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WEIGHTED INEQUALITIES FOR FRACTIONAL INTEGRAL OPERATORS WITH KERNEL SATISFYING HÖRMANDER TYPE CONDITIONS

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Abstract. In this paper we study inequalities with weights for fractional operators T_α given by convolution with a kernel K_α which is supposed to satisfy some size condition and a fractional Hörmander type condition. As it is done for singular integrals, the conditions on the kernel have been generalized from the scale of Lebesgue spaces to that of Orlicz spaces. Our fractional operators include as particular cases the classical fractional integral I_α, fractional integrals associated to an homogeneous function and fractional integrals given by a Fourier multiplier.

1. Introduction

Suppose that T is a convolution integral operator with kernel K which satisfies some regularity condition and suppose that we know of some behavior on T with respect to Lebesgue measure. Sometimes, if one wants to know the behavior of T when we change the measure, i.e., when we consider the measure $w(x)dx$ where w is a weight, i.e., $0 \leqslant w \in L^1_{loc}(\mathbb{R}^n)$, we get an inequality of the type

$$\int |Tf|^p w \leqslant C \int [\mathscr{M}_T f]^p w, \tag{1.1}$$

for all $0 < p < \infty$ and $w \in A_\infty$, where $\mathscr{M}_T$ is a maximal operator related to the operator T which is normally easier to deal with. In general, $\mathscr{M}_T$ is strongly related with the kernel K and it will be bigger as much rough will be the kernel.

For T a Calderón-Zygmund singular integral operator (i.e., $K \in H^*_\infty$ using the notation in [11]) inequality (1.1) holds with $\mathscr{M}_T = M$, where M is the Hardy-Littlewood maximal operator (see [5]). If T is a singular integral operator with less regular kernel as in [10], then inequality (1.1) holds with $\mathscr{M}_T = M_r$, where $M_r f = [M(|f|^r)]^{1/r}$ for some $1 \leqslant r < \infty$ (see [20]). The value of the exponent r is determined by the smoothness of the kernel, namely, the kernel satisfies an $L^{r'}$-Hörmander condition (see the precise definition in section 3). In [12], the L^r-Hörmander condition is generalized to the scale of the Orlicz spaces. For a Young function A, the L^A-Hörmander condition is introduced in that paper (for $A(t) = t^r$ we get the L^r-Hörmander condition) and it is proved that if the kernel satisfies the L^A-Hörmander condition, then inequality (1.1)

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holds with $\mathscr{M}_T = M_{\overline{A}}$, where $\overline{A}$ is the complementary function of A and $M_{\overline{A}}$ is the Orlicz maximal operator associated to $\overline{A}$ (see the definition in section 2). The corresponding inequality (1.1) for commutators (with symbol $b \in BMO$) of Calderón-Zygmund singular integrals appears in [16]. For commutators of generalized singular integrals, associated to a kernel satisfying a Hörmander type condition given by a Young function $\mathscr{A}$, the corresponding inequality (1.1) appears in [11].

In 1974, Muckenhoupt and Wheeden [13] proved inequality (1.1) for T the classical Riesz potential I_α and $\mathscr{M}_T$ the fractional maximal function M_α, defined for $0 < \alpha < n$ and locally integrable function f by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy \quad \text{and} \quad M_\alpha f(x) = \sup_{x\in B} \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B |f(y)|\, dy.$$

For the commutators of fractional integral operators, inequalities of the type (1.1) can be found in [6] and [1].

There are fractional integrals with less regular kernel than the Riesz transform. In [9], Kurtz stated a fractional L^r-Hörmander type condition and he applied it to study the boundedness with weights of fractional integrals given by a multiplier (see Section 4 for definitions). Other generalization of fractional integrals are those whose kernel is associated to an homogeneous function Ω. Suppose that Ω is homogeneous of degree zero and $\Omega \in L^1(S^{n-1})$, where S^{n-1} denotes the unit sphere on $\mathbb{R}^n$. Define the fractional integral associated to Ω by

$$T_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(y/|y|)}{|y|^{n-\alpha}} f(x-y)\, dy.$$

With suitable integrability conditions on the function Ω, the boundedness of this operator was studied for several authors (see for example [4] and [7]). In a more general context and with an aditional condition on Ω, that is, Ω satisfying the $L^s(S^{n-1})$-Dini smoothness condition, Segovia and Torrea [21], studied the good weights for this operator and its commutators. The $L^s(S^{n-1})$-Dini smoothness condition on Ω provides a fractional L^s-Hörmander type condition on the kernel $K_\alpha(x) = \frac{\Omega(x/|x|)}{|x|^{n-\alpha}}$.

In this paper we study convolution operators $T_\alpha f = K_\alpha * f$ which include as particular cases the operators $T_{\Omega,\alpha}$ and the fractional integrals associated to a multiplier as in [9]. We shall obtain inequalities of the type (1.1) for these operators when its kernels K_α satisfy a size condition and a fractional L^A-Hörmander condition (we will denote it by $H_{\alpha,\mathscr{A}}$). We would like to point out that no boundedness of the operator T_α is used to derive (1.1). Moreover, if we know some boundedness of the operator T_α and K_α satisfies a size condition and a suitable fractional $L^{\mathscr{A}}$-Hörmander condition we shall prove (1.1) for the commutators of T_α.

These results will allow us to obtain, for general operators T_α and its commutators, two-weight inequalities of the type

$$\int |Tf|^p w \leqslant C \int |f|^p \widetilde{\mathscr{M}_T} w, \tag{1.2}$$

for $1 < p < \infty$ and with no assumptions on the weight w. The operators $\widetilde{\mathscr{M}_T}$ are again suitable maximal operators related with T and not necessarily the same for inequalities (1.1) and (1.2).

There is a great amount of works that deal with inequalities of the type (1.2). When T is a Calderón-Zygmund operator (with kernel $K \in H^*_\infty$), inequality (1.2) holds with $\mathscr{M}_T = M^{[p]+1}$, where $[p]$ is the integer part of p and, for $k \in \mathbb{N}$, M^k denotes the Hardy-Littlewood maximal function iterated k times (see [15]). The corresponding result for commutators (with symbol $b \in BMO$) of Calderón-Zygmund singular integrals was proved in [16]. For T a singular integral associated to a kernel K satisfying a general Hörmander's condition given by a Young function A, the corresponding results, that include as particular cases those of C. Pérez, have been proved in [11]. When T is the Riesz fractional integral I_α, inequality (1.2) holds with $\mathscr{M}_T = M_{\alpha p}(M^{[p]})$ (this result is also due to C. Pérez, see [18]). For commutators (with symbol $b \in BMO$) of I_α see [1] and [2].

The paper is organized as follows. Section 2 contains preliminaries and definitions that are needed to state the results. In Section 3 we state and prove the Coifman type inequality (1.1) for generalized fractional integrals and their commutators. Section 4 is devoted to give some applications. In Section 5 we state a strong type two-weight norm inequality from the Coifman type inequality and we apply it to the examples in Section 4.

