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SOME INEQUALITIES FOR GENERALIZED FRACTIONAL INTEGRAL OPERATORS ON GENERALIZED MORREY SPACES

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Abstract. Boundedness of generalized fractional integral operators on generalized Morrey spaces and their related results were shown by many authors. We consider one of their results in a wider framework. Moreover, we show some inequalities for another operator on generalized fractional integrals on generalized Morrey spaces.

1. Introduction

For a function $\rho : (0,+\infty) \to (0,+\infty)$, the gerenarlized fractional integral operator is defined by

$$T_\rho f(x) = \int_{\mathbf{R}^n} \frac{\rho(|x-y|)f(y)}{|x-y|^n}dy$$

for any suitable function on f on $\mathbf{R}^n$. For the case $\rho(r) = r^\alpha$, $(0 < \alpha < n)$, we use the notation $I_\alpha f(x)$ for $T_\rho f(x)$. Morrey spaces can reflect the boundedness of I_α. Here in this paper we are concerned with generalized Morrey spaces. For $1 \leqslant p < +\infty$ and a function $\phi : (0,+\infty) \to (0,+\infty)$, the generalized Morrey norm is defined by

$$\|f\|_{L_p^\phi} = \sup_{\substack{x\in\mathbf{R}^n \\ r>0}} \left(\frac{1}{\phi(r)} \int_{B(x,r)} |f(y)|^p dy \right)^{1/p},$$

where $B(x,r)$ stands for the open ball centered at x with radius r. And the generalized Morrey space L_p^ϕ is defined by $L_p^\phi = \{f \in L_{loc}^p(\mathbf{R}^n) \mid \|f\|_{L_p^\phi} < +\infty\}$. For the case $\phi(r) = r^\lambda$, $(0 \leqslant \lambda < n)$, we use the notation M_p^λ for L_p^ϕ.

There is a huge amount of literatures dealing with boundedness of fractional integral operators on Morrey spaces. In this paper, we deal with the boundedness of T_ρ and its variant T_ρ^K which is defined by

$$T_\rho^K f(x) = \int_{\mathbf{R}^n} \frac{\rho(|x-y|)f(y)}{|x-y|^n(1+|x-y|)^K}dy$$

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for some $K \geqslant 0$. We shall give a deep insight of T_ρ and T_ρ^K via subtle property of generalized Morrey spaces. In [7], the authors investigated the operators whose integral kernel $\Gamma(x,y)$ satisfies $0 \leqslant \Gamma(x,y) \leqslant C/|x-y|^{n-\alpha}(1+|x-y|)^K$ for some constant C in connection with elliptic partial differential equations.

The plan of this paper is as follows. In Section 2, we describe known results on fractional integral operators on Morrey spaces with their generalized versions. In Section 3, we formulate and prove our theorem on T_ρ. In Section 4, we state and prove our theorems on T_ρ^K.

Throughout this paper the letter C stands for a constant not necessarily the same at each occurrence.

2. Known Results

In this section, we describe some known results on boundedness of fractional integral operators on Morrey spaces with their generalized version and describe some inequalities which hold for the function $g \cdot T_\rho f$ (or $g \cdot I_\alpha f$) for some function g.

Boundedness of fractional integral operators on Morrey spaces was shown in [14] as Spanne's unpublished result as the following Theorem 2.1 states.

THEOREM 2.1. ([14, Theorem 5.4]) *Let* $0 \leqslant \lambda < n$, $1 < p < n/\alpha$, $1/s = 1/p - \alpha/n$, $\lambda_1 = n\lambda/(n-\alpha p)$. *Then there exists a positive constant* C *such that*

$$\|I_\alpha f\|_{M_s^{\lambda_1}} \leqslant C\|f\|_{M_p^\lambda}, \tag{1}$$

where $f \in M_p^\lambda$.

Adams strengthened Theorem 2.1 ([1]).

THEOREM 2.2. ([1, Theorem 3.1]) *Let* $0 \leqslant \lambda < n-\alpha p$, $1 < p(< (n-\lambda)/\alpha)$, $1/s = 1/p - \alpha/(n-\lambda)$. *Then there exists a positive constant* C *such that*

$$\|I_\alpha f\|_{M_s^\lambda} \leqslant C\|f\|_{M_p^\lambda}, \tag{2}$$

where $f \in M_p^\lambda$.

REMARK 2.3. By Hölder's inequality, if $p_1 \leqslant p$ and $p_1 = (n-\lambda_1)p/(n-\lambda)$, then $\|\cdot\|_{M_{p_1}^{\lambda_1}} \leqslant \|\cdot\|_{M_p^\lambda}$. Hence Theorem 2.2 extends Theorem 2.1. In [2], the authors reproved Theorem 2.2 by using the Hardy-Littlewood maximal operator and proved Theorem 2.1 as its corollary.

We note Hölder's inequality on Morrey spaces.

LEMMA 2.4. (Hölder's inequality on Morrey spaces) *Let* $0 < s < +\infty$, $0 < q < +\infty$, $0 < u < +\infty$, $0 \leqslant \nu < n$, $0 \leqslant \mu < n$, *and* $0 \leqslant \lambda < n$. *Assume* $1/q = 1/u + 1/s$ *and that* $\nu/q = \mu/u + \lambda/s$. *Then*

$$\|gf\|_{M_q^\nu} \leqslant \|g\|_{M_u^\mu}\|f\|_{M_s^\lambda}, \tag{3}$$

where $f \in M_s^{\lambda}$ *and* $g \in M_u^{\mu}$.

For some function g, we obtain some inequalities for $g \cdot I_\alpha f$. Using Theorem 2.1 and applying Lemma 2.4 to $g \cdot I_\alpha f$, we have

THEOREM 2.5. *Let* $0 \leqslant \lambda < n$, $1 < p < n/\alpha$, $1/q = 1/u + 1/p - \alpha/n$, $\nu/q = \mu/u + \{n\lambda/(n-\alpha p)\}(1/p - \alpha/n)$, *and assume* $g \in M_u^{\mu}$. *Then there exists a positive constant* C *such that*

$$\|g \cdot I_\alpha f\|_{M_q^\nu} \leqslant C\|g\|_{M_u^\mu}\|f\|_{M_p^\lambda}, \tag{4}$$

where $f \in M_p^{\lambda}$.

As a special case $q = p$ in Theorem 2.5, we have

COROLLARY 2.6. *Let* $0 \leqslant \lambda < n$, $1 < p < n/\alpha$, $g \in M_{n/\alpha}^0$. *Then there exists a positive constant* C *such that*

$$\|g \cdot I_\alpha f\|_{M_p^\lambda} \leqslant C\|g\|_{M_{n/\alpha}^0}\|f\|_{M_p^\lambda}, \tag{5}$$

where $f \in M_p^{\lambda}$.

REMARK 2.7. We remark that $M_{n/\alpha}^0$ is the $L^{n/\alpha}$ space by the definition. If we use Theorem 2.2 for Theorem 2.1, we can obtain an inequality for $g \cdot I_\alpha f$ with g which belong to the Morrey space (see Corollary 2.9).

