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WEAK TYPE INEQUALITY FOR THE ONE–DIMENSIONAL DYADIC DERIVATIVE

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Abstract. It is shown that the maximal operator of the one-dimensional dyadic derivative of the dyadic integral is bounded from the dyadic Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$.

1. Martingale Hardy Spaces

Let $\mathbf{P}$ denote the set of positive integers, $\mathbf{N} := \mathbf{P} \cup \{0\}$. Denote Z_2 the discrete cyclic group of order 2, that is $Z_2 = \{0,1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given such that the measure of a singleton is 1/2. Let G be the complete direct product of the countable infinite copies of the compact groups Z_2. The elements of G are of the form $x = (x_0, x_1, \ldots, x_k, \ldots)$ with $x_k \in \{0,1\}\ (k \in \mathbf{N})$. The group operation on G is the coordinate-wise addition, the measure (denote by μ) and the topology are the product measure and topology. The compact Abelian group G is called the Walsh group. A base for the neighborhoods of G can be given in the following way:

$$\begin{aligned} I_0(x) :&= G,\ I_n(x) := I_n(x_0, \ldots, x_{n-1}) \\ :&= \{y \in G : y = (x_0, \ldots, x_{n-1}, y_n, y_{n+1}, \ldots)\}, \\ & (x \in G, n \in \mathbf{N}). \end{aligned}$$

These sets are called the dyadic intervals. Let $0 = (0 : i \in \mathbf{N}) \in G$ denote the null element of G, $I_n := I_n(0)\ (n \in \mathbf{N})$. Set $e_n := (0, \ldots, 0, 1, 0, \ldots) \in G$ the n th coordinate of which is 1 and the rest are zeros $(n \in \mathbf{N})$. Let $\overline{I}_n := G \backslash I_n$.

The norm (or quasinorm) of the space $L_p(G)$ is defined by

$$\|f\|_p := \left(\int_G |f(x)|^p \, d\mu(x) \right)^{1/p} \qquad (0 < p < +\infty).$$

The space weak-$L_p(G)$ consists of all measurable functions f for which

$$\|f\|_{\text{weak}-L_p(G)} := \sup_{\lambda > 0} \lambda \mu(|f| > \lambda)^{1/p} < +\infty.$$

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The σ-algebra generated by the I_k dyadic intervale of measure 2^{-k} will be denoted by $F_k\,(k \in \mathbf{N})$.

Denote by $f = \left(f^{(n)}, n \in \mathbf{N}\right)$ martingale with respect to $(F_n, n \in \mathbf{N})$ (for details see, e. g. [8, 10]). The maximal function of a martingale f is defined by

$$f^* = \sup_{n \in \mathbf{N}} \left| f^{(n)} \right|.$$

In case $f \in L_1(G)$, the maximal function can also be given by

$$f^*(x) = \sup_{n \in \mathbf{N}} \frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u)\, d\mu(u) \right|, \; x \in G.$$

For $0 < p < \infty$ the Hardy martingale space $H_p(G)$ consists all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f \in L_1(G)$ then it is easy to show that the sequence $(S_{2^n} f : n \in \mathbf{N})$ is a martingale. If f is a martingale, that is $f = (f^{(0)}, f^{(1)}, \ldots)$ then the Walsh-Fourier coefficients must be defined in a little bit different way:

$$\widehat{f}(i) = \lim_{k \to \infty} \int_G f^{(k)}(x)\, w_i(x)\, d\mu(x).$$

The Walsh-Fourier coefficients of $f \in L_1(G)$ are the same as the ones of the martingale $(S_{2^n} f : n \in \mathbf{N})$ obtained from f.

A bounded measurable function a is a p-atom, if there exists a dyadic interval I, such that

a) $\int_I a\, d\mu = 0$;

b) $\|a\|_\infty \leqslant \mu(I)^{-1/p}$;

c) supp $(a) \subset I$.