2. Preliminaries

A function $\mathscr{A} : [0,\infty) \to [0,\infty)$ is said to be a Young function if it is continuous, convex, increasing and satisfies $\mathscr{A}(0) = 0$ and $\mathscr{A}(t) \to \infty$ as $t \to \infty$.

Given a Young function $\mathscr{A}$, define the $\mathscr{A}$-mean Luxemburg norm of a function f on a ball (or a cube) B by

$$||f||_{\mathscr{A},B} = \inf\left\{\lambda > 0 : \frac{1}{|B|}\int_B \mathscr{A}\left(\frac{|f|}{\lambda}\right) \leqslant 1\right\}. \tag{2.1}$$

It is well known that if $\mathscr{A}(t)$ and $\mathscr{B}(t)$ are Young functions and $\mathscr{A}(t) \leqslant C\mathscr{B}(t)$ for all $t \geqslant t_0$ then $||f||_{\mathscr{A},B} \leqslant C||f||_{\mathscr{B},B}$, for all balls B and functions f. Thus, the behavior of $\mathscr{A}(t)$ for $t \leqslant t_0$ is not important. If $\mathscr{A} \approx \mathscr{B}$, i.e., there are constants $t_0, c_1, c_2 > 0$ such that $c_1\mathscr{A}(t) \leqslant \mathscr{B}(t) \leqslant c_2\mathscr{A}(t)$ for $t \geqslant t_0$, then $||f||_{\mathscr{A},B} \approx ||f||_{\mathscr{B},B}$.

Each Young function $\mathscr{A}$ has an associated complementary Young function $\overline{\mathscr{A}}$ satisfying

$$t \leqslant \mathscr{A}^{-1}(t)\overline{\mathscr{A}}^{-1}(t) \leqslant 2t, \qquad t > 0.$$

There is a generalization of Hölder's inequality

$$\frac{1}{|B|}\int_B |fg| \leqslant ||f||_{\mathscr{A},B}||g||_{\overline{\mathscr{A}},B}, \tag{2.2}$$

and even another one that will be used later (see [14]): If $\mathscr{A}, \mathscr{B}$ and $\mathscr{C}$ are Young functions and

$$\mathscr{A}^{-1}(t)\mathscr{B}^{-1}(t) \leqslant \mathscr{C}^{-1}(t)$$

then

$$\|fg\|_{\mathscr{C},B} \leqslant 2\|f\|_{\mathscr{A},B}\|g\|_{\mathscr{B},B}. \tag{2.3}$$

When $\mathscr{A}(t) = t$ set $\overline{\mathscr{A}}(t) = 0$ if $0 \leqslant t \leqslant 1$ and $\overline{\mathscr{A}}(t) = \infty$ otherwise. Observe that $\overline{\mathscr{A}}$ is not a Young function, still $L^{\overline{\mathscr{A}}}$ can be identified with L^∞. Also, writing $\overline{\mathscr{A}}^{-1}(t) \equiv 1$, the previous Hölder inequalities make sense if one of the functions is $\mathscr{A}$ or $\overline{\mathscr{A}}$.

For a complete account on Young functions and Orlicz spaces see [19] and [14].

For each locally integrable function f and $0 \leqslant \alpha < n$, the Orlicz fractional maximal operator associated to the Young function $\mathscr{A}$ is defined as

$$M_{\alpha,\mathscr{A}} f(x) = \sup_{B \ni x} |B|^{\alpha/n} ||f||_{\mathscr{A},B}.$$

For $\alpha = 0$, write $M_{\mathscr{A}}$ instead of $M_{0,\mathscr{A}}$. When $\mathscr{A}(t) = t^r$, $r > 1$, then we write $M_{\alpha,\mathscr{A}} = M_{\alpha,r}$ and if $r = 1$ we simply write $M_{\alpha,\mathscr{A}} = M_\alpha$ which is the classical fractional maximal operator. For $\alpha = 0$ and $\mathscr{A}(t) = t$, then $M_{0,\mathscr{A}} = M$ is the Hardy-Littlewood maximal operator.

For $1 < p < \infty$, a Young function $\mathscr{A}$ is said to belong to B_p if there exists $c > 0$ such that $\int_c^\infty \frac{\mathscr{A}(t)}{t^p}\frac{dt}{t} < \infty$. This condition appears first in [17] and it was shown that $\mathscr{A} \in B_p$ if and only if $M_{\mathscr{A}}$ is bounded on $L^p(dx)$.

We shall work with weights in the Muckenhoupt classes A_p, $1 \leqslant p \leqslant \infty$, which are defined as follows. Let w be a non-negative locally integrable function and $1 \leqslant p < \infty$. We say that $w \in A_p$ if there exists $C_p < \infty$ such that for every ball $B \subset \mathbb{R}^n$

$$\left(\frac{1}{|B|}\int_B w(x)\,dx\right)\left(\frac{1}{|B|}\int_B w(x)^{1-p'}\,dx\right)^{p-1} \leqslant C_p.$$

when $1 < p < \infty$, and for $p = 1$,

$$\frac{1}{|B|}\int_B w(y)\,dy \leqslant C_1\, w(x), \qquad \text{for a.e. } x \in B,$$

which can be equivalently written as $Mw(x) \leqslant C_1\, w(x)$ for a.e. $x \in \mathbb{R}^n$. Finally we set $A_\infty = \cup_{p \geqslant 1} A_p$. It is well known that the Muckenhoupt classes characterize the boundedness of the Hardy-Littlewood maximal function on weighted Lebesgue spaces. Namely, $w \in A_p$, $1 < p < \infty$, if and only if M is bounded on $L^p(w)$ and $w \in A_1$ if and only if M is of weak type $(1,1)$ with respect to the measure $w(x)dx$.

3. The Coifman type inequalities

In this paper, we shall consider convolution operators $T_\alpha f = K_\alpha * f$, $0 < \alpha < n$, where the kernels K_α are supposed to satisfy conditions that ensure certain control on their size and their smoothness. From now on, we adopt the following convention: $|x| \sim s$ will stand for the set $\{s < |x| \leqslant 2s\}$ and $\|f\|_{\mathscr{A},|x|\sim s}$ will stand for $\|f\,\chi_{\{|x|\sim s\}}\|_{\mathscr{A},B(0,2s)}$.

