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ON AN EXTENDED HADAMARD MAXIMUM DETERMINANT PROBLEM

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Abstract. Motivated by the Hadamard maximum determinant problem we study the quantities $a_{m,n} = \max\det(AA^T)$ where A is a $m\times n$ matrix with entries 1 and -1. We find the exact values of $a_{2,n}$ and $a_{3,n}$ and for a general m we give upper and lower bounds for $a_{m,n}$.

1. Introduction

Let m and n be positive integers. We denote by $\mathscr{M}_{m,n}(-1,1)$ the set of all $m\times n$ matrices whose elements are 1 or -1. We will write as usual $\mathscr{M}_n(-1,1)$ instead of $\mathscr{M}_{n,n}(-1,1)$. By Hadamard's classical inequality we have that $\det(A)^2 \leqslant n^n$ for every $A\in\mathscr{M}_n(-1,1)$. It easy to see that this upper bound can be reached only if $n=2$ or $n\equiv 0 \pmod 4$. A matrix $A\in\mathscr{M}_n(-1,1)$ such that $(\det A)^2 = n^n$ is called a Hadamard matrix. It is a long standing conjecture that $n\times n$ Hadamard matrices exist for every $n\equiv 0 \pmod 4$.

One can show that if $A\in\mathscr{M}_m(-1,1)$ and $B\in\mathscr{M}_n(-1,1)$ are Hadamard matrices and $\otimes$ denotes the Kronecker product then $A\otimes B\in\mathscr{M}_{mn}(-1,1)$ is again a Hadamard matrix and therefore the set of Hadamard matrices is a graded sub-semigroup of the graded semigroup $\left(\bigsqcup_{n\geqslant 1}\mathscr{M}_n(-1,1),\otimes\right)$.

The Hadamard maximum determinant problem (see Brenner and Cummings [1]) asks to find $\max\{|\det(A)| : A\in\mathscr{M}_n(-1,1)\}$. H. Ehlich in [3] and [4] gave upper bounds for $n\equiv 1,2 \pmod 4$ and $n\equiv 3 \pmod 4$ respectively. Moreover, it is shown in [6] that for $n\equiv 1,2 \pmod 4$ the upper bounds obtained by Ehlich are attained for infinitely many values of n.

For up-to-date information regarding this difficult problem we are referring to the web page `http://www.indiana.edu/~ maxdet/` which maintained by W. Orrick and B. Solomon. The reader will find known bounds on maximal determinants at `http://www.indiana.edu/~ maxdet/bounds.html`

In this paper we would like to address a similar question for non-square matrices. Namely we are interested in giving upper bounds for the following quantities:

$$a_{m,n} = \max\{\det(AA^T) : A\in\mathscr{M}_{m,n}(-1,1)\}$$

where $m\leqslant n$ are positive integers and A^T denotes the transpose of A. For $m=2$ and $m=3$ we will find the exact values of $a_{2,n}$ and $a_{3,n}$. For a general m we give upper and lower bounds for $a_{m,n}$.

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2. The Results

Suppose that $A \in \mathscr{M}_{m,n}(-1,1)$, $m \leqslant n$, and $\mathscr{A}_m$ is the set of all $m \times m$ submatrices of A. By the Binet-Cauchy formula $\det(AA^T) = \sum_{B \in \mathscr{A}_m} \det(B)^2$. It is easy to see that if $B \in \mathscr{M}_{2,2}(-1,1)$ then $\det(B) = 0$ or $|\det(B)| = 2$ and if $B \in \mathscr{M}_{3,3}(-1,1)$ then $\det(B) = 0$ or $|\det(B)| = 4$. Therefore in order to determine $a_{2,n}$ and $a_{3,n}$ we have to determine the maximum number of 2×2 and 3×3 non-zero minors that a $2 \times n$ and, respectively, $3 \times n$ matrix can have.