Following the works of Weisz [10] the base of the proof of Theorem 1 is the following theorem

THEOREM A. (Weisz) *Suppose that an operator V is sublinear and, for some* $0 < p < 1$

$$\sup_{\rho > 0} \rho^p \mu\left\{x \in \overline{I} : |Va(x)| > \rho\right\} \leqslant c_p < \infty, \tag{1}$$

for every $p-$atom a, where I denotes the support of the atom. If V is bounded from L_{p_1} to L_{p_1} for a fixed $1 < p_1 \leqslant \infty$, then

$$\|Vf\|_{weak\text{-}L_p(G)} \leqslant c_p \|f\|_{H_p(G)}.$$

2. The one-dimensional dyadic derivative

For $k \in \mathbf{N}$ and $x \in G$ denote

$$r_k(x) := (-1)^{x_k}$$

the k-th Rademacher function. If $n \in \mathbf{N}$, then $n = \sum_{i=0}^{\infty} n_i 2^i$, where $n_i \in \{0,1\}$ $(i \in \mathbf{N})$, i. e. n is expressed in the number system of base 2. Denote $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$, that is, $2^{|n|} \leqslant n < 2^{|n|+1}$.

The Walsh-Paley system is defined as the sequence of Walsh-Paley functions:

$$w_0(x) = 1, w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \quad (x \in G,\ n \in \mathbf{P}).$$

The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{if } x \in \overline{I}_n. \end{cases} \tag{2}$$

The partial sums of the Walsh-Fourier series are defined as follows:

$$S_M(f;x) := \sum_{i=0}^{M-1} \widehat{f}(i) w_i(x).$$

Butzer and Wagner [1] introduced the concept of the dyadic derivative as follows. For each function f defined on G set

$$(\mathbf{d}_n f)(x) := \sum_{j=0}^{n-1} 2^{j-1} (f(x) - f(x + e_j)),\ \ x \in G.$$

Then f is said to be differentiable at $x \in G$ if $(\mathbf{d}_n f)(x)$ converges as $n \to \infty$. It was verified by Butzer and Wagner [1] that every Walsh function is dyadically differentiable and

$$\lim_{n\to\infty} (\mathbf{d}_n w_j)(x) = j w_j(x), \tag{3}$$

moreover it is known that (see e. g. Schipp, Wade, Simon, Pal [5])

$$(\mathbf{d}_n w_j)(x) = \left(\sum_{k=0}^{n-1} j_k 2^k \right) w_j(x) \quad (n \in \mathbb{P}). \tag{4}$$

It is easy to show that for large n from (3) we obtain (2).

Let W be the function whose Walsh-Fourier coefficients satisfy

$$\widehat{W}(j) := \int_G W(x) w_j(x)\, d\mu(x) = \begin{cases} 1, & j = 0 \\ 1/j, & j \in \mathbb{P} \end{cases}. \tag{5}$$

The dyadic integral of $f \in L_1(G)$ is introduced by

$$\mathbf{I}f(x) := (f * W)(x) := \int_G f(t) W(x+t)\, dt. \tag{6}$$

For the function f we consider the maximal operator

$$\mathbf{I}^* f := \sup_{n \in \mathbb{P}} |\mathbf{d}_n(\mathbf{I}f)|.$$

The boundedness of $\mathbf{I}^*$ from $L_p(G)$ to $L_p(G)$ $(1 < p \leqslant \infty)$ and the weak type $(L_1(G), L_1(G))$ inequality

$$\sup_{\alpha > 0} \alpha \mu(\mathbf{I}^* f > \alpha) \leqslant c \|f\|_1 \quad (f \in L_1(G))$$

are due to Schipp [6]. The dyadic analogue of the differentiation follows from the weak type inequality:

$$\lim_{n \to \infty} \mathbf{d}_n(\mathbf{I}f) = f \quad \text{a.e.}$$

if $f \in L_1(G)$ is of mean zero (see Schipp[6]). This result for the Vilenkin derivative can be found in Pál and Simon [4]

Schipp and Simon [7] verified that

$$\|\mathbf{I}^* f\|_1 \leqslant c \, \||f|\|_{H_1.}$$

Recall that $|f| \in H_1(G)$ is equivalent to $f \in L \log L(G) \, (\subset H_1(G))$.