DEFINITION 3.1. Let $\mathscr{A}$ be a Young function and let $0 \leqslant \alpha < n$. The kernel K_α is said to satisfy the $S_{\alpha,\mathscr{A}}$ condition, denote it by saying $K_\alpha \in S_{\alpha,\mathscr{A}}$, if there exists a constant $C > 0$ such that

$$\|K_\alpha\|_{\mathscr{A},|x|\sim s} \leqslant Cs^{\alpha-n}.$$

When $\alpha = 0$ we simply write $S_{0,\mathscr{A}} = S_{\mathscr{A}}$ and when $\mathscr{A}(t) = t$ we write $S_{\alpha,\mathscr{A}} = S_\alpha$. Observe that if $K_\alpha \in S_\alpha$, that is, there exists a constant $C > 0$ such that

$$\int_{|x|\sim s} |K_\alpha(x)|dx \leqslant Cs^\alpha,$$

then the operator T_α is well defined for example for L_c^∞ functions, i.e., bounded functions with compact support.

In [9] a sort of fractional Hörmander condition appears in the scale of L^r spaces. As mentioned in the introduction, the L^r-Hörmander conditions for singular integrals and its commutators have been generalized to the scale of Orlicz spaces (see [12] and [11]). In this way, the same can be done for fractional operators and its commutators.

DEFINITION 3.2. Let $\mathscr{A}$ be a Young function and let k be an integer number, $k \geqslant 0$. We say that the kernel K_α satisfies the $L^{\alpha,\mathscr{A},k}$-Hörmander condition, we write $K \in H_{\alpha,\mathscr{A},k}$, if there exist $c \geqslant 1$ and $C > 0$ (depending on $\mathscr{A}$ and k) such that for all $y \in \mathbb{R}^n$ and $R > c\,|y|$

$$\sum_{m=1}^{\infty} (2^m R)^{n-\alpha} m^k \|K_\alpha(\cdot - y) - K_\alpha(\cdot)\|_{\mathscr{A},|x|\sim 2^m R} \leqslant C.$$

The exponent k in the condition $H_{\alpha,\mathscr{A},k}$ is related with the order of the commutator. If $\mathscr{A}$ gives rise to L^r, $1 \leqslant r \leqslant \infty$, we recover the fractional L^r-Hörmander condition and simply write $H_{\alpha,r,k}$ instead of $H_{\alpha,\mathscr{A},k}$. For fractional operators we shall use the condition $H_{\alpha,\mathscr{A},0}$ that we shall simply denote by $H_{\alpha,\mathscr{A}}$.

The following condition is related to the classical Lipschitz condition (H_∞^* in [11]).

DEFINITION 3.3. The kernel K_α is said to satisfy the $H_{\alpha,\infty}^*$ condition if there exist $c \geqslant 1$ and $C > 0$ such that

$$|K_\alpha(x-y) - K_\alpha(x)| \leqslant C\frac{|y|}{|x|^{n+1-\alpha}}, \qquad |x| > c|y|.$$

It is easy to see that for any $k \geqslant 0$, $H_{\alpha,\infty}^* \subset H_{\alpha,\infty,k} \subset H_{\alpha,\mathscr{A},k}$ for every Young function $\mathscr{A}$.

Next we shall state and prove one of the main results in this paper, a Coifman type estimate for the operator T_α.

THEOREM 3.4. *Let $\mathscr{A}$ be a Young function and $0 \leqslant \alpha < n$. Let $T_\alpha = K_\alpha * f$ with $K_\alpha \in S_\alpha \cap H_{\alpha,\mathscr{A}}$. Then for any $0 < p < \infty$ and any $w \in A_\infty$,*

$$\int_{\mathbb{R}^n} |T_\alpha f(x)|^p w(x)\, dx \leqslant C \int_{\mathbb{R}^n} [M_{\alpha,\overline{\mathscr{A}}} f(x)]^p w(x)\, dx, \qquad f \in L_c^\infty, \tag{3.1}$$

whenever the left-hand side is finite.

Observe that in Theorem 3.4 we do not assume any boundedness on the operator T_α.

Proof. For simplicity we may assume that $c = 1$ in the condition $H_{\alpha,\mathscr{A}}$. First we shall prove that if $w \in A_\infty$, $0 < p < \infty$, $0 < \delta < \min\{1, p\}$ and $f \in L_c^\infty$, then

$$M_\delta^\sharp (T_\alpha f)(x) \leqslant C M_{\alpha,\overline{\mathscr{A}}} f(x), \tag{3.2}$$

where $M_\delta^\sharp f = (M^\sharp |f|^\delta)^{1/\delta}$ with

$$M^\sharp f(x) = \sup_{x \in B} \inf_{a \in \mathbb{R}} \frac{1}{|B|} \int_B |f(y) - a|\, dy.$$

Fix $x \in \mathbb{R}^n$, and a ball $B = B(x_B, R)$ containing x. For $\widetilde{B} = B(x_B, 2R)$, set $f_1 = f \chi_{\widetilde{B}}$ and $f_2 = f - f_1$. Choose $a = |T_\alpha f_2(x_B)|^\delta$. Then, by Jensen's inequality and an easy inequality $|t^\delta - s^\delta|^{1/\delta} \leqslant |t - s|$,

$$\begin{aligned}
\left(\frac{1}{|B|} \int_B ||T_\alpha f|^\delta (y) - a| dy \right)^{1/\delta} &\leqslant \frac{1}{|B|} \int_B |T_\alpha f(y) - T_\alpha f_2(x_B)| dy \\
&\leqslant \frac{1}{|B|} \int_B |T_\alpha f_1(y)| dy + \frac{1}{|B|} \int_B |T_\alpha f_2(y) - T_\alpha f_2(x_B)| dy \\
&\leqslant I + II.
\end{aligned}$$

Since $K_\alpha \in S_\alpha$ we have

$$\begin{aligned}
I &\leqslant \frac{1}{|B|} \int_B \left(\int_{\widetilde{B}} |K_\alpha(y - z) f(z)| dz \right) dy \\
&= \frac{1}{|B|} \int_{\widetilde{B}} |f(z)| \left(\int_B |K_\alpha(y - z)| dy \right) dz \\
&\leqslant \frac{1}{|B|} \int_{\widetilde{B}} |f(z)| \left(\int_{|y-z| \leqslant 3R} |K_\alpha(y - z)| dy \right) dz \\
&\leqslant C \frac{(3R)^\alpha}{|B|} \int_{\widetilde{B}} |f(z)| dz \leqslant C M_\alpha f(x).
\end{aligned}$$

Notice that $M_\alpha f(x) \leqslant CM_{\alpha,\overline{\mathscr{A}}}f(x)$. To estimate II observe that using (2.2) with $\mathscr{A}$ and $\overline{\mathscr{A}}$ and the fact that $K_\alpha \in H_{\alpha,\mathscr{A}}$, we get

$$\begin{aligned}
&|T_\alpha f_2(y) - T_\alpha f_2(x_B)| \\
&= \left| \int_{\mathbb{R}^n \setminus \widetilde{B}} K_\alpha(y-z) f(z)dz - \int_{\mathbb{R}^n \setminus \widetilde{B}} K_\alpha(x_B - z) f(z)dz \right| \\
&\leqslant \int_{|z-x_B|>2R} |f(z)||K_\alpha(y-z) - K_\alpha(x_B-z)|dz \\
&\leqslant \sum_{m=1}^{\infty} \int_{2^m R < |z-x_B| \leqslant 2^{m+1}R} |f(z)||K_\alpha(y-z) - K_\alpha(x_B-z)|dz \\
&\leqslant C \sum_{m=1}^{\infty} (2^m R)^n ||f||_{\overline{\mathscr{A}},|z-x_B|\sim 2^m R} ||K_\alpha(y-\cdot) - K_\alpha(x_B-\cdot)||_{\mathscr{A},|z-x_B|\sim 2^m R} \\
&\leqslant CM_{\alpha,\overline{\mathscr{A}}}f(x),
\end{aligned}$$

so its integral average over B with respect to the y-variable gives the same bound for II.

