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SINGULAR INTEGRODIFFERENTIAL INEQUALITIES AND APPLICATION TO FRACTIONAL DIFFERENTIAL PROBLEMS

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Abstract. In this paper we are concerned with an integrodifferential problem which arises for instance when we study a Cauchy-type fractional differential equation. This problem involves a convolution of a kernel with a nonlinear function of the solution together with its derivatives up to order two. For ordinary (third order) differential equations the kernel is regular while in our case it is singular and nonintegrable. Combining a desingularization technique due to the second author with some other estimations, we find bounds for solutions of the problem with different nonlinearities. Our results are illustrated by an application to fractional differential equations.

1. Introduction

In this work we are interested in the following integro-differential problem with initial value data

$$\begin{cases} x''(t) \leqslant a(t) + \int_{t_0}^{t} (t-s)^{\alpha-1} F(s,x,x',x'')ds, \; t > t_0 \\ x(t_0) = k_0, \; x'(t_0) = k_1, \end{cases} \tag{1}$$

where k_0, k_1 are given constants, $a(t)$ is a nonnegative continuous function and $0 < \alpha < 1$. F here is a nonlinear function of the time variable, the solution x and its first and second derivatives. Two different classes of functions will be specified in the sequel. A similar problem was studied by E. Kurpinar [8] but with a regular kernel. He generalized an inequality given by Pachpatte ([17,18]) and found some continuous bounds for some classes of nonlinearities and used his results to obtain bounds on solutions of differential equations of the form $x'''(t) = f(t,x,x',x'')$ with the initial conditions $x(t_0) = k_0$, $x'(t_0) = k_1$, $x''(t_0) = k_2$, $t_0 > 0$, where f is a continuous function and k_0, k_1 and k_2 are real constants. The integral term in (1) may be regarded as a convolution of the singular kernel $1/t^{1-\alpha}$ with the nonlinear function $F(t,x,x',x'')$. Our work may then be considered as the singular version of the work of E. Kurpinar. It is worthmentioning here that because of the singularity (and nonintegrability) of the kernel the standard methods used in the case of a regular kernel are useless in our situation. To overcome these difficulties we shall combine a desingularization technique

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due to M. Kirane and N.-e. Tatar [7] which is in turn based on an idea by M. Medved (see [11,12]). This technique has been proved to be very efficient for different kinds of problems (see [21]). We refer the reader to the papers [3,4,5,9,10,13] and the references therein for some other treatements of desingularization and applications to fractional differential equations.

Our paper is organized as follows. In Section 2 we prepare some tools needed to prove our theorems. Section 3 contains the statements and proofs of our main results and in Section 4 we give an application to fractional differential problems.

2. Preliminaries

In this section we state some definitions and lemmas. We denote by $\Gamma(.)$ the usual Gamma function.

DEFINITION 1. let $[a,b]$ $(-\infty < a < b < \infty)$ be a finite interval and let $AC[a,b]$ be the space of functions f which are absolutely continuous on $[a,b]$. For n positive integer and $D = \frac{d}{dt}$, we denote by

$$AC^n[a,b] = \left\{ f : [a,b] \longrightarrow \mathbf{R},\ \left(D^{n-1}f\right)(t) \in AC[a,b] \right\}.$$

In particular $AC^1[a,b] = AC[a,b]$.

DEFINITION 2. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a real valued Lebesgue-summable function f is defined by

$$I_{t_0}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s)ds$$

provided that the integral exists.

DEFINITION 3. The fractional derivative (in the sense of Riemann-Liouville) of order $\alpha > 0$ of a real valued function f is defined as the left inverse of the fractional integral of f

$$D_{t_0}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_{t_0}^{t} \frac{f(s)ds}{(t-s)^{\alpha-n+1}},\ n = [\alpha]+1$$

provided that the right hand side exists.

The first lemma will be used in Section 4. It will help us get rid of a fractional derivative in the left hand side of an integro-fractional-differential inequality.

LEMMA 1. *Let $\alpha > 0$, $n = [\alpha]+1$ and let $f_{n-\alpha}(t) = (I_{t_0}^{n-\alpha} f)(t)$ be the fractional integral in Definition 2 of order $n-\alpha$.*

(a) If $1 \leqslant p \leqslant \infty$ *and* $f(t) \in I_{t_0}^{\alpha}(L_p)$, *then*

$$(I_{t_0}^{\alpha} D_{t_0}^{\alpha} f)(t) = f(t).$$

(b) If $f(t) \in L_1[t_0, T]$ *and* $f_{n-\alpha}(t) \in AC^n[t_0, T]$, *then the equality*

$$(I_{t_0}^{\alpha} D_{t_0}^{\alpha} f)(t) = f(t) - \sum_{j=1}^{n} \frac{f_{n-\alpha}^{(n-j)}(t_0)}{\Gamma(\alpha - j + 1)} (t - t_0)^{\alpha - 1},$$

holds almost everywhere on $[t_0, T]$. *In particular, if* $0 < \alpha < 1$, *then*

$$(I_{t_0}^{\alpha} D_{t_0}^{\alpha} f)(t) = f(t) - \frac{f_{1-\alpha}(t_0)}{\Gamma(\alpha)} (t - t_0)^{\alpha - 1},$$

where $f_{1-\alpha}(t) = (I_{t_0}^{1-\alpha} f)(t)$.