Using Theorem 2.2 and applying Lemma 2.4 to $g \cdot I_\alpha f$, we have

THEOREM 2.8. *Let* $0 \leqslant \lambda < n - \alpha p$, $1 < p(< (n-\lambda)/\alpha)$, $1/q = 1/u + 1/p - \alpha/(n-\lambda)$, $\nu/q = \mu/u + \lambda\{1/p - \alpha/(n-\lambda)\}$, *and assume* $g \in M_u^{\mu}$. *Then there exists a positive constant* C *such that*

$$\|g \cdot I_\alpha f\|_{M_q^\nu} \leqslant C\|g\|_{M_u^\mu}\|f\|_{M_p^\lambda}, \tag{6}$$

where $f \in M_p^{\lambda}$.

As a special case $q = p$ in Theorem 2.8, we have

COROLLARY 2.9. *Let* $0 \leqslant \lambda < n - \alpha p$, $1 < p(< (n-\lambda)/\alpha)$, $g \in M_{(n-\lambda)/\alpha}^{\lambda}$. *Then there exists a positive constant* C *such that*

$$\|g \cdot I_\alpha f\|_{M_p^\lambda} \leqslant C\|g\|_{M_{(n-\lambda)/\alpha}^\lambda}\|f\|_{M_p^\lambda}, \tag{7}$$

where $f \in M_p^{\lambda}$.

In [13], Olsen obtained another inequality for $g \cdot I_\alpha f$. His notations of fractional integral operators and Morrey spaces are different from ours. However, we can rewrite it under our notations as follows:

THEOREM 2.10. [13, Theorem 2] *Let* Ω *be a bounded domain. Suppose that the non-negative parameters* p, q, u, λ, μ*, and* ν *satisfy* $1 < p \leqslant q < u$, $\lambda < n - \alpha p$, $1/q = (1/u)(1 - \mu/n) + 1/p - \alpha/n$*, and* $\nu/q = \lambda/p$. *Then there exists a constant* $C = C_\Omega$ *depending on* Ω *and the parameters above such that*

$$\|g \cdot I_\alpha f\|_{M_q^\nu} \leqslant C\|g\|_{M_u^\mu} \|f\|_{M_p^\lambda}, \tag{8}$$

for all positive and measureble functions f *and* g *such that the support of* f *is contained in* Ω.

As a special case $q = p$ in Theorem 2.10, we have

COROLLARY 2.11. ([13, Corollary 3]) *Let* Ω *be a bounded domain. Suppose that the non-negative parameters* p, λ*, and* μ *satisfy* $1 < p < \min\{(n-\lambda)/\alpha, (n-\mu)/\alpha\}$. *Then there exists a constant* $C = C_\Omega$ *dependeing on* Ω *and the parameters above such that*

$$\|g \cdot I_\alpha f\|_{M_p^\lambda} \leqslant C\|g\|_{M^\mu_{(n-\mu)/\alpha}} \|f\|_{M_p^\lambda}, \tag{9}$$

for all positive and measureble functions f *and* g *such that the support of* f *is contained in* Ω.

REMARK 2.12. We compare Corollary 2.9 with Corollary 2.11. In (7), the Morrey norm of g is determined by λ which appears in the Morrey norms of f and $g \cdot I_\alpha f$. On the other hand, (9) holds for g which belong to $M^\mu_{(n-\mu)/\alpha}$, where μ is not necessarily the same as λ.

On generalizations of boundedness of fractional integral operators on Morrey spaces, they studied generalized Morrey spaces earier than for generalized fractional integral operators. Nakai passed Theorem 2.1 to gerenalized Morrey spaces. Here, to compare Nakai's result with ours, we present a precise formulation.

THEOREM 2.13. ([8, Theorem 3]) *Let* $0 < \alpha < n$, $1 < p < n/\alpha$, $1/s = 1/p - \alpha/n$. *For* $\phi(r)$ *assume that there exist positive constants* C_1 *and* A *such that*

$$\frac{1}{C_1} \leqslant \frac{\phi(t)}{\phi(r)} \leqslant C_1 \qquad \text{for} \qquad r \leqslant t \leqslant 2r, \tag{10}$$

$$\int_r^{+\infty} \frac{\phi(t)}{t^{n-\alpha p+1}} dt \leqslant A \frac{\phi(r)}{r^{n-\alpha p}}, \tag{11}$$

for every $r > 0$. *Then there exists a positive constant* C *such that*

$$\|I_\alpha f\|_{L_s^{\phi^{s/p}}} \leqslant C\|f\|_{L_p^\phi}, \tag{12}$$

where $f \in L_p^\phi$.

REMARK 2.14. The condition (10) is called the doubling condition. Under the assumptions (10) and (11), the Hardy-Littlewood maximal operator is known to be bounded from L_p^ϕ to itself, where $1 < p < +\infty$ (see [8, Theorem 1]).

We note Hölder's inequality on generalized Morrey spaces.

LEMMA 2.15. (Hölder's inequality on generalized Morrey spaces) *Let* $0 < s < +\infty$, $0 < q < +\infty$, *and* $0 < u < +\infty$, *and let* $\phi(r)$, $\psi(r)$, *and* $\eta(r)$ *be functions which satisfy* $0 < \phi(r) < +\infty$, $0 < \psi(r) < +\infty$, *and* $0 < \eta(r) < +\infty$ *for* $r > 0$. *Assume* $1/q = 1/u + 1/s$ *and* $\psi(r) = \eta(r)^{q/u}\phi(r)^{q/s}$. *Then*

$$\|gf\|_{L_q^\psi} \leqslant \|g\|_{L_u^\eta}\|f\|_{L_s^\phi}, \tag{13}$$

where $f \in L_s^\phi$, $g \in L_u^\eta$.

Using Theorem 2.13 and applying Lemma 2.15 to $g \cdot I_\alpha f$, we have

THEOREM 2.16. *Let* $1 < p < n/\alpha$ *and* $1/q = 1/u + 1/p - \alpha/n$. *Assume that* $\phi(r)$ *satisfies* (10) *and* (11). *Assume also that* $\psi(r) = \eta(r)^{q/u}\phi(r)^{q/p}$ *and* $g \in L_u^\eta$. *Then there exists a positive constant* C *such that*

$$\|g \cdot I_\alpha f\|_{L_q^\psi} \leqslant C\|g\|_{L_u^\eta}\|f\|_{L_p^\phi}, \tag{14}$$

where $f \in L_p^\phi$.

As a special case $q = p$ in Theorem 2.16, we have

COROLLARY 2.17. *Let* $1 < p < n/\alpha$. *Assume that* $\phi(r)$ *satisfies* (10) *and* (11) *and assume* $g \in L_{n/\alpha}^1$. *Then there exists a positive constant* C *such that*

$$\|g \cdot I_\alpha f\|_{L_p^\phi} \leqslant C\|g\|_{L_{n/\alpha}^1}\|f\|_{L_p^\phi}, \tag{15}$$

where $f \in L_p^\phi$.