The reasoning to complete the proof follows the same arguments given in [11]. Here we include them for the sake of completeness. By the extrapolation results obtained in [6], the inequality (3.1) will hold for all $0 < p < \infty$ and all $w \in A_\infty$ if, and only if, it holds for some fixed exponent $0 < p_0 < \infty$ and all $w \in A_\infty$. Therefore, fix $p_0 \in (1,\infty)$, $w \in A_\infty$ and $f \in L_c^\infty$, and assume without loss of generality that $\|M_{\alpha,\overline{\mathscr{A}}}f\|_{L^{p_0}(w)}$ and $\|T_\alpha f\|_{L^{p_0}(w)}$ are both finite. Since $w \in A_\infty$, then there exists $r > 1$ (that can be taken greater than p_0) such that $w \in A_r$. Observe that for all $0 < \delta < p_0/r < 1$, we have that $1 < r < p_0/\delta$ and thus, $w \in A_{p_0/\delta}$. Then

$$\|M_\delta(T_\alpha f)\|_{L^{p_0}(w)} = \left\|M\big(|T_\alpha f|^\delta\big)\right\|_{L^{\frac{p_0}{\delta}}(w)}^{\frac{1}{\delta}} \leqslant C\|T_\alpha f\|_{L^{p_0}(w)} < \infty.$$

The right hand side is finite by assumption. Then, applying the Fefferman-Stein inequality (see [8]) and (3.2),

$$\int_{\mathbb{R}^n} |T_\alpha f|^{p_0} w \leqslant C \int_{\mathbb{R}^n} [M_\delta(T_\alpha f)]^{p_0} w \leqslant C \int_{\mathbb{R}^n} [M_\delta^\sharp(T_\alpha f)]^{p_0} w \leqslant C \int_{\mathbb{R}^n} [M_{\alpha,\overline{\mathscr{A}}} f]^{p_0} w. \qquad \square$$

REMARK 3.5. If $K_\alpha \in S_\alpha \cap H_{\alpha,\infty}$ we get (3.1) for any $0 < p < \infty$ and for all $w \in A_\infty$ with $\overline{\mathscr{A}}(t) = t$, which means M_α on the right hand side.

3.1. Commutators

Now we are going to state and prove a Coifman type inequality for the commutators of the operator T_α.

Recall that a locally integrable functions b is said to belong to BMO if

$$\|b\|_{\mathrm{BMO}} = \sup_B \frac{1}{|B|} \int_B |b(x) - b_B|\, dx < \infty,$$

where the sup runs over all balls (or cubes) $B \subset \mathbb{R}^n$ and b_B denotes the integral average of b over B.

Given $b \in \mathrm{BMO}$, define the k-th order commutator of T_α, $k \geqslant 0$, by

$$T^k_{\alpha,b} f(x) = \int_{\mathbb{R}^n} [b(x) - b(y)]^k K_\alpha(x-y) f(y)\, dy.$$

Note that for $k = 0$, $T^k_{\alpha,b} = T_\alpha$. Also observe that $T^k_{\alpha,b} f = b T^{k-1}_{\alpha,b} f - T^{k-1}_{\alpha,b}(bf)$, $k \geqslant 1$.

THEOREM 3.6. *Let T_α be a convolution operator with kernel K_α and suppose that T_α is bounded from $L^{q_0}(dx)$ to $L^{p_0}(dx)$, for some $1 < p_0, q_0 < \infty$. Let $b \in \mathrm{BMO}$ and $k \in \mathbb{N}$. Let $\mathscr{A}$ and $\mathscr{B}$ be Young functions such that $\overline{\mathscr{A}}^{-1}(t)\mathscr{B}^{-1}(t) \leqslant \mathscr{C}_k^{-1}(t)$ with $\mathscr{C}_k(t) = t(1+\log^+ t)^k$. If $K_\alpha \in S_\alpha \cap H_{\alpha,\mathscr{B},k}$ then, for any $0 < p < \infty$ and any $w \in A_\infty$,*

$$\int_{\mathbb{R}^n} |T^k_{\alpha,b} f(x)|^p w(x)\, dx \leqslant C \|b\|^{pk}_{\mathrm{BMO}} \int_{\mathbb{R}^n} [M_{\alpha,\overline{\mathscr{A}}} f(x)]^p w(x)\, dx, \qquad f \in L^\infty_c, \tag{3.3}$$

whenever the left-hand side is finite.

Proof. First we shall prove the following inequality

$$M^\sharp_\delta\left(T^k_{\alpha,b} f\right)(x) \leqslant C \sum_{j=0}^{k-1} \|b\|^{k-j}_{\mathrm{BMO}} M_\varepsilon(T^j_{\alpha,b} f)(x) + C\|b\|^k_{\mathrm{BMO}} M_{\alpha,\overline{\mathscr{A}}} f(x), \tag{3.4}$$

for $0 < \delta < \varepsilon < 1$ and $k \geqslant 1$. As in the proof of Lemma 5.1 in [11] we write inequality (5.4) in [11] for $T^m_{\alpha,b}$ instead of T^m_b, $0 \leqslant m \leqslant k$. To obtain the estimates of the terms I and III in (5.4) for our operators we proceed as in [11], by using the conditions $H_{\alpha,\mathscr{B}}$ and $H_{\alpha,\mathscr{A},k}$ instead of the corresponding ones with $\alpha = 0$. Notice that by the relation among the funtions $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}_k$ we get that $H_{\alpha,\mathscr{B},k} \subset H_{\alpha,\mathscr{A},k}$. In order to estimate the term II for the operator T_α, we proceed as in the proof of Theorem 3.4. In fact, using that $K_\alpha \in S_\alpha$, inequality (2.2) with $\overline{\mathscr{C}}_k$ and $\mathscr{C}_k$ and inequality (2.3) for $\overline{\mathscr{A}}$, $\mathscr{B}$, $\mathscr{C}_k$,