The following lemma is part of a desingularization technique used in this paper.

LEMMA 2. *If* $\theta, \delta, \omega > 0$, *then for any* $t > t_0 > 0$ *we have*

$$t^{1-\theta} \int_{t_0}^{t} (t-s)^{\theta-1} s^{\delta-1} e^{-\omega s} ds \leqslant K$$

where $K = K(\theta, \delta, \omega)$ *is a positive constant independent of* t. *In fact,*

$$K = \max\left\{1, 2^{1-\theta}\right\} \Gamma(\delta)(1 + \frac{\delta}{\theta})\omega^{-\delta}.$$

Our last lemma is a well-known result which gives an upper bound for solutions of a differential inequality.

LEMMA 3. *Let* $V(t)$ *be a positive differentiable function satisfying the inequality*

$$V'(t) \leqslant f(t)V(t) + g(t)V^p(t),\ t \in [t_0, \infty),\ t_0 > 0$$

where the functions $f(t), g(t)$ *are continuous in* $[t_0, \infty)$, *and* $p \geqslant 0$, $p \neq 1$ *is a constant, then*

$$V(t) \leqslant \exp\left(\int_{t_0}^{t} f(\xi) d\xi\right) \left[V^q(t_0) + q \int_{t_0}^{t} g(\xi) \exp\left(-q \int_{t_0}^{\xi} f(\eta) d\eta\right) d\xi\right]^{\frac{1}{q}}$$

for $t, \xi \in [t_0, \rho)$, *where* $q = 1 - p$ *and* ρ *is chosen so that the expression*

$$\left[V^q(t_0) + q \int_{t_0}^{t} g(\xi) \exp\left(-q \int_{t_0}^{\xi} f(\eta) d\eta\right) d\xi\right]$$

is positive in the subinterval $[t_0, \rho)$.

For the proof of Lemma 1 and more on fractional calculus we refer to [6,15,16,19,20]. Lemma 2 is in [14]. The proof of Lemma 3 may be found in [1,2].

3. The Results

The first result concerns a nonlinearity F satisfying for $\gamma > 0,\ \nu > 0,\ \mu > 0$,

$$(\mathbf{F}_1)\quad |F(t,x,x',x'')| \leqslant |x''(t)| \left[p(t)\,|x(t)|^{\gamma} + q(t)\,|x'(t)|^{\nu} + r(t)\,|x''(t)|^{\mu}\right]$$

where $p(t),\ q(t),\ r(t)$ are real-valued continuous functions defined on $I = [t_0,T]$. To enlighten the statement of our first result let us denote

$\sigma = q\max(\gamma,\nu,\mu)$ for some $q > \frac{1}{\alpha}$,

$M = 6^{q-1}K^{\frac{q}{p}}$, where K comes from the appplication of Lemma 2,

$Q(t) = \max\left(|p(t)|^q, |q(t)|^q, |r(t)|^q\right)$, $C(T) = 2^{1-\frac{1}{q}} \max\limits_{t_0 \leqslant t \leqslant T} a(t)t^{1-\alpha} > 0$,

$v(t_0) = 3 + |k_0| + |k_1| + C(T)t_0^{\alpha-1}$ and

$$G_1(t) = Mt_0^{q(\alpha-1)}e^{qt+\sigma(t-t_0)}Q(t)\left[v^{-\sigma}(t_0) - \frac{\sigma M}{q}t_0^{q(\alpha-1)}e^{-\sigma t_0}\int_{t_0}^{t} e^{(q+\sigma)\xi}Q(\xi)d\xi\right]^{-1}.$$

It is clear that we may assume, without loss of generality, that $C(T) > 0$.

THEOREM 1. *Assume that ($\boldsymbol{F}_1$) is satisfied. Let $x(t)$ be a twice continuously differentiable real-valued function defined on $I = [t_0,T]$, $t_0 > 0$, $x''(t)$ and $a(t)$ be nonnegative continuous functions. Then for solutions of (1) we have*

(i) If $\sigma \geqslant 1$, then

$$x(t) \leqslant k_0 + k_1(t-t_0) + C(T)t_0^{\alpha-1}\frac{(t-t_0)^2}{2}\exp\left(\frac{1}{q}\int_{t_0}^{t} G_1(s)\,ds\right),\ t \in [t_0,\rho_1), \tag{2}$$

where $C(T)$ is defined above and ρ_1 is chosen so that the expression

$$v^{-\sigma}(t_0) - \frac{\sigma M}{q}t_0^{q(\alpha-1)}e^{-\sigma t_0}\int_{t_0}^{t} e^{(q+\sigma)\xi}Q(\xi)d\xi \tag{3}$$

is positive in the subinterval $[t_0,\rho_1)$.

(ii) If $0 < \sigma < 1$, we have a similar result as in (i) with σ replaced by the value 1 in the estimation.