Corollary 2.17 was stated as [7, Corollary 2.1]. In [7], the authors obtained the counterpart of Corollary 2.9 on generalized Morrey spaces.

THEOREM 2.18. ([7, Theorem 2.1]) *Let* $1 < p < +\infty$. *For* $\phi(r)$ *assume that there exist positive constants* C_1, A, *and* α *and non-negative constants* λ *and* δ *with* $\alpha p + \lambda + \delta < n$ *such that* (10), (11), *and*

$$\phi(r) \leqslant C_2 r^\lambda(1 + r^\delta) \tag{16}$$

for every $r > 0$. *Assume also that* $g \in L_{(n-\lambda)/\alpha}^\phi \cap L_{(n-\lambda-\delta)/\alpha}^\phi$. *Then there exists a positive constant* C *such that*

$$\|g \cdot I_\alpha f\|_{L_p^\phi} \leqslant C(\|g\|_{L_{(n-\lambda)/\alpha}^\phi} + \|g\|_{L_{(n-\lambda-\delta)/\alpha}^\phi})\|f\|_{L_p^\phi}, \tag{17}$$

where $f \in L_p^\phi$.

When $\phi(r) = r^{\lambda}(1+r^{\delta})$, the generalized Morrey space $L^{\phi}_{(n-\lambda)/\alpha}$ is the same as $M^{\lambda}_{(n-\lambda)/\alpha}$ $\cap M^{\lambda+\delta}_{(n-\lambda)/\alpha}$. As the special case in $\delta = 0$ in Theorem 2.18, we have Corollary 2.9.

In [7], the authors considered a related operator on fractional integrals and showed an inequality on generalized Morrey spaces. And in [17], the authors proved Theorem 2.10 on generalized Morrey spaces and as its corollary they extended Theorem 2.2 to generalized Morrey spaces. Their definition of generalized Morrey spaces is different from ours. In this paper, we shall extend and generalize Theorem 2.18. Hence the details of the theorems stated in another definition of generalized Morrey spaces are omitted.

It is known that the results on the generalization of fractional integral operators. In [9], the author introduced generalized fractional integral operators and studied on generalization of Hardy-Littlewood-Sobolev's inequality. Eridani, Gunawan, and Nakai proved Theorem 2.2 for generalized fractional integral operators on generalized Morrey spaces ([4], [5], and [6]). Their definition of generalized Morrey spaces is the same as in the one in [17].

Moreover, Nakai proved boundedness of generalized fractional integral operators on Orlicz-Morrey spaces ([11, Theorem 7.1], also [10, Theorem 2.2] and [11, Theorem 7.3]). In [11], the author showed Hölder's inequality on Orlicz-Morrey spaces ([11, Theorem 4.1]). Recently, under another definition of generalized Morrey spaces, which is the same as in [17], we have some inequalities for generalized fractional integral operators on generalized Morrey spaces ([15] and [16]).

3. Our theorem for T_ρ

In this section, we state and prove our theorem on T_ρ. For simplicity, we write $\|\cdot\|_{p,\phi}$ for $\|\cdot\|_{L^{\phi}_p}$ below.

THEOREM 3.1. *Let $1 < p < +\infty$, $1 < q < +\infty$. For $\phi(r)$ and $\rho(r)$ assume that there exist positive constants C_1, C_2, C_3, and α and non-negative constants λ, δ, and ε with $(\alpha+\varepsilon)p+\lambda+\delta < n$ such that (10), (16), and*

$$\rho(r) \leqslant C_3 r^{\alpha}(1+r^{\varepsilon}) \tag{18}$$

for every $r > 0$. Assume also that

$$\psi(r) = \eta_1(r)^{\alpha p/(n-\lambda)}\phi(r)^{1-\alpha p/(n-\lambda)} = \eta_2(r)^{(\alpha+\varepsilon)p/(n-\lambda-\delta)}\phi(r)^{1-(\alpha+\varepsilon)p/(n-\lambda-\delta)}$$

and $g \in L^{\eta_1}_{(n-\lambda)q/\alpha p} \cap L^{\eta_2}_{(n-\lambda-\delta)q/(\alpha+\varepsilon)p}$. Then there exists a positive constant C such that

$$\|g\cdot T_\rho f\|_{q,\psi} \leqslant C\Big(\|g\|_{(n-\lambda)q/\alpha p,\eta_1}\|f\|_{q,\phi}^{1-\alpha p/(n-\lambda)}\|f\|_{p,\phi}^{\alpha p/(n-\lambda)} + \|g\|_{(n-\lambda-\delta)q/(\alpha+\varepsilon)p,\eta_2}\|f\|_{q,\phi}^{1-(\alpha+\varepsilon)p/(n-\lambda-\delta)}\|f\|_{p,\phi}^{(\alpha+\varepsilon)p/(n-\lambda-\delta)}\Big), \tag{19}$$

where $f \in L^{\phi}_p \cap L^{\phi}_q$.

REMARK 3.2. The function $\phi(r) = r^{\lambda}\log(2+r)$ with $0 \leqslant \lambda < n-\alpha p$ satisfies (10) and (16) for every $\delta > 0$ and the function $\rho(r) = r^{\alpha}\log(2+r)$ with $0 < \alpha < n$ satisfies (18) for every $\varepsilon > 0$.

REMARK 3.3. In Theorem 3.1, the case $\rho(r) = r^{\alpha}$, $q = p$, $\eta_1 = \eta_2 = \phi(=\psi)$ is the same as Theorem 2.18.

As a special case $q = p$ in Theorem 3.1, we have

COROLLARY 3.4. *Let $1 < p < +\infty$. For $\phi(r)$ and $\rho(r)$ assume that there exist positive constants C_1, C_2, C_3, and α and non-negative constants λ, δ, and ε with $(\alpha+\varepsilon)p+\lambda+\delta < n$ such that (10), (16), and (18) for every $r > 0$. Assume also that*

$$\psi(r) = \eta_1(r)^{\alpha p/(n-\lambda)}\phi(r)^{1-\alpha p/(n-\lambda)} = \eta_2(r)^{(\alpha+\varepsilon)p/(n-\lambda-\delta)}\phi(r)^{1-(\alpha+\varepsilon)p/(n-\lambda-\delta)}$$

and $g \in L^{\eta_1}_{(n-\lambda)/\alpha} \cap L^{\eta_2}_{(n-\lambda-\delta)/(\alpha+\varepsilon)}$. Then there exists a positive constant C such that

$$\|g\cdot T_\rho f\|_{p,\psi} \leqslant C\left(\|g\|_{(n-\lambda)/\alpha,\eta_1} + \|g\|_{(n-\lambda-\delta)/(\alpha+\varepsilon),\eta_2}\right)\|f\|_{p,\phi}, \tag{20}$$

where $f \in L^{\phi}_p$.