$$\begin{aligned}
II &= \left(\frac{1}{|B|}\int_B |T_\alpha((b_{\widetilde{B}} - b)^k f_1)(y)|^\delta\, dy\right)^{\frac{1}{\delta}} \\
&\leqslant \frac{1}{|B|}\int_B \left(\int_{\widetilde{B}} |K_\alpha(y-z)(b(z)-b_{\widetilde{B}})^k f(z)| dz\right) dy \\
&\leqslant \frac{1}{|B|}\int_{\widetilde{B}} |b(z)-b_{\widetilde{B}}|^k |f(z)| \left(\int_{|y-z|\leqslant 3R} |K_\alpha(y-z)| dy\right) dz \\
&\leqslant C\frac{(3R)^\alpha}{|B|}\int_{\widetilde{B}} |b(z)-b_{\widetilde{B}}|^k |f(z)| dz \\
&\leqslant C|B|^{\frac{\alpha}{n}} \|(b-b_{\widetilde{B}})^k\|_{\overline{\mathscr{C}}_k,\widetilde{B}} \|f\|_{\overline{\mathscr{A}},\widetilde{B}} \leqslant C||b||^k_{BMO} M_{\alpha,\overline{\mathscr{A}}} f(x).
\end{aligned}$$

Observe that we do not use any boundedness of the operator T_α to obtain inequality (3.4).

To finish the proof observe that, by extrapolation, it suffices to obtain the theorem for some fixed exponent $0 < p_0 < \infty$ and all $w \in A_\infty$. Therefore, choose p_0 such that the operator T_α is bounded from $L^{q_0}(dx)$ to $L^{p_0}(dx)$. As in [11] we first consider w and $b \in L^\infty$ and we proceed by induction. Notice that for $f \in L^\infty_c$ and $j \in \mathbb{N}$,

$$||T^j_{\alpha,b}f||_{L^{p_0}} = \left\| \sum_{m=0}^{j} C_{m,j} b^{j-m} T_\alpha(b^m f) \right\|_{L^{p_0}} \leqslant C||b||^j_{L^\infty}||f||_{L^{q_0}} < \infty.$$

Then, since $w \in L^\infty$, we have that $||T^j_{\alpha,b}f||_{L^{p_0}(w)} < \infty$ for all $0 \leqslant j \leqslant k-1$ and $f \in L^\infty_c$. The rest of the proof follows the same steps as in Theorem 3.3, part (a) in [11]. □

REMARK 3.7. As in [11], changing in Theorem 3.6 the hypothesis on the kernel by the condition $K_\alpha \in S_\alpha \cap H_{\alpha,\infty,k}$, then (3.3) still holds with $\overline{\mathscr{A}}(t) = \mathscr{C}_k(t) = t(1 + \log^+ t)^k$.

4. Applications

We start this section with a proposition which shows the relation between the conditions $S_{\alpha,\mathscr{A}}$ and $H_{\alpha,\mathscr{A},k}$ with the corresponding ones for $\alpha = 0$.

PROPOSITION 4.1. *If $K_\alpha(x) = |x|^\alpha K(x)$ with $K \in H_{\mathscr{A},k} \cap S_{\mathscr{A}}$ then $K_\alpha \in H_{\alpha,\mathscr{A},k} \cap S_{\alpha,\mathscr{A}}$.*

Proof. It is clear that $K_\alpha \in S_{\alpha,\mathscr{A}}$ is equivalent to $K \in S_{\mathscr{A}}$. To prove that $K_\alpha \in H_{\alpha,\mathscr{A},k}$ let $|x| \sim s$ and $|y| < s/2$ then $s/2 < |x-y| < 5s/2$ and therefore, by the mean value theorem,

$$\begin{aligned} |K_\alpha(x-y) - K_\alpha(x)| &\leqslant |x-y|^\alpha |K(x-y) - K(x)| + |K(x)|||x-y|^\alpha - |x|^\alpha| \\ &\leqslant Cs^\alpha \left[|K(x-y) - K(x)| + \frac{|y|}{s}|K(x)| \right]. \end{aligned}$$

Let $R > 0$ and $s = 2^m R$. Then, for $|y| < R$ and $|x| \sim 2^m R$, we have

$$|K_\alpha(x-y) - K_\alpha(x)| \leqslant C(2^m R)^\alpha \left[|K(x-y) - K(x)| + 2^{-m}|K(x)| \right].$$

Therefore, since $K \in H_{\mathscr{A},k} \cap S_{\mathscr{A}}$,

$$\begin{aligned} &\sum_{m=1}^{\infty} (2^m R)^{n-\alpha} m^k \|K_\alpha(\cdot - y) - K_\alpha(\cdot)\|_{\mathscr{A},|x|\sim 2^m R} \\ &\qquad \leqslant C \sum_{m=1}^{\infty} (2^m R)^n m^k \|K(\cdot - y) - K(\cdot)\|_{\mathscr{A},|x|\sim 2^m R} \\ &\qquad\quad + C \sum_{m=1}^{\infty} 2^{-m} m^k (2^m R)^n \|K\|_{\mathscr{A},|x|\sim 2^m R} \leqslant C. \end{aligned}$$ □

The fractional integral operator. Note that the kernel of the fractional integral I_α, $K_\alpha(x)=\frac{1}{|x|^{n-\alpha}}$, belongs to $S_\alpha\cap H^*_{\alpha,\infty}\subset S_\alpha\cap H_{\alpha,\infty,k}$, for all $k\geqslant 0$. Consequently, by Remarks 3.5 and 3.7 we get that (3.3) holds for all $k\geqslant 0$ with $\overline{\mathscr{A}}(t)=t(1+\log^+t)^k$.

The fractional integral operator with rough kernel. Denote by S^{n-1} the unit sphere of $\mathbb{R}^n$. For $x\neq 0$, we write $x'=x/|x|$. Consider a function Ω defined on S^{n-1}. This function can be extended to $\mathbb{R}^n\setminus\{0\}$ as $\Omega(x)=\Omega(x')$ (notice the abuse in also calling the extension Ω). Thus Ω is a homogeneous function of degree 0. Given a Young function $\mathscr{B}$ we define the $L^{\mathscr{B}}$-modulus of continuity of Ω as

$$\varpi_{\mathscr{B}}(t)=\sup_{|y|\leqslant t}\|\Omega(\cdot+y)-\Omega(\cdot)\|_{\mathscr{B},S^{n-1}}.$$

Set $K_\alpha(x)=\Omega(x)/|x|^{n-\alpha}$ and let $T_{\Omega,\alpha}$ be the corresponding operator with kernel K_α. We can then prove the following proposition for K_α.

PROPOSITION 4.2. *Let $\Omega\in L^{\mathscr{B}}(S^{n-1})$ and $k\geqslant 0$. If*

$$\int_0^1\left(1+\log\frac{1}{t}\right)^k\varpi_{\mathscr{B}}(t)\,\frac{dt}{t}<\infty, \tag{4.1}$$

then $K_\alpha\in S_\alpha\cap H_{\alpha,\mathscr{B},k}$.