Weisz [9] proved that

$$\|\mathbf{I}^* f\|_p \leqslant c_p \|f\|_{H_p}, \quad (p > 1/2).$$

The aim of this paper is to prove that in the case $p = 1/2$ the maximal operator $\mathbf{I}^*$ has weak $\left(H_{1/2}, L_{1/2}\right)$-type. In particular, the following is true

THEOREM 1. *The maximal operator $\mathbf{I}^*$ of dyadic derivative is bounded from the Hardy space $H_{1/2}(G)$ to the space weak-$L_{1/2}(G)$.*

COROLLARY 1. (Weisz [9]) *Let $p > 1/2$. Then the maximal operator $\mathbf{I}^*$ of dyadic derivative is bounded from the Hardy space $H_p(G)$ to the space $L_p(G)$.*

Proof of Theorem 1. Set

$$W_N(x) := \sum_{n=2^N}^{\infty} \frac{w_n(x)}{n}.$$

It is proved in [5, 9] that

$$\left|\left(\mathbf{d}_n W_N\right)(x)\right| \leqslant \left|D_{2^n}(x)-D_{2^N}(x)\right| \mathbf{1}_{\{n>N\}}(x) + 2F^1_{N,n}(x)+2F^2_{N,n}(x)+\frac{2}{2^{N-n}\vee 1}V_n(x),\ n\in\mathbf{N}, \tag{7}$$

where

$$F^1_{N,n}(x):=\sum_{j=0}^{n-1}2^j\sum_{i=n}^{\infty}2^{-i}\frac{D_{2^i}(x+e_j)}{2^{N-i}\vee 1}, \tag{8}$$

$$F^2_{N,n}(x):=\sum_{k=0}^{\infty}2^{-k}\frac{D_{2^{n+k}}(x)}{2^{N-n-k}\vee 1}, \tag{9}$$

$$V_n(x)\leqslant\frac{c}{2^{2n}}\sum_{j=1}^{2^n-1}j\left|K_j(x)\right|, \tag{10}$$

$$\left|K_n(x)\right|\leqslant c\sum_{s=0}^{m-1}2^{s-m}\sum_{l=s}^{m-1}D_{2^l}(x+e_s),\quad 2^m\leqslant n<2^{m+1}. \tag{11}$$

From (11) we can write

$$\begin{aligned} V_n(x) &\leqslant \frac{c}{2^{2n}}\sum_{m=1}^{n}\sum_{j=2^{m-1}}^{2^m-1}\left(\sum_{s=0}^{m-1}2^s\sum_{l=s}^{m-1}D_{2^l}(x+e_s)\right) \\ &\leqslant \frac{c}{2^{2n}}\sum_{m=1}^{n}2^m\sum_{s=0}^{m-1}2^s\sum_{l=s}^{m-1}D_{2^l}(x+e_s) \\ &\leqslant \frac{c}{2^n}\sum_{s=0}^{n-1}2^s\sum_{l=s}^{n-1}D_{2^l}(x+e_s). \end{aligned} \tag{12}$$

Combining (7)–(12) we obtain that

$$\left|\left(\mathbf{d}_n W_N\right)(x)\right| \leqslant \left|D_{2^n}(x)-D_{2^N}(x)\right| \mathbf{1}_{\{n>N\}}(x) +2F^1_{N,n}(x)+2F^2_{N,n}(x)+F^3_{N,n}(x), \tag{13}$$

where

$$F^3_{N,n}(x):=\frac{c}{2^N\vee 2^n}\sum_{s=0}^{n-1}2^s\sum_{l=s}^{n-1}D_{2^l}(x+e_s). \tag{14}$$

First we assume that $f\in H_{1/2}\bigcap L_1$. By theorem A the proof of Theorem 1 will be complete if we show that operator I^* satisfies (1) and is bounded from L_∞ to L_∞. Since ([9])

$$\sup_n\left\|\mathbf{d}_n W\right\|_1\leqslant c<\infty$$

we conclude that I^* is bounded from L_∞ to L_∞.