As a special case $\delta = \varepsilon = 0$ in Theorem 3.1, we have

COROLLARY 3.5. *Let $0 \leqslant \lambda < n-\alpha p$, $1 < p(< (n-\lambda)/\alpha)$, $q = \alpha p u/(n-\lambda)$, $\nu = \alpha p\mu/(n-\lambda) + \lambda\{1-\alpha p/(n-\lambda)\}$, and assume $g \in M^{\mu}_{u}$. Then there exists a positive constant C such that*

$$\|g\cdot I_\alpha f\|_{M^{\nu}_{q}} \leqslant C\|g\|_{M^{\mu}_{u}}\|f\|^{1-\alpha p/(n-\lambda)}_{M^{\lambda}_{q}}\|f\|^{\alpha p/(n-\lambda)}_{M^{\lambda}_{p}}, \tag{21}$$

where $f \in M^{\lambda}_{p} \cap M^{\lambda}_{q}$.

We compare Corollary 3.5 with Theorem 2.8. Let λ, p, α, q, u, μ, and ν satisfy the assumption of Corollary 3.5. If we use Theorem 2.8 for $f \in M^{\lambda}_{p}$ and $g \in M^{\mu}_{u}$, we obtain $g\cdot I_\alpha f \in M^{\nu_p}_{p^{\#}}$, where $p^{\#}$ and ν_p satisfy $1/p^{\#} = 1/u + 1/p - \alpha/(n-\lambda)$, $\nu_p/p^{\#} = \mu/u + \lambda\{1/p - \alpha/(n-\lambda)\}$. Similarly, if we use Theorem 2.8 for $f \in M^{\lambda}_{q}$ and $g \in M^{\mu}_{u}$, we obtain $g\cdot I_\alpha f \in M^{\nu_p}_{q^{\#}}$, where $q^{\#}$ and ν_q satisfy $1/q^{\#} = 1/u + 1/q - \alpha/(n-\lambda)$, $\nu_q/q^{\#} = \mu/u + \lambda\{1/q - \alpha/(n-\lambda)\}$. If $q = p$, then $q^{\#} = p^{\#}$ and $\nu_q = \nu_p$. Hence the case $q = p$ in Corollary 3.5 follows from Theorem 2.8. We consider the case $q < p$. If $q < p$ then $q^{\#} < p^{\#}$ and since

$$\frac{n-\nu_q}{q^{\#}} - \frac{n-\nu_p}{p^{\#}} = \frac{n-\lambda}{q} - \frac{n-\lambda}{p} > 0,$$

it follows that $(n-\nu_q)/q^{\#} > (n-\nu_p)/p^{\#}$. Hence $M^{\nu_p}_{p^{\#}}$ is not a proper subset of $M^{\nu_q}_{q^{\#}}$. On the other hand, if $q < p$ then $q < p^{\#}$. Indeed,

$$\begin{aligned}\frac{1}{q}-\frac{1}{p^{\#}} &= \frac{1}{q}-\frac{\alpha p}{q(n-\lambda)}-\frac{1}{p}+\frac{\alpha}{n-\lambda}\\ &= \frac{1}{q}\cdot\frac{n-\lambda-\alpha p}{n-\lambda}+\frac{\alpha p-n+\lambda}{p(n-\lambda)}\\ &= \frac{p}{q}\cdot\frac{n-\lambda-\alpha p}{p(n-\lambda)}-\frac{n-\lambda-\alpha p}{p(n-\lambda)}\\ &= \frac{n-\lambda-\alpha p}{p(n-\lambda)}\left(\frac{p}{q}-1\right) > 0,\end{aligned}$$

where we have used $1/u = \alpha p/q(n-\lambda)$, $q < p$, and $p < (n-\lambda)/\alpha$. Then we obtain

$$\begin{aligned}\frac{n-\nu}{q}-\frac{n-\nu_p}{p^{\#}} &= (n-\lambda)\left(\frac{1}{q}-\frac{1}{p}\right)+\frac{\alpha}{n-\lambda}\cdot\frac{p}{q}(\lambda-n)+\frac{\alpha}{n-\lambda}(n-\lambda)\\ &= (n-\lambda)\cdot\frac{p-q}{pq}+\alpha\cdot\frac{q-p}{q}\\ &= \frac{p-q}{q}\left(\frac{n-\lambda}{p}-\alpha\right) > 0.\end{aligned}$$

Hence it follows that $(n-\nu)/q > (n-\nu_p)/p^{\#}$ and $M^{\nu_p}_{p^{\#}}$ is not a proper subset of M^{ν}_{q}. Moreover, if $q < p$ then $q^{\#} < q$. Indeed, combining $1/q^{\#} = 1/u + 1/q - \alpha/(n-\lambda)$ with $1/u = \alpha p/(n-\lambda)q$, we have $1/q^{\#} = \alpha(p/q-1)/(n-\lambda)+1/q$. Since $p/q > 1$, it follows that $q^{\#} < q$. Then we obtain

$$\frac{n-\nu_q}{q^{\#}}-\frac{n-\nu}{q} = \frac{\alpha p(n-\lambda)}{q(n-\lambda)}-\frac{\alpha(n-\lambda)}{n-\lambda} = \alpha\left(\frac{p}{q}-1\right) > 0.$$

Since $(n-\nu_q)/q^{\#} > (n-\nu)/q$, M^{ν}_{q} is not a proper subset of $M^{\nu_q}_{q^{\#}}$. Considering the fact mentioned above, if $q < p$ then we have $q^{\#} < q < p^{\#}$, similarly, if $q > p$ then we have $q^{\#} > q > p^{\#}$, and neither $M^{\nu_p}_{p^{\#}}$ nor $M^{\nu_q}_{q^{\#}}$ is contained in M^{ν}_{q}. Hence we arrive at

REMARK 3.6. We can not obtain Corollary 3.5 combining boundedness of fractional integral operators on Morrey spaces and Hölder's inequality on Morrey spaces as we obtained Theorem 2.8. We also remark that Corollary 3.5 does not follow from combining boundedness of generalized fractional integral operators on Orlicz-Morrey spaces with Hölder's inequality on Orlicz-Morrey spaces obtained in [10] and [11]. (See also Remark 3.13.)

Let $L^{\infty}_{c} = \{f \in L^{\infty}(\mathbf{R}^n) | \operatorname{supp} f \text{ is compact}\}$. By the monotone convergence theorem, we may assume $f \in L^{\phi}_{p} \cap L^{\phi}_{q} \cap L^{\infty}_{c}$ without loss of generality in the proof. (See [7, p.1129] for the details.) To prove Theorem 3.1, we use a pointwise estimate by the Hardy-Littlewood maximal operator M the following Lemma 3.7 states.

LEMMA 3.7. *Under the assumptions* (10), (16), *and* (18), *we have*

$$|T_\rho f(x)| \leqslant C\left(Mf(x)^{1-\alpha p/(n-\lambda)}\|f\|_{p,\phi}^{\alpha p/(n-\lambda)} + Mf(x)^{1-(\alpha+\varepsilon)p/(n-\lambda-\delta)}\|f\|_{p,\phi}^{(\alpha+\varepsilon)p/(n-\lambda-\delta)}\right), \tag{22}$$

where $f \in L_p^\phi \cap L_c^\infty$.