Proof. (i) From (1) and ($\mathbf{F}_1$) we have

$$x''(t) \leqslant a(t) + \int_{t_0}^{t}(t-s)^{\alpha-1}x''(s)\left[p(s)\,|x(s)|^{\gamma} + q(s)\left|x'(s)\right|^{\nu} + r(s)\left(x''(s)\right)^{\mu}\right]ds.$$

Applying Hölder inequality we obtain

$$x''(t) \leqslant a(t) + \left(\int_{t_0}^{t}(t-s)^{p(\alpha-1)}e^{-ps}ds\right)^{\frac{1}{p}}$$

$$\times\left[\int_{t_0}^{t} e^{qs}\left(x''(s)\right)^q\left(p(s)\,|x(s)|^{\gamma} + q(s)\left|x'(s)\right|^{\nu} + r(s)\left(x''(s)\right)^{\mu}\right)^q ds\right]^{\frac{1}{q}},$$

for $q>1$ such that $0<\frac{1}{q}<\alpha<1$ and p such that $\frac{1}{p}+\frac{1}{q}=1$. Since $p(\alpha-1)>-1$, we can apply Lemma 2 with $\delta=1$ to get

$$x''(t)\leqslant a(t)+K^{\frac{1}{p}}t^{\alpha-1} \times\left[\int_{t_0}^{t}e^{qs}\left(x''(s)\right)^q\left(p(s)\left|x(s)\right|^{\gamma}+q(s)\left|x'(s)\right|^{\nu}+r(s)\left(x''(s)\right)^{\mu}\right)^q ds\right]^{\frac{1}{q}}. \tag{4}$$

By using the following well known (consequence of the Jensen) inequality

$$(a_1+a_2+...+a_n)^r\leqslant n^{r-1}(a_1^r+a_2^r+...+a_n^r),\ \ a_1,a_2,...,a_n\geqslant 0,\ r>1$$

twice (once with $n=2$ and once with $n=3$), $r=q$ and we find

$$(t^{1-\alpha}x''(t))^q\leqslant C(T)^q +M\int_{t_0}^{t}e^{qs}Q(s)\left(x''(s)\right)^q\left(\left|x(s)\right|^{q\gamma}+\left|x'(s)\right|^{q\nu}+\left(x''(s)\right)^{q\mu}\right)ds. \tag{5}$$

Define a function $y(t)$ to be equal to the right hand side of the inequality (5), that is

$$y(t):=C^q+M\int_{t_0}^{t}e^{qs}Q(s)\left(x''(s)\right)^q\left(\left|x(s)\right|^{q\gamma}+\left|x'(s)\right|^{q\nu}+\left(x''(s)\right)^{q\mu}\right)ds. \tag{6}$$

Therefore

$$x''(t)\leqslant t^{\alpha-1}\sqrt[q]{y(t)}\leqslant t_0^{\alpha-1}\sqrt[q]{y(t)},\ t\geqslant t_0>0. \tag{7}$$

An integration of both sides of (7) from t_0 to t and taking into account that $y(t)$ is nonnegative and nondecreasing, yields

$$\left|x'(t)\right|\leqslant|k_1|+t_0^{\alpha-1}\int_{t_0}^{t}\sqrt[q]{y(s)}ds\leqslant|k_1|+t_0^{\alpha-1}(t-t_0)\sqrt[q]{y(t)}, \tag{8}$$

and an integration again gives

$$\begin{aligned}|x(t)|&\leqslant|k_0|+|k_1|(t-t_0)+t_0^{\alpha-1}\int_{t_0}^{t}(s-t_0)\sqrt[q]{y(s)}ds\\&\leqslant|k_0|+|k_1|(t-t_0)+t_0^{\alpha-1}\frac{(t-t_0)^2}{2}\sqrt[q]{y(t)}.\end{aligned} \tag{9}$$

Now differentiating (6) and using (7)-(9) we get

$$\begin{aligned}y'(t)\leqslant Me^{qt}Q(t)t_0^{q(\alpha-1)}y(t)\Bigg[&\left(1+|k_0|+|k_1|(t-t_0)+t_0^{\alpha-1}\frac{(t-t_0)^2}{2}\sqrt[q]{y(t)}\right)^{q\gamma}\\&+\left(1+|k_1|+t_0^{\alpha-1}(t-t_0)\sqrt[q]{y(t)}\right)^{q\nu}+\left(1+t_0^{\alpha-1}\sqrt[q]{y(t)}\right)^{q\mu}\Bigg].\end{aligned} \tag{10}$$

By the definition of σ, we infer that

$$\begin{aligned} y'(t) \leqslant Me^{qt}Q(t)t_0^{q(\alpha-1)}y(t)\Bigg[\left(1+|k_0|+|k_1|(t-t_0)+t_0^{\alpha-1}\frac{(t-t_0)^2}{2}\sqrt[q]{y(t)}\right)^{\sigma} \\ +\left(1+|k_1|+t_0^{\alpha-1}(t-t_0)\sqrt[q]{y(t)}\right)^{\sigma}+\left(1+t_0^{\alpha-1}\sqrt[q]{y(t)}\right)^{\sigma}\Bigg]. \end{aligned} \tag{11}$$