Proof. First, notice that $\Omega\in L^{\mathscr{B}}(S^{n-1})$ implies that $K(x)=\frac{\Omega(x)}{|x|^n}\in S_{\mathscr{B}}$. In fact, since

$$\begin{aligned}\frac{1}{|B(0,s)|}\int_{|x|\sim s}\mathscr{B}\left(\frac{|K(x)|}{\lambda}\right)dx&=\frac{C}{s^n}\int_{S^{n-1}}\int_s^{2s}\mathscr{B}\left(\frac{|\Omega(x')|}{\lambda\rho^n}\right)\rho^{n-1}d\rho\, d\sigma(x')\\&\leqslant C\int_{S^{n-1}}\mathscr{B}\left(\frac{|\Omega(x')|}{\lambda s^n}\right)d\sigma(x'),\end{aligned}$$

then $||K||_{\mathscr{B},|x|\sim s}\leqslant Cs^{-n}||\Omega||_{\mathscr{B},S^{n-1}}$.

On the other hand, it was proved in [11] that if $\Omega\in L^{\mathscr{B}}(S^{n-1})$ and if Ω satisfies (4.1), then $K\in H_{\mathscr{B},k}$. Now, from Proposition 4.1, $K_\alpha\in H_{\alpha,\mathscr{B},k}\cap S_{\alpha,\mathscr{B}}\subset H_{\alpha,\mathscr{B},k}\cap S_\alpha$. □

An immediate consequence of this proposition is that if $\Omega\in L^{\mathscr{A}}(S^{n-1})$ and (4.1) holds with $k=0$ and $\varpi_{\mathscr{A}}$ in place of $\varpi_{\mathscr{B}}$ then $T_{\Omega,\alpha}$ verifies (3.1). In the particular case that $\mathscr{A}(t)=t^r$, inequality (3.1) holds with $M_{\alpha,r'}$ in the right hand side.

On the other hand, if $k>0$, let Ω be such that $T_{\Omega,\alpha}$ is bounded from $L^p(dx)$ to $L^q(dx)$ for some $1<p,q<\infty$ (for example, any Ω in $L^{\frac{n}{n-\alpha}}(S^{n-1})$ works fine). Let $\mathscr{A}$, $\mathscr{B}$ be Young functions such that $\overline{\mathscr{A}}^{-1}(t)\mathscr{B}^{-1}(t)\leqslant\mathscr{C}_k^{-1}(t)$ with $\mathscr{C}_k(t)=t(1+\log^+t)^k$. Then, if Ω verifies the hypothesis of the above proposition, the commutators of $T_{\Omega,\alpha}$ satisfy (3.3). In the particular case that $\mathscr{B}(t)=t^r$ inequality (3.3) holds with $M_{\alpha,L^{r'}(\log L)^{kr'}}$; if $\mathscr{B}(t)=t^r(1+\log^+t)^{kr}$, (3.3) holds with $M_{\alpha,r'}$; if $\mathscr{B}(t)=$

$t^r(1+\log^+ t)^k$, (3.3) holds with $M_{\alpha,L^{r'}(\log L)^k}$ (see table 2 in [11]).

Multipliers. Given a function m defined in $\mathbb{R}^n$ consider the multiplier operator T_m defined *a priori* for functions f in the Schwartz class by $\widehat{T_m f}(\xi) = m(\xi)\widehat{f}(\xi)$. Let $\beta = (\beta_1,\cdots,\beta_n)$ denote a multi-index of non-negative integers and $|\beta| = \beta_1+\cdots+\beta_n$. As in [9], given $1 \leqslant s < \infty$, $l \in \mathbb{N}$ and $0 < \alpha < n$, we say that $m \in M(s,l,\alpha)$ if there exists a constant B such that $|m(x)| \leqslant B|x|^{-\alpha}$ and

$$\sup_{R>0} R^{|\beta|+\alpha}\|D^\beta m\|_{L^s,|\xi|\sim R} < +\infty, \qquad \text{for all } |\beta| \leqslant l.$$

As a consequence of Theorem 3.6 we obtain the following result.

COROLLARY 4.3. *Assume that $m \in M(s,l,\alpha)$, with $1 < s \leqslant 2$ and $\frac{n}{s} < l \leqslant n$. Then, for all $k \geqslant 0$ and any $\varepsilon > 0$ we have that for all $0 < p < \infty$ and $w \in A_\infty$,*

$$\int_{\mathbb{R}^n} |T^k_{\alpha,b}f(x)|^p w(x)dx \leqslant C\int_{\mathbb{R}^n} [M_{\alpha,\frac{n}{l}+\varepsilon}f(x)]^p w(x)dx, \qquad f \in L^\infty_c, \tag{4.2}$$

whenever the left-hand side is finite.

Proof. Decompose the operator T_m as in [10]. To do that, let $\phi \in C^\infty$ be a nonnegative function supported in $\{\xi : 1/2 < |\xi| < 2\}$ so that $\sum_{j\in\mathbb{Z}}\phi_j(\xi) = \sum_{j\in\mathbb{Z}}\phi(2^{-j}\xi) = 1$, $\xi \neq 0$. Write $m_j(\xi) = \phi_j(\xi)m(\xi)$ and so $m(\xi) = \sum_{j\in\mathbb{Z}} m_j(\xi)$ for $\xi \neq 0$. Set $K_{\alpha,j} = (m_j)^\vee$ and

$$m^N(\xi) = \sum_{|j|\leqslant N} m_j(\xi), \qquad K^N_\alpha(x) = (m^N)^\vee(x) = \sum_{|j|\leqslant N} K_{\alpha,j}(x).$$

Proceeding as in the final part of the proof of Lemma 1 of [10], only that working with K^N_α instead of K_N, $d = l$, $t = s$ and $p = 1$, we obtain that since $l > \frac{n}{s}$, then

$$\int_{|x|\sim R} |K^N_\alpha(x)|\,dx \leqslant CR^\alpha,$$

where C does not depend on N. This implies that $K^N_\alpha \in S_\alpha$.