We remark that, since $(\alpha+\varepsilon)p+\lambda+\delta < n$, there exists a constant C such that

$$\int_r^{+\infty} \frac{\phi(t)}{t^{n-(\alpha+\varepsilon)p+1}}dt \leqslant C\frac{\phi(r)}{r^{n-(\alpha+\varepsilon)p}}. \tag{23}$$

To prove Lemma 3.7, we invoke the following Lemma 3.8 from [8].

LEMMA 3.8. ([8, Lemma 2]) *Suppose* $\phi : (0,+\infty) \to (0,+\infty)$ *satisfies* (23). *Then there exist positive constants* C_0 *and* μ *such that*

$$\int_r^{+\infty} \frac{\phi(t)}{t^{n-(\alpha+\varepsilon)p+1-\mu}}dt \leqslant C_0\frac{\phi(r)}{r^{n-(\alpha+\varepsilon)p-\mu}}. \tag{24}$$

Proof of Lemma 3.7. We follow the argument in [2]. We write, for $\theta > 0$ which will be determined later,

$$T_\rho f(x) = \int_{|x-y|<\theta} \frac{\rho(|x-y|)f(y)}{|x-y|^n}\,dy + \int_{|x-y|\geqslant\theta} \frac{\rho(|x-y|)f(y)}{|x-y|^n}\,dy = I_1 + I_2.$$

By using (18), we can estimate I_1 by

$$\begin{aligned} |I_1| &\leqslant C\sum_{k=-\infty}^{-1}\int_{2^k\theta\leqslant|x-y|<2^{k+1}\theta} \frac{|x-y|^\alpha(1+|x-y|^\varepsilon)|f(y)|}{|x-y|^n}\,dy \\ &\leqslant C(\theta^\alpha+\theta^{\alpha+\varepsilon})Mf(x). \end{aligned} \tag{25}$$

Let

$$|I_2| \leqslant C\left(\int_{|x-y|\geqslant\theta} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy + \int_{|x-y|\geqslant\theta} \frac{|f(y)|}{|x-y|^{n-\alpha-\varepsilon}}\,dy\right) = C(I_3+I_4).$$

First we estimate I_3. Let $p' = p/(p-1)$ and we take σ_1 to satisfy $n-(\alpha+\varepsilon)p-\mu < \sigma_1 < p(n-\alpha-n/p')$, where μ is a constant in (24). Since $n-\alpha = \sigma_1/p + \{-(\sigma_1/p+\alpha-n)\}$, Hölder's inequality yields

$$I_3 \leqslant \left(\int_{|x-y|\geqslant\theta} \frac{|f(y)|^p}{|x-y|^{\sigma_1}}\,dy\right)^{1/p}\left(\int_{|x-y|\geqslant\theta} |x-y|^{(\sigma_1/p+\alpha-n)p'}\,dy\right)^{1/p'} = I_5 I_6.$$

For I_5, we have by (10)

$$\begin{aligned} I_5 &\leqslant \left(\sum_{k=0}^{+\infty}\int_{2^k\theta<|x-y|\leqslant 2^{k+1}\theta} \frac{|f(y)|^p}{(2^k\theta)^{\sigma_1}}\,dy\right)^{1/p} \\ &\leqslant C\left(\sum_{k=0}^{+\infty} \frac{\phi(2^k\theta)}{(2^k\theta)^{\sigma_1}}\right)^{1/p}\|f\|_{p,\phi}. \end{aligned}$$

We obtain

$$\begin{aligned}\sum_{k=0}^{+\infty}\frac{\phi(2^k\theta)}{(2^k\theta)^{\sigma_1}} &= \sum_{k=0}^{+\infty}\frac{(2^k\theta)^{n-(\alpha+\varepsilon)p-\mu}}{(2^k\theta)^{\sigma_1}}\cdot\frac{\phi(2^k\theta)}{(2^k\theta)^{n-(\alpha+\varepsilon)p-\mu}}\\ &\leqslant C\sum_{k=0}^{+\infty}\frac{(2^k\theta)^{n-(\alpha+\varepsilon)p-\mu}}{(2^k\theta)^{\sigma_1}}\int_{2^k\theta}^{2^{k+1}\theta}\frac{\phi(t)}{t^{n-(\alpha+\varepsilon)p+1-\mu}}dt\\ &\leqslant C\frac{\theta^{n-(\alpha+\varepsilon)p-\mu}}{\theta^{\sigma_1}}\int_{\theta}^{+\infty}\frac{\phi(t)}{t^{n-(\alpha+\varepsilon)p+1-\mu}}dt.\end{aligned}$$

For the last inequality, since $\sigma_1 > n-(\alpha+\varepsilon)p-\mu$, we used $(2^k)^{n-(\alpha+\varepsilon)p-\mu-\sigma_1}\leqslant 1$ for every $k\geqslant 0$. By Lemma 3.8, we conclude

$$\sum_{k=0}^{+\infty}\frac{\phi(2^k\theta)}{(2^k\theta)^{\sigma_1}}\leqslant C\frac{\phi(\theta)}{\theta^{\sigma_1}}.$$

On the other hand, it is easy to obtain

$$I_6 = C\theta^{-n/p+\sigma_1/p+\alpha}.$$

Hence, it follows that

$$I_3\leqslant C\phi(\theta)^{1/p}\theta^{\alpha-n/p}\|f\|_{p,\phi}.$$

Next we estimate I_4. We take σ_2 to satisfy $n-(\alpha+\varepsilon)p-\mu<\sigma_2<p(n-\alpha-\varepsilon-n/p')$. Since $n-\alpha-\varepsilon=\sigma_2/p+\{-(\sigma_2/p+\alpha-n+\varepsilon)\}$, by Hölder's inequality we have

$$I_4\leqslant\left(\int_{|x-y|\geqslant\theta}\frac{|f(y)|^p}{|x-y|^{\sigma_2}}dy\right)^{1/p}\left(\int_{|x-y|\geqslant\theta}|x-y|^{(\sigma_2/p+\alpha-n+\varepsilon)p'}dy\right)^{1/p'}=I_7I_8.$$

Since $\sigma_2>n-(\alpha+\varepsilon)p-\mu$, by a way similar to the estimate I_5 we obtain

$$\begin{aligned}I_7&\leqslant C\left(\sum_{k=0}^{+\infty}\frac{\phi(2^k\theta)}{(2^k\theta)^{\sigma_2}}\right)^{1/p}\|f\|_{p,\phi}\\ &\leqslant C\left(\frac{\phi(\theta)}{\theta^{\sigma_2}}\right)^{1/p}\|f\|_{p,\phi}.\end{aligned}$$

On the other hand, we have

$$I_8 = C\theta^{-n/p+\sigma_2/p+\alpha+\varepsilon}.$$

Hence, it follows that

$$I_4\leqslant C\phi(\theta)^{1/p}\theta^{\alpha-n/p+\varepsilon}\|f\|_{p,\phi}.$$