Observe that in case $|k_0|$ and $|k_1|$ are greater than or equal to 1, then there is no need to add 1 to the first two terms in the right hand side of (10). Now using the elementary inequality

$$a^{\sigma}+b^{\sigma}+c^{\sigma} \leqslant (a+b+c)^{\sigma}, \ \ a,b,c \geqslant 0, \ \ \sigma \geqslant 1$$

and the fact that $y(t)$ is positive $(C(T)>0)$ we obtain from (11) that

$$\begin{aligned} \frac{y'(t)}{y(t)} &\leqslant Me^{qt}Q(t)t_0^{q(\alpha-1)} \\ &\times\left[3+|k_0|+|k_1|(1+t-t_0)+t_0^{\alpha-1}\left(1+t-t_0+\frac{(t-t_0)^2}{2}\right)\sqrt[q]{y(t)}\right]^{\sigma}. \end{aligned} \tag{12}$$

Let us denote

$$v(t) := 3+|k_0|+|k_1|(1+t-t_0)+t_0^{\alpha-1}\left(1+t-t_0+\frac{(t-t_0)^2}{2}\right)\sqrt[q]{y(t)}.$$

Notice that $v(t)>0, \ v(t_0)=3+|k_0|+|k_1|+C(T)t_0^{\alpha-1}$ and

$$\begin{aligned} v'(t) &= |k_1|+t_0^{\alpha-1}\left[(1+t-t_0)\sqrt[q]{y(t)}+\frac{1}{q}\frac{y'(t)}{y(t)}\sqrt[q]{y(t)}\left(1+t-t_0+\frac{(t-t_0)^2}{2}\right)\right] \\ &\leqslant v(t)+\frac{1}{q}Me^{qt}Q(t)t_0^{q(\alpha-1)}v(t)^{\sigma+1}. \end{aligned} \tag{13}$$

By Lemma 3 we deduce from (13) the estimate

$$v(t) \leqslant e^{t-t_0}\left[v^{-\sigma}(t_0)-\frac{\sigma M}{q}t_0^{q(\alpha-1)}e^{-\sigma t_0}\int_{t_0}^{t}e^{(q+\sigma)\xi}Q(\xi)d\xi\right]^{\frac{1}{-\sigma}} \tag{14}$$

for $t \in [t_0,\rho_1)$ and ρ_1 is chosen so that the expression between brackets is positive in the subinterval $[t_0,\rho_1)$. Next, the relations (12) in (14) imply

$$\frac{y'(t)}{y(t)} \leqslant Me^{qt+\sigma(t-t_0)}Q(t)t_0^{q(\alpha-1)}\left[v^{-\sigma}(t_0)-\frac{\sigma M}{q}t_0^{q(\alpha-1)}e^{-\sigma t_0}\int_{t_0}^{t}e^{(q+\sigma)\xi}Q(\xi)d\xi\right]^{-1}. \tag{15}$$

Changing t into s and integrating the inequality (15) from t_0 to t we find

$$y(t) \leqslant y(t_0)\exp\left(\int_{t_0}^{t}G_1(s)ds\right) \tag{16}$$

where $G_1(t)$ is defined just before the theorem. Using (16) in (9) we obtain the desired inequality in (2).

The second result concerns a nonlinearity F satisfying

$(\mathbf{F}_2)$
$$\left|F(t,x,x',x'')\right| \leqslant x''(t) \times \left(\left|x(t)+x'(t)+x''(t)\right| + \int_{t_0}^{t} b(t,s)\left|x(s)+x'(s)+x''(s)\right| ds\right)$$

First, let us denote

$$N = 2^{q-1}K^{\frac{q}{p}},\ E(t) = \max\left(1, b^q(t,t)\right),$$
$$F(t_0) = |k_0| + |k_1| + Ct_0^{\alpha-1},\ C(T) = 2^{1-\frac{1}{q}} \max_{t_0 \leqslant t \leqslant T} a(t)t^{1-\alpha},$$
$$G_3(t) = Ne^{3qt-2qt_0}t_0^{q(\alpha-1)}E(t)\left[F^{-q}(t_0) - Nt_0^{q(\alpha-1)}e^{-2qt_0}\int_{t_0}^{t} e^{3q\xi}E(\xi)d\xi\right]^{-1}.$$

THEOREM 2. *Assume that F satisfies ($\boldsymbol{F}_2$). Suppose that $x(t)$, $x'(t)$ are real-valued continuous functions defined on $I = [t_0,T]$ and $x''(t)$, $a(t)$ are nonnegative continuous functions defined on I. The function b is nonnegative and continuous on $\triangle = \left\{(t,s) \in I^2 : s \leqslant t\right\}$. Moreover, we assume that $b(t,s)$ is nondecreasing in s. Then solutions of (1) satisfy*

$$x(t) \leqslant |k_0| + |k_1|(t-t_0) + Ct_0^{\alpha-1}\frac{(t-t_0)^2}{2}\exp\left(\frac{1}{q}\int_{t_0}^{t} G_3(s)ds\right) \tag{17}$$

for $t \in [t_0,\rho_2)$, where ρ_2 is chosen so that the expression

$$F^{-q}(t_0) - Nt_0^{q(\alpha-1)}e^{-2qt_0}\int_{t_0}^{t} e^{3q\xi}E(\xi)d\xi$$

is positive in the subinterval $[t_0,\rho_2)$.