Let a be an arbitrary 1/2-atom with support I and $\mu(I) = 2^{-N}$ $(n \in \mathbb{N})$. Without loss of generality we can suppose that $I = I_N := I_N(0)$. For $k < 2^N$, w_k is constant on I_N and so

$$Ia(x) = \int_G a(t) W_N(x+t) d\mu(t).$$

Moreover,

$$|\mathbf{d}_n(\mathbf{I}a)(x)| \leqslant |a| * |(\mathbf{d}_n W_N)(x)|.$$

By Theorem A, to verify (1) we have to investigate

$$\mu\left\{x \in \overline{I}_N : \sup_n |a| * |(\mathbf{d}_n W_N)(x)| \geqslant c2^\lambda\right\}.$$

From (13) we can write

$$\begin{aligned}
&\mu\left\{x \in \overline{I}_N : \sup_n |a| * |(\mathbf{d}_n W_N)(x)| \geqslant c2^\lambda\right\} \\
&\leqslant \sum_{i=1}^{3} \mu\left\{x \in \overline{I}_N : \sup_{1 \leqslant n \leqslant N} |a| * F_{N,n}^{(i)} \geqslant c2^\lambda\right\} \\
&\quad + \sum_{i=1}^{3} \mu\left\{x \in \overline{I}_N : \sup_{n>N} |a| * F_{N,n}^{(i)} \geqslant c2^\lambda\right\}.
\end{aligned}$$

Weisz [9] proved that

$$\int_{\overline{I}_N} \sup_{n>N} \left(|a| * F_{N,n}^{(i)}\right)^p d\mu \leqslant c_p, \quad 0 < p \leqslant 1, \quad i = 1,2,3.$$

Hence

$$\begin{aligned}
&2^{\lambda/2}\mu\left\{x \in \overline{I}_N : \sup_{n>N} |a| * F_{N,n}^{(i)} \geqslant c2^\lambda\right\} \\
&\leqslant \int_{\overline{I}_N} \sup_{n>N} \left(|a| * F_{N,n}^{(i)}\right)^{1/2} d\mu \leqslant c < \infty, \quad i = 1,2,3.
\end{aligned} \tag{15}$$

Since $\|a\|_\infty \leqslant c2^{2N}$, from (8) we can write

$$\begin{aligned}
&|a| * F_{N,n}^{(1)}(x) \\
&\leqslant c2^{2N} \int_{I_N} \sum_{k=0}^{n-1} 2^k \sum_{i=n}^{\infty} 2^{-i} \frac{D_{2^i}(x+t+e_k)}{2^{N-i} \vee 1} d\mu(t) \\
&= c2^{2N} \int_{I_N} \sum_{k=0}^{n-1} 2^k \sum_{i=n}^{N} 2^{-i} \frac{D_{2^i}(x+t+e_k)}{2^{N-i} \vee 1} d\mu(t) \\
&\quad + c2^{2N} \int_{I_N} \sum_{k=0}^{n-1} 2^k \sum_{i=N+1}^{\infty} 2^{-i} \frac{D_{2^i}(x+t+e_k)}{2^{N-i} \vee 1} d\mu(t)
\end{aligned}$$

$$= c\sum_{k=0}^{n-1} 2^k \sum_{i=n}^{N} D_{2^i}(x+e_k)$$
$$+c2^{2N}\int_{I_N}\sum_{k=0}^{n-1} 2^k \sum_{i=N+1}^{\infty} 2^{-i} D_{2^i}(x+t+e_k)\,d\mu(t).$$