By the assumption (16), we have

$$\begin{aligned}|I_2|&\leqslant C\phi(\theta)^{1/p}(1+\theta^{\varepsilon})\theta^{\alpha-n/p}\|f\|_{p,\phi}\\ &\leqslant C(1+\theta^{\delta/p}+\theta^{\varepsilon}+\theta^{\delta/p+\varepsilon})\theta^{\alpha+(\lambda-n)/p}\|f\|_{p,\phi}.\end{aligned}\tag{26}$$

Combining (25) with (26), we obtain

$$|T_\rho f(x)| \leqslant C\left\{(\theta^\alpha+\theta^{\alpha+\varepsilon})Mf(x)+\left(\theta^{\alpha+(\lambda-n)/p}+\theta^{\alpha+\varepsilon+(\lambda-n+\delta)/p}\right)\|f\|_{p,\phi}\right\}.$$

We consider two cases. For the case $\|f\|_{p,\phi}/Mf(x)\leqslant 1$, i.e. $\|f\|_{p,\phi}\leqslant Mf(x)$, we choose $\theta=(\|f\|_{p,\phi}/Mf(x))^{p/(n-\lambda)}$. Since $\theta\leqslant 1$, it follows that

$$\begin{aligned}|T_\rho f(x)| &\leqslant C\left\{\left(\frac{\|f\|_{p,\phi}}{Mf(x)}\right)^{\alpha p(n-\lambda)}Mf(x)+\left(\frac{\|f\|_{p,\phi}}{Mf(x)}\right)^{\{p/(n-\lambda)\}\{\alpha+(\lambda-n)/p\}}\|f\|_{p,\phi}\right\}\\ &= CMf(x)^{1-\alpha p/(n-\lambda)}\|f\|_{p,\phi}^{\alpha p/(n-\lambda)}.\end{aligned}$$

For the case $\|f\|_{p,\phi}/Mf(x)\geqslant 1$, we choose $\theta=(\|f\|_{p,\phi}/Mf(x))^{p/(n-\lambda-\delta)}$. In this case, it follows in a similar way that

$$|T_\rho f(x)|\leqslant CMf(x)^{1-(\alpha+\varepsilon)p/(n-\lambda-\delta)}\|f\|_{p,\phi}^{(\alpha+\varepsilon)p/(n-\lambda-\delta)}.$$

Hence, we obtain the desired estimate. □

Now we are ready to give

Proof of Theorem 3.1. Let

$$\begin{aligned}|T_\rho f(x)| &\leqslant C\left(Mf(x)^{1-\alpha p/(n-\lambda)}\|f\|_{p,\phi}^{\alpha p/(n-\lambda)}+Mf(x)^{1-(\alpha+\varepsilon)p/(n-\lambda-\delta)}\|f\|_{p,\phi}^{(\alpha+\varepsilon)p/(n-\lambda-\delta)}\right)\\ &= C(J_1(x)+J_2(x)).\end{aligned}$$

Define $t\in(1,+\infty)$ by $1/t=1-\alpha p/(n-\lambda)$. By Hölder's inequality, we have

$$\begin{aligned}&\left(\frac{1}{\psi(r)}\int_{B(x,r)}|g(y)J_1(y)|^q\,dy\right)^{1/q}\\ &\leqslant C\left(\frac{1}{\psi(r)}\int_{B(x,r)}|Mf(y)|^{q/t}|g(y)|^q\,dy\right)^{1/q}\|f\|_{p,\phi}^{\alpha p/(n-\lambda)}\\ &\leqslant C\left(\frac{1}{\phi(r)}\int_{B(x,r)}|Mf(y)|^q\,dy\right)^{1/tq}\left(\frac{1}{\eta_1(r)}\int_{B(x,r)}|g(y)|^{(n-\lambda)q/\alpha p}dy\right)^{\alpha p/(n-\lambda)q}\|f\|_{p,\phi}^{\alpha p/(n-\lambda)}\\ &\leqslant C\|g\|_{(n-\lambda)q/\alpha p,\eta_1}\|f\|_{q,\phi}^{1-\alpha p/(n-\lambda)}\|f\|_{p,\phi}^{\alpha p/(n-\lambda)}.\end{aligned}$$

In the last inequality we used the fact that $\|Mf\|_{q,\phi}\leqslant C\|f\|_{q,\phi}$ holds under the assumptions (10) and (11) (see Remark 2.14). This implies

$$\|gJ_1\|_{q,\psi}\leqslant C\|g\|_{(n-\lambda)q/\alpha p,\eta_1}\|f\|_{q,\phi}^{1-\alpha p/(n-\lambda)}\|f\|_{p,\phi}^{\alpha p/(n-\lambda)}.$$

In a similar way we have

$$\|gJ_2\|_{q,\psi}\leqslant C\|g\|_{(n-\lambda-\delta)q/(\alpha+\varepsilon)p,\eta_2}\|f\|_{q,\phi}^{1-(\alpha+\varepsilon)p/(n-\lambda-\delta)}\|f\|_{p,\phi}^{(\alpha+\varepsilon)p/(n-\lambda-\delta)}.$$

This completes the proof of Theorem 3.1. □

REMARK 3.9. If we use Lemma 3.7 and the boundedness of the Hardy-Littlewood maximal operator on generalized Morrey spaces, we obtain the boundedness of T_ρ on generalized Morrey spaces. In addition, combining it with Hölder's inequality on generalized Morrey spaces, we obtain an inequality for $g \cdot T_\rho f$. However, the result follows from Nakai's theorem [11, Theorem 7.3], which describes the boundedness of T_ρ on Orlicz-Morrey spaces ([12]).

For reference, we describe the result which we mentioned in Remark 3.9.

THEOREM 3.10. ([12]) *Let $1 < p < +\infty$, $1 < q < +\infty$. For $\phi(r)$ and $\rho(r)$ assume that there exist positive constants C_1, C_2, C_3, and α and non-negative constants λ, δ, and ε with $(\alpha+\varepsilon)p+\lambda+\delta < n$ such that (10), (16), and (18). Let $1 < p_1 < +\infty$, $p_1 = p\{1/p-(\alpha+\varepsilon)/(n-\lambda-\delta)\}/\{1/p-\alpha/(n-\lambda)\}$, and $1/q = 1/p-\alpha/(n-\lambda)$. Then there exists a positive constant C such that*

$$\|T_\rho f\|_{q,\phi} \leqslant C\left(\|f\|_{p,\phi} + \|f\|_{p_1,\phi}^{1-(\alpha+\varepsilon)p/(n-\lambda-\delta)} \|f\|_{p,\phi}^{(\alpha+\varepsilon)p/(n-\lambda-\delta)}\right), \tag{27}$$

where $f \in L_p^\phi \cap L_{p_1}^\phi$.