The proof of this theorem is similar to that of Theorem 1. Nevertheless, for completeness, it is helpful to sketch few details before refereeing to the proof of Theorem 1.

Proof. Clearly we have

$$x''(t) \leqslant a(t) + \int_{t_0}^{t} (t-s)^{\alpha-1}x''(s) \times \left(\left|x(s)+x'(s)+x''(s)\right| + \int_{t_0}^{s} b(s,\tau)\left|x(\tau)+x'(\tau)+x''(\tau)\right| d\tau\right) ds.$$

By Hölder inequality we obtain

$$x''(t) \leqslant a(t) + \left(\int_{t_0}^{t} (t-s)^{p(\alpha-1)}e^{-ps}ds\right)^{\frac{1}{p}} \left[\int_{t_0}^{t} e^{qs}x''^q(s)\left(\left|x(s)+x'(s)+x''(s)\right| + \int_{t_0}^{s} b(s,\tau)\left|x(\tau)+x'(\tau)+x''(\tau)\right| d\tau\right)^q ds\right]^{\frac{1}{q}}$$

for $q>1$ such that $0<\frac{1}{q}<\alpha<1$ and p its conjuguate exponent i.e. $\frac{1}{p}+\frac{1}{q}=1$. As $p(\alpha-1)>-1$, we can use Lemma 2 with $\delta=1$, to get

$$x''(t) \leqslant a(t)+K^{\frac{1}{p}}t^{\alpha-1}\left[\int_{t_0}^{t} e^{qs}x''^{q}(s)\left(\left|x(s)+x'(s)+x''(s)\right|\right.\right.$$
$$\left.\left.+\int_{t_0}^{s} b(s,\tau)\left|x(\tau)+x'(\tau)+x''(\tau)\right|d\tau\right)^{q} ds\right]^{\frac{1}{q}}.$$

By the discrete Jensen's inequality with $n=2,\ r=q$ and the fact that $b(t,s)$ is nondecreasing in s we obtain

$$\begin{aligned}\left(t^{1-\alpha}x''(t)\right)^{q} \leqslant C^{q}+N\int_{t_0}^{t} e^{qs}x''^{q}(s)E(s)\left(\left|x(s)+x'(s)+x''(s)\right|\right.\\ \left.+\int_{t_0}^{s}\left|x(\tau)+x'(\tau)+x''(\tau)\right|d\tau\right)^{q} ds,\end{aligned} \tag{18}$$

where C and N are as above. If we denote the right hand side of (18) by $z(t)$, we easily see that $z(t)>0$ (because $C(T)>0$)), $z(t)$ is nondecreasing and $z(t_0)=C^q$. Further, from (18), we have for $t\geqslant t_0>0$

$$x''(t)\leqslant t^{\alpha-1}\sqrt[q]{z(t)}\leqslant t_0^{\alpha-1}\sqrt[q]{z(t)}. \tag{19}$$

An integration of both sides of (19) from t_0 to t (after replacing t by s) gives

$$\left|x'(t)\right|\leqslant |k_1|+t_0^{\alpha-1}\int_{t_0}^{t}\sqrt[q]{z(s)}ds\leqslant |k_1|+t_0^{\alpha-1}(t-t_0)\sqrt[q]{z(t)}, \tag{20}$$

and another one gives

$$\begin{aligned}|x(t)| &\leqslant |k_0|+|k_1|(t-t_0)+t_0^{\alpha-1}\int_{t_0}^{t}(s-t_0)\sqrt[q]{z(s)}ds\\ &\leqslant |k_0|+|k_1|(t-t_0)+t_0^{\alpha-1}\frac{(t-t_0)^2}{2}\sqrt[q]{z(t)}.\end{aligned} \tag{21}$$

Next, a differentiation of $z(t)$ followed by the use of (19)-(21) implies

$$\begin{aligned}&\frac{z'(t)}{z(t)}\leqslant Ne^{qt}t_0^{q(\alpha-1)}E(t)\\ &\times\left(|k_0|+|k_1|(1+t-t_0)+t_0^{\alpha-1}(1+t-t_0+\frac{(t-t_0)^2}{2})\sqrt[q]{z(t)}\right.\\ &\left.+\int_{t_0}^{t}\left(|k_0|+|k_1|(1+s-t_0)+t_0^{\alpha-1}(1+s-t_0+\frac{(s-t_0)^2}{2})\sqrt[q]{z(s)}\right)ds\right)^{q}.\end{aligned}$$

Now we put

$$\begin{aligned}F(t) &:= |k_0|+|k_1|(1+t-t_0)+t_0^{\alpha-1}(1+t-t_0+\frac{(t-t_0)^2}{2})\sqrt[q]{z(t)}\\ &+\int_{t_0}^{t}\left(|k_0|+|k_1|(1+s-t_0)+t_0^{\alpha-1}(1+s-t_0+\frac{(s-t_0)^2}{2})\sqrt[q]{z(s)}\right)ds\end{aligned}$$

and complete the proof as in part (i) of Theorem 1.