LEMMA 1. *a) Let* $A \in \mathscr{M}_n(-1,1)$ *where* $n \in \{2,3\}$. *Then* $\det(A) = 0$ *if and only if two columns of* A *are proportional.*

b) If $A \in \mathscr{M}_4(-1,1)$ *and* $\det(A) = 0$ *then* A *has two proportional columns or two proportional rows.*

REMARK. For $n = 2$ or $n = 3$, replacing A with A^T we get that $\det(A) = 0$ if and only if two rows of A are proportional. Hence we get that A has two proportional rows if and only if it has two proportional columns.

Proof. a) The "if" part is obvious. Also the "only if" part for $n = 2$ is straightforward. We will prove the "only if" part for $n = 3$. Let $A = (a_{ij}) \in \mathscr{M}_3(-1,1)$ with $\det A = 0$. For $j = 1,2,3$ let $\mathbf{a}^j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \end{bmatrix}$ be the columns of A. If A has rank 1 then any two columns are proportional and there is nothing to prove. We will assume then that A has rank 2. For example we assume that $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$

Since $\det(A) = 0$ there exist three real numbers, $\alpha, \beta, \gamma \in \mathbb{R}$, not all equal to zero, such that $\alpha \mathbf{a}^1 + \beta \mathbf{a}^2 + \gamma \mathbf{a}^3 = 0$. If $\gamma = 0$ then $\mathbf{a}^1$ and $\mathbf{a}^2$ are proportional and we are done. Suppose that $\gamma \neq 0$. Dividing by γ we can assume that $\gamma = 1$. Hence we have that

$$\alpha a_{11} + \beta a_{12} = -a_{13}$$
$$\alpha a_{21} + \beta a_{22} = -a_{23}.$$

As we assumed that $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$ and $a_{ij} \in \{-1,1\}$ we must have that $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \in \{-2,2\}$. At the same time $\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \in \{-2,0,2\}$ and $\begin{vmatrix} a_{13} & a_{12} \\ a_{23} & a_{22} \end{vmatrix} \in \{-2,0,2\}$. It follows that $\alpha, \beta \in \{-1,0,1\}$. If $\alpha, \beta \in \{-1,1\}$ then $\alpha a_{11} + \beta a_{12}$ is an even integer which is impossible since a_{13} is odd. It follows then that at least one of α and β must be zero and part a) follows.

b) Suppose that $A = (a_{ij}) \in \mathscr{M}_4(-1,1)$ is a matrix with $\det(A) = 0$. Let $\mathscr{P}$ be the set of all matrices $X \in \mathscr{M}_4(-1,1)$ that have either two proportional columns or two proportional rows. We have to show that $A \in \mathscr{P}$. Note that $\mathscr{P}$ is invariant under the following operations: interchanging two rows, interchanging two columns, multiplying a column by -1, multiplying a row by -1. We can assume then that $a_{i1}=1$ for $1=1,2,3,4$. We distinguish two cases:

Case 1: there exists $j\in\{2,3,4\}$ such that $\sum_{i=1}^4 a_{ij}\neq 0$. If $\sum_{i=1}^4 a_{ij}=\pm 4$ then either $a_{ij}=1\ \forall i$ or $a_{ij}=-1\ \forall i$ and the column j will be proportional to the first column. If $\sum_{i=1}^4 a_{ij}=\pm 2$ then the j-th column will contain three entries equal to 1 and one entry equal to -1 or three entries equal to -1 and one entry equal to 1. Without loss of generality (since $\mathscr{P}$ is invariant under the operations mentioned above) we can assume then that $a_{14}=a_{24}=a_{34}=-1$ and $a_{44}=1$. Adding the first column to the last one we get that

$$\det A = 2\det\begin{bmatrix}1 & a_{12} & a_{13}\\ 1 & a_{22} & a_{23}\\ 1 & a_{32} & a_{33}\end{bmatrix}$$

and hence the determinant of this last matrix is equal to 0. By part a) it must have two proportional columns. If $\begin{bmatrix}a_{1i}\\ a_{2i}\\ a_{3i}\end{bmatrix}=\varepsilon\begin{bmatrix}1\\1\\1\end{bmatrix}$ where $\varepsilon\in\{-1,1\}$ and $i\in 2,3$ then either $a_{4i}=\varepsilon$ in which case the i-th column of A is proportional to the first one, or $a_{4i}=-\varepsilon$ and then the i-th column is proportional to the fourth one. Suppose now that $\begin{bmatrix}a_{12}\\ a_{22}\\ a_{32}\end{bmatrix}$ is proportional to $\begin{bmatrix}a_{13}\\ a_{23}\\ a_{33}\end{bmatrix}$. Since $a_{12},a_{22},a_{32}\in\{-1,1\}$ two of them must be equal. Say that $a_{12}=a_{22}$. By proportionality we must have that $a_{13}=a_{23}$ as well. It follows then that the first two rows of A are equal.