Combining Theorem 3.10 with Hölder's inequality on generalized Morrey spaces, we have

THEOREM 3.11. *Let $1<p<+\infty$, $1<q<+\infty$, $1<u<+\infty$. For $\phi(r)$ and $\rho(r)$ assume that there exist positive constants C_1, C_2, C_3, and α and non-negative constants λ, δ, and ε with $(\alpha+\varepsilon)p+\lambda+\delta < n$ such that (10), (16), and (18) for every $r>0$. Let $1<p_1<+\infty$, $p_1 = p\{1/p-(\alpha+\varepsilon)/(n-\lambda-\delta)\}/\{1/p-\alpha/(n-\lambda)\}$, and $1/q = 1/u+1/p-\alpha/(n-\lambda)$. Assume further that $\psi(r) = \eta(r)^{q/u}\phi(r)^{q\{1/p-\alpha/(n-\lambda)\}}$ and $g \in L_u^\eta$. Then there exists a positive constant C such that*

$$\|g \cdot T_\rho f\|_{q,\psi} \leqslant C\|g\|_{u,\eta}\left(\|f\|_{p,\phi} + \|f\|_{p_1,\phi}^{1-(\alpha+\varepsilon)p/(n-\lambda-\delta)} \|f\|_{p,\phi}^{(\alpha+\varepsilon)p/(n-\lambda-\delta)}\right), \tag{28}$$

where $f \in L_p^\phi \cap L_{p_1}^\phi$.

As a special case $q = p$ in Theorem 3.11, we have

COROLLARY 3.12. *Let $1<p<+\infty$. For $\phi(r)$ and $\rho(r)$ assume that there exist positive constants C_1, C_2, C_3, and α and non-negative constants λ, δ, and ε with $(\alpha+\varepsilon)p+\lambda+\delta < n$ such that (10), (16), and (18) for every $r>0$. Let $1<p_1<+\infty$ and $p_1 = p\{1/p-(\alpha+\varepsilon)/(n-\lambda-\delta)\}/\{1/p-\alpha/(n-\lambda)\}$. Assume further that $\psi(r) = \eta(r)^{\alpha p/(n-\lambda)}\phi(r)^{1-\alpha p/(n-\lambda)}$ and $g \in L_{(n-\lambda)/\alpha}^\eta$. Then there exists a positive constant C such that*

$$\|g \cdot T_\rho f\|_{p,\psi} \leqslant C\|g\|_{(n-\lambda)/\alpha,\eta}\left(\|f\|_{p,\phi} + \|f\|_{p_1,\phi}^{1-(\alpha+\varepsilon)p/(n-\lambda-\delta)} \|f\|_{p,\phi}^{(\alpha+\varepsilon)p/(n-\lambda-\delta)}\right), \tag{29}$$

where $f \in L_p^\phi \cap L_{p_1}^\phi$.

REMARK 3.13. As a special case $\delta = \varepsilon = 0$ in Theorem 3.11, we have Theorem 2.8.

4. Our theorems for T_ρ^K

In this section, we show some inequalities for the operator T_ρ^K on generalized Morrey spaces.

THEOREM 4.1. *Let $1 < p < +\infty$, $1 < q < +\infty$. For $\phi(r)$ and $\rho(r)$ assume that there exist positive constants C_1, C_2, C_3, and α and non-negative constants λ, δ, and ε with $(\alpha+\varepsilon)p+\lambda+\delta < n$ such that (10), (16), and (18).*

(1) *Let $K \geqslant (\lambda+\delta)/p+\varepsilon$ and assume $\psi(r) = \eta(r)^{\alpha p/n}\phi(r)^{1-\alpha p/n}$ and $g \in L^\eta_{nq/\alpha p}$. Then there exists a positive constant C such that*

$$\|g \cdot T_\rho^K f\|_{q,\psi} \leqslant C\|g\|_{nq/\alpha p,\eta}\|f\|_{q,\phi}^{1-\alpha p/n}\|f\|_{p,\phi}^{\alpha p/n}, \tag{30}$$

where $f \in L_p^\phi \cap L_q^\phi$.

(2) Let $K \geqslant \delta/p+\varepsilon$ and assume $\psi(r) = \eta(r)^{\alpha p/(n-\lambda)}\phi(r)^{1-\alpha p/(n-\lambda)}$ and $g \in L^\eta_{(n-\lambda)q/\alpha p}$. Then there exists a positive constant C such that

$$\|g \cdot T_\rho^K f\|_{q,\psi} \leqslant C\|g\|_{(n-\lambda)q/\alpha p,\eta}\|f\|_{q,\phi}^{1-\alpha p/(n-\lambda)}\|f\|_{p,\phi}^{\alpha p/(n-\lambda)}, \tag{31}$$

where $f \in L_p^\phi \cap L_q^\phi$.

REMARK 4.2. In Theorem 4.1 (1), the case $\rho(r) = r^\alpha$, $q = p$, and $\eta = \phi(= \psi)$ was shown in [7, Theorem 2.2].

To prove Theorem 4.1, we need the following Lemma 4.3.

LEMMA 4.3. *Assume (10), (16), and (18).*

(1) *If $K \geqslant (\lambda+\delta)/p+\varepsilon$, then we have*

$$|T_\rho^K f(x)| \leqslant CMf(x)^{1-\alpha p/n}\|f\|_{p,\phi}^{\alpha p/n}, \tag{32}$$

where $f \in L_p^\phi \cap L_c^\infty$.

(2) *If $K \geqslant \delta/p+\varepsilon$, then we have*

$$|T_\rho^K f(x)| \leqslant CMf(x)^{1-\alpha p/(n-\lambda)}\|f\|_{p,\phi}^{\alpha p/(n-\lambda)}, \tag{33}$$

where $f \in L_p^\phi \cap L_c^\infty$.

Proof. We prove only (1), since we can prove (2) by the same method as in the proof of (1). We write, for $\theta > 0$ which will be determined later,

$$\begin{aligned} T_\rho^K f(x) &= \int_{|x-y|<\theta} \frac{\rho(|x-y|)f(y)}{|x-y|^n(1+|x-y|)^K}dy \\ &\quad + \int_{|x-y|\geqslant\theta} \frac{\rho(|x-y|)f(y)}{|x-y|^n(1+|x-y|)^K}dy = I_1' + I_2'. \end{aligned}$$

By using (18), as in the proof of Lemma 3.7, we can estimate I_1' by

$$
\begin{aligned}
|I_1'| &\leqslant C\theta^{\alpha} \min\left\{1+\theta^{\varepsilon}, \frac{1+\theta^{\varepsilon}}{\theta^K}\right\} Mf(x) \\
&\leqslant C\theta^{\alpha} Mf(x),
\end{aligned}
$$

since $K \geqslant \varepsilon$. For I_2', let

$$
\begin{aligned}
|I_2'| \leqslant C &\left(\int_{|x-y|\geqslant\theta} \frac{|f(y)|}{|x-y|^{n-\alpha}(1+|x-y|)^K} dy \right. \\
&\left. + \int_{|x-y|\geqslant\theta} \frac{|f(y)|}{|x-y|^{n-\alpha-\varepsilon}(1+|x-y|)^K} dy \right) = C(I_3' + I_4').
\end{aligned}
$$