REMARK 1. 1- It is clear that we have not tried to push the estimates to the extreme. Indeed, for instance in (10) we could have rather used the algebraic inequality $x^a \leqslant 1+x^b,\ a \leqslant b$ (see the argument in the Application section).

2- Using (16) in (8) and (7) we can have estimates on $x'(t)$ and $x''(t)$, respectively.

3- Other nonlinearities can be treated by the same technique. Notice for instance that there is still room for other nonlinearities involving polynomials in the variable t. This is due to the applicability of Lemma 2 for values of $\delta \neq 1$.

4- Our argument works also for higher order inequalities. That is we can obtain similar results for

$$\begin{cases} x^{(n)}(t) \leqslant a(t) + \int_{t_0}^{t} (t-s)^{\alpha-1} F\left(s, x(s), x'(s), \ldots, x^{(n)}(s)\right) ds \\ x(t_0) = k_0,\ x'(t_0) = k_1,\ \ldots,\ x^{(n-1)}(t_0) = k_{n-1} \end{cases}$$

with $n > 3$.

4. Application

In this section we present an application illustrating our findings in the previous section. We will see how to obtain bounds on solutions of some fractional differential problems. In particular, in case solutions exist locally then these bounds may help proving global existence in time.

Let us consider the Cauchy-type problem

$$\begin{cases} D_{t_0}^{\alpha}\left(x''(t)\right) = F(t, x, x', x''),\ 0 < \alpha < 1,\ t \in I = [t_0, T],\ t_0 > 0, \\ x(t_0) = k_0,\ x'(t_0) = k_1,\ \lim\limits_{t \to t_0} \left[(t-t_0)^{1-\alpha} x''(t)\right] = k_2, \end{cases} \tag{22}$$

where D^{α} denotes the fractional derivative (in the sense of Riemann-Liouville) of order α, $(0 < \alpha < 1)$, and F is a continuous function satisfying the assumption

$$\left|F(t, x, x', x'')\right| \leqslant x''(t)\left[p(t)\left|x(t)\right|^{\gamma} + q(t)\left|x'(t)\right|^{\nu} + r(t)\left|x''(t)\right|^{\mu}\right]$$

for $\gamma > 0$, $\nu > 0$, $\mu > 0$, and $p(t)$, $q(t)$, $r(t)$ are real-valued continuous functions defined on $I = [t_0, T]$.

The weighted initial condition

$$\lim_{t \to t_0} \left[(t-t_0)^{1-\alpha} x''(t)\right] = k_2$$

is a natural one. It corresponds to $(I_{t_0}^{1-\alpha} x'')(t_0^+) = k_2 \Gamma(\alpha)$ or by a notation convention to $(D_{t_0}^{\alpha-1} x'')(t_0^+) = k_2 \Gamma(\alpha)$, see the following lemma which can be found in [3, Lemma 3.2 p. 151]

LEMMA 4. *Let $0<\alpha<1$ and $y(t)$ be a Lebesgue measurable function on $[a,b]$. If there exists a.e. a limit*

$$\lim_{t\to a^+}\left[(t-a)^{1-\alpha}y(t)\right]=c$$

then there exists a.e. a limit

$$(I_{a^+}^{1-\alpha}y)(a^+):=\lim_{t\to a^+}(I_{a^+}^{1-\alpha}y)(a^+)=c\Gamma(\alpha).$$

Assuming that $x''_{1-\alpha}(t)=(I_{t_0}^{1-\alpha}x'')(t)\in AC[t_0,T]$, we can apply I^α to both sides of the equation in (22) and use Lemma 1 to find

$$x''(t)-\frac{(I_{t_0}^{1-\alpha}x'')(t_0^+)}{\Gamma(\alpha)}(t-t_0)^{\alpha-1}=I^\alpha F(t,x,x',x'')$$

or

$$x''(t)=k_2(t-t_0)^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_{t_0}^t(t-s)^{\alpha-1}F(s,x,x',x'')ds. \tag{23}$$

Although the same argument as in the proof of Theorem 1 is applicable here, the equation (23) presents some differences and therefore some points need to be clarified. In particular, our function $a(t)$ here is equal to $k_2(t-t_0)^{\alpha-1}$ which is singular at t_0. Moreover, the function $v(t)$ we get by a direct application of the argument in the proof of Theorem 1 is not defined at t_0. We shall briefly sketch the proof (of how to obtain bounds on solutions of (23)) below.