By the same Lemma 1 of [10] and the same replacements as above, we get that if $m \in M(s_0,l_0,\alpha)$ and $\frac{n}{s_0} < l_0 < \frac{n}{s_0}+1$ then

$$\|K^N_\alpha(\cdot-y) - K^N_\alpha(\cdot)\|_{L^{s'_0},|x|\sim R} \leqslant CR^{-n+\alpha}\left(\frac{|y|}{R}\right)^{l_0-\frac{n}{s_0}}, \qquad |y| < \frac{R}{2}, \tag{4.3}$$

where C does not depend on N. This implies that $K^N_\alpha \in H_{\alpha,s'_0,k}$ for all $k \geqslant 0$ and this happens uniformly on N: for all $R > 0$ and $|y| < R$,

$$\begin{aligned}\sum_{j=1}^\infty (2^jR)^{n-\alpha} j^k\|K^N_\alpha(\cdot-y) - K^N_\alpha(\cdot)\|_{L^{s'_0},|x|\sim 2^jR} &\leqslant C\sum_{j=1}^\infty j^k\left(\frac{|y|}{2^jR}\right)^{l_0-\frac{n}{s_0}}\\ &\leqslant C\sum_{j=1}^\infty j^k 2^{-j(l_0-\frac{n}{s_0})} \leqslant C,\end{aligned}$$

where C does not depend on N. Observe that by the same arguments than in Proposition 6.2 of [11], $K_\alpha^N \in H_{\alpha,\mathscr{B},k}$ with $\mathscr{B}(t) = t^r(1+\log^+ t)^{kr}$ uniformly in N, for all $1 < r < (n/l)'$.

To finish the proof of the corollary, take $N > 1$ and consider the operator T_m^N whose kernel is K_α^N. Since $K_\alpha^N \in S_\alpha \cap H_{\alpha,r}$ for all $1 < r < (\frac{n}{l})'$, then T_m^N verify (3.1) with $M_{\alpha,\overline{\mathscr{A}}} = M_{\alpha,\frac{n}{l}+\varepsilon}$ for all $\varepsilon > 0$ with a constant independent of N.

Set now $\mathscr{A}(t) = t^r$ $(1 < r < (\frac{n}{l})')$ and $\mathscr{B}(t) = t^r(1+\log^+ t)^{kr}$ as before. Then $\overline{\mathscr{A}}^{-1}(t)\,\mathscr{B}^{-1}(t) \leqslant C\mathscr{C}_k^{-1}(t)$. Since $K_\alpha^N \in H_{\alpha,\mathscr{B},k}$ and T_m^N map $L^p(dx)$ to $L^q(dx)$, $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$ (see [9]), then Theorem 3.6 applies and therefore (3.3) holds with $M_{\alpha,\overline{\mathscr{A}}} = M_{\alpha,r'}$ with a constant independent of N. A standard approximation argument as in [10] leads to the desired estimate for T_m and $T_{m,b}^k$. $\square$

5. Two weights inequalities

For operators such that their adjoints satisfy a Coifman type inequality it is possible to obtain two-weight norm inequalities, using a duality argument (see for example [15] and [11]).

THEOREM 5.1. *Let $\mathscr{A}$ be a Young function, $0 \leqslant \alpha < n$ and $1 < p < n/\alpha$. Suppose that there exist Young functions $\mathscr{E}$ and $\mathscr{D}$ such that $\mathscr{E} \in B_{p'}$, $\mathscr{E}^{-1}(t)\,\mathscr{F}^{-1}(t) \leqslant \overline{\mathscr{A}}^{-1}(t)$ with $\mathscr{F}(t) = \mathscr{D}(t^p)$ and the function $\Phi(t) = t\mathscr{D}'(t) - \mathscr{D}(t)$ for $t > 1$ is also a Young function. If T is a linear operator such that its adjoint T^* satisfies that for all $w \in A_\infty$,*

$$\int_{\mathbb{R}^n} |T^*f(x)|^{p'}\, w(x)\, dx \leqslant C \int_{\mathbb{R}^n} [M_{\alpha,\overline{\mathscr{A}}} f(x)]^{p'}\, w(x)\, dx, \qquad f \in L_c^\infty \tag{5.1}$$

then, for any weight u,

$$\begin{aligned}\int_{\mathbb{R}^n} |Tf(x)|^p\, u(x)\, dx &\leqslant C \int_{\mathbb{R}^n} |f(x)|^p\, M_{\alpha p,\mathscr{D}} u(x)\, dx, \\ &= C \int_{\mathbb{R}^n} |f(x)|^p\, M_\alpha(M_\Phi u)(x)\, dx, \qquad f \in L_c^\infty.\end{aligned} \tag{5.2}$$

REMARK 5.2. For the applications below, and since all the operators considered here are of convolution type, proving (5.1) for T^* or T turns out be equivalent.

Proof of Theorem 5.1. Fix a weight u. The key point here is to prove that $(M_{\alpha p,\mathscr{D}} u)^\delta \in A_1$, for all $0 < \delta < 1$. Then, the rest of the proof will follow standard arguments. Therefore let us start observing that the conditions on the Young functions $\mathscr{D}$ and Φ, and Theorem 1.1 in [3] give that $M_{\alpha p,\mathscr{D}} u \approx M_\alpha(M_\Phi u)$. On the other hand, we can restrict ourselves to the set $\{x : M_{\alpha p,\mathscr{D}} u(x) < \infty\}$ or suppose that the functions f have support contained in this set. By duality, (5.2) turns out to be equivalent to

$$\int_{\mathbb{R}^n} |T^*f(x)|^{p'}\, [M_{\alpha p,\mathscr{D}} u(x)]^{1-p'}\, dx \leqslant C \int_{\mathbb{R}^n} |f(x)|^{p'}\, [u(x)]^{1-p'}\, dx, \qquad f \in L_c^\infty.$$

Again observe that $M_{\alpha p,\mathscr{D}}u(x)=\infty$ would imply $[M_{\alpha p,\mathscr{D}}u(x)]^{1-p'}=0$. Therefore, by Corollary 1.2 and Remark 1.3 in [3] we know that $(M_{\alpha p,\mathscr{D}}u)^{\delta}$ belongs to A_1 for all $0<\delta<1$. Thus, choosing $r>p'$ and $\delta=(p'-1)/(r-1)$, $(M_{\alpha p,\mathscr{D}}u)^{1-p'}=[(M_{\alpha p,\mathscr{D}}u)^{\delta}]^{1-r}\in A_r\subset A_\infty$, and so (5.1) can be applied. This and the generalized Hölder's inequality for $\overline{\mathscr{A}}$, $\mathscr{E}$ and $\mathscr{F}$, yields

$$\begin{aligned}\int_{\mathbb{R}^n}|T^*f(x)|^{p'}[M_{\alpha p,\mathscr{D}}u(x)]^{1-p'}\,dx &\leqslant C\int_{\mathbb{R}^n}[M_{\alpha,\overline{\mathscr{A}}}f(x)]^{p'}[M_{\alpha p,\mathscr{D}}u(x)]^{1-p'}\,dx\\ &\leqslant C\int_{\mathbb{R}^n}[M_{\mathscr{E}}(fu^{-\frac{1}{p}})(x)]^{p'}[M_{\alpha,\mathscr{F}}(u^{\frac{1}{p}})(x)]^{p'}[M_{\alpha p,\mathscr{D}}u(x)]^{1-p'}\,dx\\ &=C\int_{\mathbb{R}^n}[M_{\mathscr{E}}(fu^{-\frac{1}{p}})(x)]^{p'}[M_{\alpha p,\mathscr{D}}u(x)]^{\frac{p'}{p}}[M_{\alpha p,\mathscr{D}}u(x)]^{1-p'}\,dx\\ &=C\int_{\mathbb{R}^n}[M_{\mathscr{E}}(fu^{-\frac{1}{p}})(x)]^{p'}\,dx\leqslant C\int_{\mathbb{R}^n}|f(x)\,u(x)^{-\frac{1}{p}}|^{p'}\,dx\\ &=C\int_{\mathbb{R}^n}|f(x)|^{p'}\,[u(x)]^{1-p'}\,dx,\end{aligned}$$

where we have used that $\mathscr{E}\in B_{p'}$ and so $M_{\mathscr{E}}$ is bounded on $L^{p'}(dx)$ (see [17]). □