As in the proof of Lemma 3.7, for I_3' we have by (16),

$$
\begin{aligned}
I_3' &\leqslant C\theta^{\alpha-n/p}\theta^{\lambda/p} \min\left\{1+\theta^{\delta/p}, \frac{1+\theta^{\delta/p}}{\theta^K}\right\} \|f\|_{p,\phi} \\
&\leqslant C\theta^{\alpha-n/p} \|f\|_{p,\phi},
\end{aligned} \tag{34}
$$

since $K \geqslant (\lambda+\delta)/p$. Similarly, for I_4', we have by (16),

$$
\begin{aligned}
I_4' &\leqslant C\theta^{\alpha-n/p}\theta^{\lambda/p+\varepsilon} \min\left\{1+\theta^{\delta/p}, \frac{1+\theta^{\delta/p}}{\theta^K}\right\} \|f\|_{p,\phi} \\
&\leqslant C\theta^{\alpha-n/p} \|f\|_{p,\phi},
\end{aligned} \tag{35}
$$

since $K \geqslant (\lambda+\delta)/p+\varepsilon$. From (34) and (35) we have

$$
|I_2'| \leqslant C\theta^{\alpha-n/p} \|f\|_{p,\phi}.
$$

Then it follows that

$$
|T_\rho^K f(x)| \leqslant C\left(\theta^{\alpha} Mf(x) + \theta^{\alpha-n/p}\|f\|_{p,\phi}\right).
$$

We choose $\theta = \left(\|f\|_{p,\phi}/Mf(x)\right)^{p/n}$ and we obtain

$$
|T_\rho^K f(x)| \leqslant CMf(x)^{1-\alpha p/n} \|f\|_{p,\phi}^{\alpha p/n}. \quad \square
$$

We prove only (1) of Theorem 4.1, since we can prove (2) by the same method as in the proof of (1) by using Lemma 4.3 (2).

Proof of Theorem 4.1 *(1).* Let $1/t = 1-\alpha p/n$. By Hölder's inequality and the boundedness of the Hardy-Littlewood maximal operator on generalized Morrey spaces, we have

$$\begin{aligned}
&\left(\frac{1}{\psi(r)}\int_{B(x,r)}|g(y)T_\rho^K f(y)|^q\,dy\right)^{1/q}\\
&\leqslant C\left(\frac{1}{\psi(r)}\int_{B(x,r)}|Mf(y)|^{q/t}|g(y)|^q\,dy\right)^{1/q}\|f\|_{p,\phi}^{\alpha p/n}\\
&\leqslant C\left(\frac{1}{\phi(r)}\int_{B(x,r)}|Mf(y)|^q\,dy\right)^{1/tq}\left(\frac{1}{\eta(r)}\int_{B(x,r)}|g(y)|^{nq/\alpha p}dy\right)^{\alpha p/nq}\|f\|_{p,\phi}^{\alpha p/n}\\
&\leqslant C\|g\|_{nq/\alpha p,\eta}\|f\|_{q,\phi}^{1-\alpha p/n}\|f\|_{p,\phi}^{\alpha p/n}.
\end{aligned}$$

This implies

$$\|g\cdot T_\rho^K f\|_{q,\psi}\leqslant C\|g\|_{nq/\alpha p,\eta}\|f\|_{q,\phi}^{1-\alpha p/n}\|f\|_{p,\phi}^{\alpha p/n}. \qquad \square$$

As we mentioned for T_ρ in Section 3, if we use Lemma 4.3, we obtain the following theorems.

THEOREM 4.4. *Let $1<p<+\infty$, $1<q<+\infty$, $1<u<+\infty$. For $\phi(r)$ and $\rho(r)$ assume that there exist positive constants C_1, C_2, C_3, and α and non-negative constants λ, δ, and ε with $(\alpha+\varepsilon)p+\lambda+\delta<n$ such that (10), (16), and (18).*

(1) Let $1/s=1/p-\alpha/n$ and $K\geqslant(\lambda+\delta)/p+\varepsilon$. Then there exists a positive constant C such that

$$\|T_\rho^K f\|_{s,\phi}\leqslant C\|f\|_{p,\phi}, \tag{36}$$

where $f\in L_p^\phi$.

(2) Let $1/s=1/p-\alpha/(n-\lambda)$ and $K\geqslant\delta/p+\varepsilon$. Then there exists a positive constant C such that

$$\|T_\rho^K f\|_{s,\phi}\leqslant C\|f\|_{p,\phi}, \tag{37}$$

where $f\in L_p^\phi$.

THEOREM 4.5. *Let $1<p<+\infty$, $1<q<+\infty$, $1<u<+\infty$. For $\phi(r)$ and $\rho(r)$ assume that there exist positive constants C_1, C_2, C_3, and α and non-negative constants λ, δ, and ε with $(\alpha+\varepsilon)p+\lambda+\delta<n$ such that (10), (16), and (18).*

(1) Let $1/q=1/u+1/p-\alpha/n$ and $K\geqslant(\lambda+\delta)/p+\varepsilon$. Assume that

$$\psi(r)=\eta(r)^{q/u}\phi(r)^{q(1/p-\alpha/n)}$$

and $g\in L_u^\eta$. Then there exists a positive constant C such that

$$\|g\cdot T_\rho^K f\|_{q,\psi}\leqslant C\|g\|_{u,\eta}\|f\|_{p,\phi}, \tag{38}$$

where $f\in L_p^\phi$.

(2) *Let* $1/q = 1/u + 1/p - \alpha/(n-\lambda)$ *and* $K \geqslant \delta/p + \varepsilon$. *Assume that*

$$\psi(r) = \eta(r)^{q/u}\phi(r)^{q\{1/p-\alpha/(n-\lambda)\}}$$

and $g \in L_u^\eta$. *Then there exists a positive constant* C *such that*

$$\|g \cdot T_\rho^K f\|_{q,\psi} \leqslant C\|g\|_{u,\eta}\|f\|_{p,\phi}, \tag{39}$$

where $f \in L_p^\phi$.

REMARK 4.6. As we mentioned in Remark 3.9, (2) of Theorems 4.4 and 4.5 follow from [11, Theorem 7.3].

We only prove (1) of Theorems 4.4 and 4.5.

Proof of Theorem 4.4 *(1).* By using (32) and the boundedness of the Hardy-Littlewood maximal operator on generalized Morrey spaces, we have

$$\begin{aligned}\|T_\rho^K f\|_{s,\phi} &\leqslant C \sup_{\substack{x\in\mathbf{R}^n \\ r>0}} \left(\frac{1}{\phi(r)}\int_{B(x,r)} |Mf(y)|^{(1-\alpha p/n)s} dy\right)^{1/s} \|f\|_{p,\phi}^{\alpha p/n} \\ &\leqslant C\|f\|_{p,\phi}^{1-\alpha p/n}\|f\|_{p,\phi}^{\alpha p/n} \\ &= C\|f\|_{p,\phi}. \quad \square\end{aligned}$$

Proof of Theorem 4.5 *(1).* Let $1 < s < +\infty$ and $1/s = 1/p - \alpha/n$. By Hölder's inequality on Morrey spaces, we have

$$\|g \cdot T_\rho^K f\|_{q,\psi} \leqslant \|g\|_{u,\eta}\|T_\rho^K f\|_{s,\phi}. \tag{40}$$

Combining (36) with (40), we arrive at the desired inequality. $\square$

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