mathscr{K}}{rr'^2}, \quad \frac{d\mathscr{E}}{dr} = \frac{\mathscr{E} - \mathscr{K}}{r},$$

and

$$\frac{d}{dr}\mu(r) = -\frac{\pi^2}{4rr'^2\mathscr{K}^2}.$$

The following Lemma 1 is from [6, Theorem 3.21(1),(7), Exercise 3.43(32); 2, Lemma 5.2(2)].

LEMMA 1. *(1) The function* $f_1(r) = (\mathscr{E} - r'^2\mathscr{K})/r^2$ *is strictly increasing and convex from* $(0,1)$ *onto* $(\pi/4, 1)$.

(2) The function $f_2(r) = r'^2\mathscr{K}$ *is strictly decreasing from* $(0,1)$ *onto* $(0, \pi/2)$.

(3) The function $f_3(r) = (\mathscr{K} - \mathscr{E})/r^2\mathscr{K}$ *is increasing from* $(0,1)$ *onto* $(1/2, 1)$.

(4)The function $f_4(r) = r'^2\mathscr{K}/\mathscr{E}$ *is strictly decreasing from* $(0,1)$ *onto* $(0,1)$.

LEMMA 2. *For* $0 < r < 1$, *let* $g(r) = \frac{\pi + 4\mathscr{K}'(\mathscr{K} - \mathscr{E})}{\pi + 2r'^2\mathscr{K}\mathscr{K}'}$. *Then* $g(r)$ *is strictly increasing from* $(0,1)$ *onto* $(0, \infty)$.

Proof. By differentiation,

$$\begin{aligned}(\pi + 2r'^2\mathscr{K}\mathscr{K}')^2 g'(r) &= 4\left[(\mathscr{K} - \mathscr{E})\frac{\mathscr{E}' - r^2\mathscr{K}'}{r'r^2}\left(-\frac{r}{r'}\right) + \mathscr{K}'\frac{r\mathscr{E}}{r'^2}\right](\pi + 2r'^2\mathscr{K}\mathscr{K}') \\ &\quad - [\pi + 4\mathscr{K}'(\mathscr{K} - \mathscr{E})]\frac{\pi - 4\mathscr{E}'\mathscr{K}}{r} \\ &= \frac{-4(\mathscr{K}\mathscr{E}' - r^2\mathscr{K}\mathscr{K}' - \mathscr{E}\mathscr{E}')}{rr'^2}(\pi + 2r'^2\mathscr{K}\mathscr{K}') \\ &\quad + \frac{4(\mathscr{K}\mathscr{E}' - \mathscr{K}'\mathscr{E} + \mathscr{K}\mathscr{K}')^2}{r} \\ &= \frac{4(\pi + 2r'^2\mathscr{K}\mathscr{K}')}{rr'^2}\mathscr{E}\left(\mathscr{E}' - \frac{r'^2\mathscr{K}}{\mathscr{E}}\frac{\mathscr{E}' - r^2\mathscr{K}'}{r'^2}\right) + \frac{(\pi - 4\mathscr{K}\mathscr{E}')^2}{r},\end{aligned}$$

for the third equality, the Legendre identity [2, 6]:

$$\mathscr{K}\mathscr{E}' + \mathscr{K}'\mathscr{E} - \mathscr{K}\mathscr{K}' = \pi/2$$

is used. Then $g'(r)$ is positive by lemma 1, and $g(0^+) = \lim_{r \to 0+} \frac{\pi + 4r^2\mathscr{K}'\mathscr{K}\frac{\mathscr{K} - \mathscr{E}}{r^2\mathscr{K}}}{\pi + 2r'^2\mathscr{K}\mathscr{K}'} = 0$, $g(1^-) = \infty$. Hence $g(r)$ is strictly increasing from $(0,1)$ onto $(0, \infty)$. □

LEMMA 3. *Let* λ *be a real number. The function* $h(r) = \frac{\mu(r)^{\lambda - 1}}{r^\lambda r'^2 \mathscr{K}^2}$ *is strictly increasing on* $(0,1)$ *if and only if* $\lambda \leqslant 0$.