Consequently,

$$\sup_{1\leqslant n\leqslant N} |a| * F_{N,n}^{(1)}(x) \tag{16}$$
$$\leqslant c \sup_{1\leqslant n\leqslant N} \sum_{k=0}^{n-1} 2^k \sum_{i=n}^{N} D_{2^i}(x+e_k)$$
$$+c \sup_{1\leqslant n\leqslant N} 2^{2N} \sum_{k=0}^{n-1} 2^k \sum_{i=N+1}^{\infty} 2^{-i} \int_{I_N} D_{2^i}(x+t+e_k)\,d\mu(t)$$
$$\leqslant c \sum_{k=0}^{N-1} 2^k \sum_{i=k}^{N} D_{2^i}(x+e_k)$$
$$+c2^{2N} \sum_{k=0}^{N-1} 2^k \sum_{i=N+1}^{\infty} 2^{-i} \int_{I_N} D_{2^i}(x+t+e_k)\,d\mu(t)$$
$$= I(N,x) + II(N,x).$$

Let

$$x \in I_N(0,\ldots,0,x_j=1,0,\ldots,0,x_l=1,x_{l+1},\ldots,x_{N-1})$$

for some $0 \leqslant j < l \leqslant N,$ where

$$I_N(0,\ldots,0,x_j=1,0,\ldots,0,x_l=1,x_{l+1},\ldots,x_{N-1})$$
$$= I_N(0,\ldots,0,x_j=1,0,\ldots,0),\text{if} \quad l=N.$$

Then we obtain that

$$I(N,x) \leqslant c2^{j+l}.$$

Let $j+l \leqslant \lambda$. Then $I(N,x) \leqslant c2^{j+l} \leqslant c2^{\lambda}$ and $\mu\left\{x \in \overline{I}_N : I(N,x) > c2^{\lambda}\right\} = 0$. Hence we can suppose that $j+l > \lambda$.

It is easy to show that

$$\mu\left\{x \in \overline{I}_N : I(N,x) > c2^{\lambda}\right\} \leqslant c \sum_{l=[\lambda/2]}^{[\lambda]} \sum_{j=[\lambda]-l}^{l} \frac{1}{2^l} + c\sum_{l=[\lambda]}^{N} \sum_{j=0}^{l} \frac{1}{2^l} \tag{17}$$
$$\leqslant c \sum_{l=[\lambda/2]}^{[\lambda]} \frac{l-\lambda/2+1}{2^l} + c\sum_{l=[\lambda]}^{N} \frac{l}{2^l}$$
$$\leqslant \frac{c}{2^{\lambda/2}}.$$

For $II(N,x)$ we have

$$II(N,x) \leqslant c2^{2N} \sum_{k=0}^{N-1} 2^k \sum_{i=N+1}^{\infty} 2^{-i} \mathbf{1}_{I_N(0,\dots,0,x_k=1,0,\dots,0)}(x)$$
$$\leqslant c2^N \sum_{k=0}^{N-1} 2^k \mathbf{1}_{I_N(0,\dots,0,x_k=1,0,\dots,0)}(x).$$

Let

$$x \in I_N(0,\dots,0,x_l=1,0,\dots,0), \quad \text{for some } 0 \leqslant l < N.$$

Then we can write

$$II(N,x) \leqslant c2^{N+l}.$$

Consequently,

$$\mu\left\{x \in \overline{I}_N : II(N,x) > c2^{\lambda}\right\} \leqslant c \sum_{l=[\lambda]-N}^{N} \frac{1}{2^N} \leqslant \frac{c(N-\lambda/2)}{2^N} \leqslant \frac{c}{2^{\lambda/2}}. \tag{18}$$

Combining (16)–(18) we conclude that

$$\mu\left\{x \in \overline{I}_N : \sup_{1\leqslant n\leqslant N} |a| * F_{N,n}^{(1)} \geqslant c2^{\lambda}\right\} \leqslant \frac{c}{2^{\lambda/2}}. \tag{19}$$

From (9) we have

$$|a| * F_{N,n}^{(2)}(x) \leqslant c2^{2N} \sum_{k=0}^{\infty} 2^{-k} \frac{1}{2^{N-n-k} \vee 1} \int_{I_N} D_{2^{n+k}}(x+t)\, d\mu(t)$$
$$\leqslant c2^n \sum_{k=0}^{N-1-n} D_{2^{n+k}}(x), \quad x \notin \overline{I}_N.$$