Case 2: $\sum_{i=1}^4 a_{ij}=0$ for $j=2,3,4$. Adding the first, second and third row of A to the last one we get that

$$\det(A) = 4\det\begin{bmatrix}a_{12} & a_{13} & a_{14}\\ a_{22} & a_{23} & a_{24}\\ a_{32} & a_{33} & a_{34}\end{bmatrix}$$

and the last matrix must have then two proportional columns. Say that $\begin{bmatrix}a_{12}\\ a_{22}\\ a_{32}\end{bmatrix}=\varepsilon\begin{bmatrix}a_{13}\\ a_{23}\\ a_{33}\end{bmatrix}$ where $\varepsilon\in\{-1,1\}$. As we assumed that $\sum_{i=1}^4 a_{i2}=0$ and $\sum_{i=1}^4 a_{i3}=0$ it follows that $a_{42}=-\sum_{i=1}^3 a_{i2}=-\varepsilon\sum_{i=1}^3 a_{i3}=\varepsilon a_{43}$. We deduce that the second and third row of A are proportional. $\square$

REMARKS. 1) If $A\in\mathscr{M}_4(-1,1)$ has rank at most 2 then it must have both two proportional rows and two proportional columns, the proof of this fact being the same as the one for part a) of the theorem above.

2) If $n=4$ then there are singular matrices that do not have two proportional columns (and, by taking the transpose, there are singular matrices that do not have two

proportional rows) and for $n \geqslant 5$ there exist singular matrices that do not have two proportional columns or two proportional rows as the following two examples show:

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

We denote by $\mathbb{N}$ the set of non-negative integers. For $n \in \mathbb{N}$, $n \geqslant 1$, we denote by $\mathbb{R}[X_1, X_2, \dots, X_n]$ the ring of polynomials in n variables with coefficients in $\mathbb{R}$. For $P \in \mathbb{R}[X_1, X_2, \dots, X_n]$ and $j \in \{1, 2, \dots, n\}$ be denote by $\deg_{X_i} P$ the degree of P when P is viewed as a polynomial in the variable X_j and having as coefficients polynomials in the remaining variables (in other words $P \in \mathbb{R}[X_1, \dots, X_{j-1}, X_j, \dots X_n][X_j]$).

LEMMA 2. *Suppose that $P \in \mathbb{R}[X_1, X_2, \dots, X_n]$ is a symmetric polynomial with positive coefficients such that $\deg_{X_i} P = 1$ for every $i \in \{1, 2, \dots, n\}$. Let $M > 0$ and $k, p \in \mathbb{N}$ be some constants such that $p \leqslant n-1$.*

a) If $A = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_j \geqslant 0 \, \forall j$ and $\sum_{j=1}^n x_j = M\}$ then
$\max\{P(x_1, x_2, \dots, x_n) : (x_1, x_2, \dots, x_n) \in A\} = P(\frac{M}{n}, \frac{M}{n}, \dots, \frac{M}{n})$.

b) If $B = \{(x_1, x_2, \dots, x_n) \in \mathbb{N}^n : \sum_{j=1}^n x_j = kn + p\}$ and we set $a_j = k$ for $1 \leqslant j \leqslant n-p$ and $a_j = k+1$ for $j > n-p$ then
$\max\{P(x_1, x_2, \dots, x_n) : (x_1, x_2, \dots, x_n) \in B\} = P(a_1, a_2, \dots, a_n)$.