Assume that $\sigma\geqslant 1$. As in (4) we have

$$x''(t)\leqslant k_2(t-t_0)^{\alpha-1}+\frac{K^{\frac{1}{p}}(t-t_0)^{\alpha-1}}{\Gamma(\alpha)}\times\left[\int_{t_0}^t e^{qs}x''^q(s)\left(p(s)|x(s)|^\gamma+q(s)\left|x'(s)\right|^\nu+r(s)\left(x''(s)\right)^\mu\right)^q ds\right]^{\frac{1}{q}}$$

or

$$(t-t_0)^{1-\alpha}x''(t)\leqslant k_2+\frac{K^{\frac{1}{p}}}{\Gamma(\alpha)}\times\left[\int_{t_0}^t e^{qs}x''^q(s)\left(p(s)|x(s)|^\gamma+q(s)\left|x'(s)\right|^\nu+r(s)\left(x''(s)\right)^\mu\right)^q ds\right]^{\frac{1}{q}}. \tag{24}$$

Then raising both sides of (24) to the power q we see that

$$\left[(t-t_0)^{1-\alpha}x''(t)\right]^q\leqslant 2^{q-1}k_2^q+\frac{2^{q-1}K^{\frac{q}{p}}}{\Gamma^q(\alpha)}\times\left[\int_{t_0}^t e^{qs}x''^q(s)\left(p(s)|x(s)|^\gamma+q(s)\left|x'(s)\right|^\nu+r(s)\left(x''(s)\right)^\mu\right)^q ds\right]. \tag{25}$$

Denoting by $y(t)$ the right hand side of (25) we entail that

$$x''(t) \leqslant (t-t_0)^{\alpha-1} \sqrt[q]{y(t)},\ t \geqslant t_0. \tag{26}$$

Notice that the coefficient of $\sqrt[q]{y(t)}$ is no more a constant (as in the first argument). The relation (26) implies that

$$\left|x'(t)\right| \leqslant |k_1| + \int_{t_0}^{t} (s-t_0)^{\alpha-1} \sqrt[q]{y(s)}ds \leqslant |k_1| + \frac{(t-t_0)^{\alpha}}{\alpha} \sqrt[q]{y(t)}, \tag{27}$$

and

$$|x(t)| \leqslant |k_0| + |k_1|(t-t_0) + \frac{(t-t_0)^{\alpha+1}}{\alpha(\alpha+1)} \sqrt[q]{y(t)}. \tag{28}$$

On the other hand, we have

$$\begin{aligned} y'(t) &= \frac{2^{q-1}K^{\frac{q}{p}}}{\Gamma^q(\alpha)} e^{qt} x''^q(t) \left(p(t)|x(t)|^{\gamma} + q(t)\left|x'(t)\right|^{\nu} + r(t)\left(x''(t)\right)^{\mu}\right)^q \\ &\leqslant R(t)y(t) \\ &\times \left[p^q(t)|x(t)|^{\gamma q} + q^q(t)\left|x'(t)\right|^{\nu q} + r^q(t)(t-t_0)^{-\mu q(\alpha-1)}\left|(t-t_0)^{1-\alpha}x''(t)\right|^{\mu q}\right] \\ &\leqslant R(t)Q(t)y(t)\left[3 + |x(t)|^{\sigma} + \left|x'(t)\right|^{\sigma} + \left|(t-t_0)^{1-\alpha}x''(t)\right|^{\sigma}\right] \\ &\leqslant R(t)Q(t)y(t)\left[3 + \left(|x(t)| + \left|x'(t)\right| + \left|(t-t_0)^{1-\alpha}x''(t)\right|\right)^{\sigma}\right] \end{aligned}$$

where

$$R(t) := \frac{6^{q-1}K^{\frac{q}{p}} e^{qt}(t-t_0)^{q(\alpha-1)}}{\Gamma^q(\alpha)} \tag{29}$$

and

$$Q(t) = \max\{p^q(t), q^q(t), r^q(t)(t-t_0)^{-\mu q(\alpha-1)}\}. \tag{30}$$

Hence, taking into account the relations (26)-(28), we find

$$\begin{aligned} y'(t) &\leqslant 3R(t)Q(t)y(t) + R(t)Q(t)y(t) \\ &\times \left[|k_0| + |k_1|(1+t-t_0) + \left(1 + \frac{1}{\alpha} + \frac{t-t_0}{\alpha} + \frac{(t-t_0)^{\alpha+1}}{\alpha(\alpha+1)}\right)\sqrt[q]{y(t)}\right]^{\sigma}. \end{aligned} \tag{31}$$

Let us designate by $v(t)$ the expression

$$v(t) := |k_0| + |k_1|(1+t-t_0) + \left(1 + \frac{1}{\alpha} + \frac{t-t_0}{\alpha} + \frac{(t-t_0)^{\alpha+1}}{\alpha(\alpha+1)}\right)\sqrt[q]{y(t)}, \tag{32}$$

then $v(t) > 0$, $v(t_0) = |k_0| + |k_1| + 2^{1-\frac{1}{q}}\left(1 + \frac{1}{\alpha}\right)k_2$ and

$$\begin{aligned} v'(t) = |k_1| &+ \frac{y'(t)}{qy(t)}\sqrt[q]{y(t)}\left(1 + \frac{1}{\alpha} + \frac{t-t_0}{\alpha} + \frac{(t-t_0)^{\alpha+1}}{\alpha(\alpha+1)}\right) \\ &+ \sqrt[q]{y(t)}\left(\frac{1}{\alpha} + \frac{(t-t_0)^{\alpha}}{\alpha}\right). \end{aligned} \tag{33}$$

In virtue of (31) we can write

$$\frac{y'(t)}{y(t)} \leqslant 3R(t)Q(t) + R(t)Q(t)(t-t_0)^{q(\alpha-1)}v^{\sigma}(t). \tag{34}$$