Next we want to apply Theorem 5.1 to the examples considered in section 4. In order to do that we shall assume hypothesis in the kernels K_α that ensure the boundedness of the operators T_α from $L^p(dx)$ to $L^q(dx)$, for all $1<p,q<\infty$ with $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$ plus the hypothesis in Theorem 3.3 (Theorem 3.1 in the case $k=0$). Notice that if T_α is bounded as before then, for all $f\in L^\infty_c$, $w\in A_\infty\cap L^\infty$ and $b\in L^\infty$,

$$||T^k_{\alpha,b}f||_{L^q(w)}\leqslant ||w||^{1/q}_{L^\infty}\left\|\sum_{m=0}^{k}C_{m,k}b^{k-m}T_\alpha(b^m f)\right\|_{L^q}\leqslant C||w||^{1/q}_{L^\infty}||b||^k_{L^\infty}||f||_{L^p}<\infty,$$

for any $k\geqslant 0$. Therefore, under the hypothesis in Theorem 3.3 (respectively Theorem 3.1) we obtain (5.1) for the operators $T^k_{\alpha,b}$ for all $w\in A_\infty\cap L^\infty$ and $b\in L^\infty$. To remove the restrictions on the functions w and b we proceed as in the proof of Theorem 3.3, part (a) in [11].

The classical fractional integral. In the case that $\overline{\mathscr{A}}(t)=t$ we have that the hypotheses on the Young functions in Theorem 5.1 hold, if $p\geqslant 2$, for the functions $\mathscr{E}(t)=t^{p'}(1+\log^+t)^{-1-\varepsilon\frac{p'}{p}}$ and $\mathscr{D}(t)=t(1+\log^+t)^{p-1+\varepsilon}$, because $p-1+\varepsilon>1$, and so Remark 1.4 of [3] can be applied. If $1<p<2$, then the hypotheses on the Young functions in Theorem 5.1 hold for $\mathscr{E}(t)=t^{p'}(1+\log^+t)^{-p'/p}$ and $\mathscr{D}(t)=t(1+\log^+t)$, because again Remark 1.4 of [3] can be applied (with $p=1$ and $\beta=1$).

Then, we obtain inequality (5.2) for I_α with $\mathscr{D}(t)=t(1+\log^+t)^{[p]}$. This was obtained by Pérez in [18]. In this case, $M_{\alpha p,\mathscr{D}}$ is equivalent to $M_{\alpha p}(M^{[p]})$. For the k-th order commutator of I_α, we begin with $\overline{\mathscr{A}}(t)=t(1+\log^+t)^k$, $k\in\mathbb{N}$ and Theorem 5.1 holds with $\mathscr{D}(t)=t(1+\log^+t)^{[(k+1)p]}$. This was first obtained in [1].

The fractional integral with rough kernel. Let us consider the fractional operator $T_{\Omega,\alpha}f = K_\alpha * f$, where $K_\alpha(x) = \frac{\Omega(x)}{|x|^{n-\alpha}}$ and Ω is as in the previous sections. If $\Omega \in L^{\mathscr{A}}(S^{n-1}) \cap L^{\frac{n}{n-\alpha}}(S^{n-1})$ and satisfies (4.1) with $k=0$ and $\mathscr{A}$ in place of $\mathscr{B}$, then we have that the Coifman type inequality holds with $M_{\alpha,\overline{\mathscr{A}}}$ on the right hand side. Thus Theorem 5.1 can be applied to this operator. In the particular case that $\mathscr{A}(t)=t^r$, $r>p$, (5.2) holds with $\mathscr{D}(t)=t^{(r/p)'}(1+\log^+ t)^{(r/p)'(p-1)+\varepsilon}$ and $\varepsilon>0$ small enough. It suffices to apply Theorem 5.1 with $\mathscr{E}(t)=t^{p'}(1+\log^+ t)^{-1-\delta}$, and $\mathscr{F}(t)=t^{\frac{rp}{r-p}}(1+\log^+ t)^{(r/p)'(p-1)+\varepsilon}$, where $\delta>0$ is some number small enough that is related with $\varepsilon>0$.

For the commutator of this operator, assume that $\Omega \in L^{\mathscr{B}}(S^{n-1}) \cap L^{\frac{n}{n-\alpha}}(S^{n-1})$ satisfies (4.1), where $\mathscr{B}(t)=t^r$, $r>p$. Then the Coifman type inequality holds with $\overline{\mathscr{A}}(t)=t^{r'}(1+\log^+ t)^{kr'}$ and, consequently, inequality (5.2) holds with $\mathscr{D}(t)=t^{(r/p)'}(1+\log^+ t)^{(r/p)'((k+1)p-1)+\varepsilon}$ and $\varepsilon>0$ small enough (see table 1 in [11]).

The fractional integral associated to a multiplier. Suppose that we are under the same hypotheses as in Corollary 4.3. For these operators we have that the Coifman type inequality holds with $M_{\alpha,\frac{n}{l}+\varepsilon}$ on the right hand side, for both, T_m and $T^k_{m,b}$. Therefore we obtain the following.

COROLLARY 5.3. *If $1<p<r<(n/l)'$, $k \geqslant 0$ and u is a weight, then*

$$\int_{\mathbb{R}^n} |T^k_{m,b}f(x)|^p u(x)dx \leqslant C \int_{\mathbb{R}^n} |f(x)|^p M_{\alpha p,\mathscr{D}}u(x)dx, \qquad f \in L^\infty_c, \tag{5.3}$$

where $\mathscr{D}(t)=t^{(r/p)'}(1+\log^+ t)^{(r/p)'(p-1)+\varepsilon}$ and $\varepsilon>0$ is small enough.

The proof is the same as for $T_{\Omega,\alpha}$ in the case $\mathscr{A}(t)=t^r$, since the Coifman type inequality holds with $M_{\alpha,r'}$, and $r'=n/l+\varepsilon$. Let us observe that, as was point out in [11], since ε is at our choice, we can consider $\mathscr{D}(t)=t^{(r/p)'}$ for any $p<r<(n/l)'$ in (5.3).

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