Proof. We rewrite $h(r) = \left(\frac{\mu(r)}{r}\right)^\lambda \frac{1}{r'^2\mathscr{K}^2\mu(r)}$. By logarithmic differentiation,

$$\begin{aligned}\frac{h'(r)}{h(r)} &= \lambda\left[\frac{1}{\mu(r)}\left(-\frac{\pi^2}{4rr'^2\mathscr{K}^2}\right) - \frac{1}{r}\right] \\ &\quad - 2\frac{1}{r'}\left(-\frac{r}{r'}\right) - 2\frac{1}{\mathscr{K}}\frac{\mathscr{E} - r'^2\mathscr{K}}{rr'^2} - \frac{1}{\mu(r)}\left(-\frac{\pi^2}{4rr'^2\mathscr{K}^2}\right) \\ &= \frac{\pi + 2r'^2\mathscr{K}\mathscr{K}'}{2rr'^2\mathscr{K}\mathscr{K}'}\left(\frac{\pi + 4\mathscr{K}'(\mathscr{K} - \mathscr{E})}{\pi + 2r'^2\mathscr{K}\mathscr{K}'} - \lambda\right),\end{aligned}$$

which is positive for all $r \in (0,1)$ if and only if $\lambda \leqslant 0$ by Lemma 2. □

3. Proof of the main result

We are now in a position to prove the main result.

Proof of theorem. We only need prove the inequality (3) for $\lambda \neq 0$. We may suppose that $x \leqslant y$. Define

$$F(x,y) = \mu\left(M^{\lambda}(x,y)\right)^{\lambda} - \frac{\mu(x)^{\lambda} + \mu(y)^{\lambda}}{2}, \quad \lambda \neq 0.$$

Let $t = M^{\lambda}(x,y)$, then $\frac{\partial t}{\partial x} = \frac{1}{2}(\frac{x}{t})^{\lambda-1}$. If $x < y$, we have that $t > x$. By differentiation, we have

$$\begin{aligned}\frac{\partial F}{\partial x} &= \frac{\lambda}{2}\mu(t)^{\lambda-1}\left(-\frac{\pi^2}{4tt'^2\mathscr{K}(t)^2}\right)\left(\frac{x}{t}\right)^{\lambda-1} - \frac{\lambda}{2}\mu(x)^{\lambda-1}\left(-\frac{\pi^2}{4xx'^2\mathscr{K}(x)^2}\right)\\ &= -\frac{\pi^2\lambda x^{\lambda-1}}{8}\left(\frac{\mu(t)^{\lambda-1}}{t^{\lambda}t'^2\mathscr{K}(t)^2} - \frac{\mu(x)^{\lambda-1}}{x^{\lambda}x'^2\mathscr{K}(x)^2}\right)\end{aligned}$$

which is positive if and only if $\lambda < 0$ by Lemma 3. Hence $F(x,y)$ is strictly increasing with respect to x and $F(x,y) \leqslant F(y,y) = 0$. We now obtain the inequality

$$\mu\left(M^{\lambda}(x,y)\right)^{\lambda} \leqslant \frac{\mu(x)^{\lambda} + \mu(y)^{\lambda}}{2},$$

that is $\mu\left(M^{\lambda}(x,y)\right) \geqslant M^{\lambda}\left(\mu(x),\mu(y)\right)$, with the equality if and only if $x = y$. □

Acknowledgements. This research is partly supported by the Foundation of the Department of Education of Zhejiang Province(Innovation Team under Grant No. T200924, and general project under Grant No. Y200908671), and the NSF of Zhejiang Province (No. Y7080106). All authors appreciate the referee's valuable comments which improve the paper greatly.

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(Received August 6, 2009)

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