Hence (33), (34) and the definition of $v(t)$ allow us to derive

$$v'(t) \leqslant \left(\frac{1}{\alpha} + \frac{3R(t)}{q}\right)v(t) + \frac{R(t)}{q}v^{\sigma+1}(t).$$

A direct application of Lemma 3 entails

$$\begin{aligned} v(t) \leqslant &\exp\left\{\int_{t_0}^{t}\left(\frac{1}{\alpha}+\frac{3R(\xi)}{q}\right)d\xi\right\} \\ &\times\left[v^{-\sigma}(t_0) - \frac{\sigma}{q}\int_{t_0}^{t}R(\xi)\exp\left\{\sigma\int_{t_0}^{\xi}\left(\frac{1}{\alpha}+\frac{3R(\eta)}{q}\right)d\eta\right\}d\xi\right]^{\frac{-1}{\sigma}}. \end{aligned}$$

Therefore

$$y(t) \leqslant y(t_0)\exp\left(\int_{t_0}^{t}G_3(s)ds\right)$$

with

$$\begin{aligned} G_3(t) := &3R(t)Q(t) + R(t)Q(t)(t-t_0)^{q(\alpha-1)}\exp\left\{\sigma\int_{t_0}^{t}\left(\frac{1}{\alpha}+\frac{3R(\xi)}{q}\right)d\xi\right\} \\ &\times\left[v^{-\sigma}(t_0) - \frac{\sigma}{q}\int_{t_0}^{t}R(\xi)\exp\left\{\sigma\int_{t_0}^{\xi}\left(\frac{1}{\alpha}+\frac{3R(\eta)}{q}\right)d\eta\right\}d\xi\right]^{-1}. \end{aligned} \tag{35}$$

Finally, with the help of the relation (28) we get an estimate for $x(t)$

$$|x(t)| \leqslant |k_0| + |k_1|(t-t_0) + \frac{2^{1-\frac{1}{q}}k_2(t-t_0)^{\alpha+1}}{\alpha(\alpha+1)}\exp\left(\frac{1}{q}\int_{t_0}^{t}G_3(s)ds\right) \tag{36}$$

as long as the expression $v^{-\sigma}(t_0) - \frac{\sigma}{q}\int_{t_0}^{t}R(\xi)\exp\left\{\sigma\int_{t_0}^{\xi}\left(\frac{1}{\alpha}+\frac{3R(\eta)}{q}\right)d\eta\right\}d\xi$ is positive.

REMARK 2. Notice that if $\frac{\sigma}{q}\int_{t_0}^{t}R(\xi)\exp\left\{\sigma\int_{t_0}^{\xi}\left(\frac{1}{\alpha}+\frac{3R(\eta)}{q}\right)d\eta\right\}d\xi < v^{-\sigma}(t_0)$ for all $t \geqslant t_0$, then solutions exist globally in time.

The above findings may be stated in the following theorem.

THEOREM 3. *Under the assumptions of Theorem 1 and* $\sigma \geqslant 1$, *we have*

$$x(t) \leqslant |k_0| + |k_1|(t-t_0) + \frac{2^{1-\frac{1}{q}}k_2(t-t_0)^{\alpha+1}}{\alpha(\alpha+1)}\exp\left(\frac{1}{q}\int_{t_0}^{t}G_3(s)ds\right)$$

as long as the expression $v^{-\sigma}(t_0) - \frac{\sigma}{q}\int_{t_0}^{t} R(\xi)\exp\left\{\sigma\int_{t_0}^{\xi}\left(\frac{1}{\alpha}+\frac{3R(\eta)}{q}\right)d\eta\right\}d\xi$ *is positive. Furthermore, if*

$$\frac{\sigma}{q}\int_{t_0}^{t} R(\xi)\exp\left\{\sigma\int_{t_0}^{\xi}\left(\frac{1}{\alpha}+\frac{3R(\eta)}{q}\right)d\eta\right\}d\xi < v^{-\sigma}(t_0)$$

for all $t \geqslant t_0$, *then any local solution exists globally in time.*

REMARK 3. Note that the assumption that x'' be nonnegative (in the first two theorems) is not really restrictive since for equations (which is often the case, as in the present application) we can work with the absolute value of this expression as in the relation (23) above.

Discussion. It is worth noting that $D_{t_0}^{\alpha}(x''(t)) \neq (D_{t_0}^{\alpha+2}x)(t)$. Indeed, from the property 2.4 p. 75 in [6] we have

$$(D_{t_0}^{\alpha+2}x)(t) = (D_{t_0}^{\alpha}D^2x)(t) + \frac{x'(t_0)(t-t_0)^{-\alpha-1}}{\Gamma(-\alpha)} + \frac{x(t_0)(t-t_0)^{-\alpha-2}}{\Gamma(-1-\alpha)}.$$

This leads to some difficulties when we apply the fractional integration operator I^{α} to both sides of the equation. Clearly, we do not have this problem if we assume that $x(t_0) = x'(t_0) = 0$ or if we assume that the last two terms in (32) are nonnegative. In the latter case we can get rid of these two terms before applying the operator I^{α}.

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