Hence

$$\sup_{1\leqslant n\leqslant N} |a| * F_{N,n}^{(2)}(x) \leqslant c \sum_{n=0}^{N} 2^n \sum_{k=0}^{N-1-n} D_{2^{n+k}}(x).$$

Let $x \in I_l \backslash I_{l+1}$, for some $l = 0,1,\dots,N-1$. Then we can write

$$\sup_{1\leqslant n\leqslant N} |a| * F_{N,n}^{(2)}(x) \leqslant c \sum_{n=0}^{l} 2^n \sum_{k=0}^{l-n} 2^{n+k} \leqslant c2^{2l},$$

$$\mu\left\{x \in \overline{I}_N : \sup_{1\leqslant n\leqslant N} |a| * F_{N,n}^{(2)} \geqslant c2^{\lambda}\right\} \leqslant c \sum_{l=[\lambda/2]}^{N} \frac{1}{2^l} \leqslant \frac{c}{2^{\lambda/2}}. \tag{20}$$

It is easy to show that

$$|a| * F_{N,n}^{(3)}(x) \leqslant \frac{c2^{2N}}{2^N} \sum_{s=0}^{n-1} 2^s \sum_{l=s}^{n-1} \int_{I_N} D_{2^l}(x+t+e_s)\, d\mu(t)$$
$$\leqslant c \sum_{s=0}^{n-1} 2^s \sum_{l=s}^{n-1} D_{2^l}(x+e_s),$$

$$\sup_{1\leqslant n\leqslant N}|a|*F_{N,n}^{(3)}(x)\leqslant c\sum_{s=0}^{N-1}2^{s}\sum_{l=s}^{N-1}D_{2^{l}}(x+e_{s})=I(N,x).$$

Consequently (see (17)),

$$\mu\left\{x\in\overline{I}_{N}:\sup_{1\leqslant n\leqslant N}|a|*F_{N,n}^{(3)}\geqslant c2^{\lambda}\right\} \tag{21}$$
$$\leqslant\mu\left\{x\in\overline{I}_{N}:I(N,x)\geqslant c2^{\lambda}\right\}\leqslant\frac{c}{2^{\lambda/2}}.$$

Combining (13), (15), (19)–(21) we conclude that

$$\mu\left\{x\in G:\mathbf{I}^{*}a(x)\geqslant c2^{\lambda}\right\}\leqslant\frac{c}{2^{\lambda/2}}$$

which proves the theorem for $f\in H_{1/2}(G)\cap L_{1/2}(G)$.

If $f\in H_{1/2}(G)$ then $f_k\in L_1(G)$ and $f_k\to f$ in $H_{1/2}(G)$. We have

$$\mu\left\{\left|\mathbf{I}^{*}f_{j}-\mathbf{I}^{*}f_{k}\right|>c\lambda\right\}$$
$$\leqslant\frac{c}{\sqrt{\lambda}}\left\|f_{j}-f_{k}\right\|_{H_{1/2}}^{1/2}\to0\text{ as }j,k\to\infty.$$

For $f\in H_{1/2}(G)$ we define $\mathbf{I}^{*}f$ by

$$\mathbf{I}^{*}f:=\lim_{n\to\infty}\mathbf{I}^{*}f_{n}\quad\text{in measure}$$

which finishes the proof of the theorem. □

Finally, we note that in [2] the author proved that the maximal operator $\mathbf{I}^{*}$ is not bounded from the Hardy space $H_p(G)$ to the Hardy space $H_p(G)$, when $0<p\leqslant1$. For the maximal operator σ^{*} of the Fejér means of Walsh-Fourier series Weisz [11] proved that σ^{*} is bounded from the Hardy space $H_{1/2}(G)$ to the space weak$-L_{1/2}(G)$. On the other hand, the author [3] proved that σ^{*} is not bounded the Hardy space $H_{1/2}(G)$ to the space $L_{1/2}(G)$.

We suspect that $\mathbf{I}^{*}$ is not bounded from the the Hardy space $H_{1/2}(G)$ to the space $L_{1/2}(G)$ though we could not find any counterexample.

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