Proof. We begin with the following simple observation: if x_1, x_2, y_1, y_2 are real numbers such that $x_1 + x_2 = y_1 + y_2$ and $|x_1 - x_2| < |y_1 - y_2|$ then $y_1 y_2 < x_1 x_2$.

a) Since A is a compact subset of $\mathbb{R}^n$ and the real-valued function $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \to P(x_1, x_2, \dots, x_n)$ is continuous, it reaches its maximum value in A. We denote by μ this maximum value and we set

$$\tilde{A} = \{(x_1, x_2, \dots, x_n) \in A : P(x_1, x_2, \dots, x_n) = \mu\}.$$

Let $f : \tilde{A} \to \mathbb{R}$, $f(x_1, x_2, \dots, x_n) = \sum_{i<j}^n (x_i - x_j)^2$. $\tilde{A}$ is a compact set and f is a continuous function. Therefore f has a minimum point in $\tilde{A}$. Let $(a_1, a_2, \dots, a_n) \in \tilde{A}$ be a minimum point for f. We claim that one must have $a_1 = a_2 = \dots = a_n$. Indeed, let's assume that there exists $i, j \in \{1, 2, \dots, n\}$, $i \neq j$, such that $a_i \neq a_j$. Without loss of generality we can assume that $a_1 \neq a_2$. Let $b_1 = b_2 = \frac{a_1 + a_2}{2}$ and let $Q : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $Q(x_1, x_2) = P(x_1, x_2, a_3, \dots, a_n)$. We have that $b_1 + b_2 = a_1 + a_2$ and $b_1 = b_2 \geqslant 0$ and therefore $(b_1, b_2, a_3, \dots, a_n) \in A$. At the same time, given the properties of P, Q is of the form $Q(x_1, x_2) = \alpha x_1 + \alpha x_2 + \beta x_1 x_2$ for some non-negative numbers α and β. Since $|b_1 - b_2| < |a_1 - a_2|$, it follows from the observation that we started the proof with that $a_1 a_2 < b_1 b_2$, hence $Q(a_1, a_2, \dots, a_n) \leqslant Q(b_1, b_2, a_3, \dots, a_n)$ and therefore $P(a_1, a_2, \dots, a_n) \leqslant P(b_1, b_2, a_3, \dots, a_n)$. As $(a_1, a_2, \dots, a_n)$ was a maximum point for P we deduce that $(b_1, b_2, a_3, \dots, a_n)$ is a maximum point as well (and $P(a_1, a_2, \dots, a_n) = P(b_1, b_2, a_3, \dots, a_n)$). That means that $(b_1, b_2, a_3, \dots, a_n) \in \tilde{A}$. On the other hand $f((b_1,b_2,a_3,\dots,a_n) < f(a_1,a_2,\dots,a_n)$ and this is a contradiction with our choice of $(a_1,a_2,\dots,a_n)$ as a minimum point for f. In this way we proved that $a_1 = a_2 = \cdots = a_n$. Because $(a_1,a_2,\dots,a_n) \in A$ we must have that $\sum_{j=1}^n a_j = M$ and therefore $a_1 = a_2 = \cdots = a_n = \frac{M}{n}$. However $(a_1,a_2,\dots,a_n)$ was a point in $\tilde{A}$ and we conclude that $P(\frac{M}{n},\frac{M}{n},\dots,\frac{M}{n}) = \mu$ which finishes the proof of part *a*).

b) Note that B is finite and therefore $(x_1,x_2,\dots,x_n) \in B \to P(x_1,x_2,\dots,x_n)$ has a maximum point in B. We set:

$$\begin{aligned}
&\nu = \max\{P(x_1,x_2,\dots,x_n) : (x_1,x_2,\dots,x_n) \in B\}\\
&\tilde{B} = \{(x_1,x_2,\dots,x_n) \in B : P(x_1,x_2,\dots,x_n) = \nu\},\\
&g : \tilde{B} \to \mathbb{R}, \quad g(x_1,x_2,\dots,x_n) = \sum_{i<j}^{n} (x_i - x_j)^2.
\end{aligned}$$

Obviously $\tilde{B}$ is finite. Let $(a_1,a_2,\dots,a_n) \in \tilde{B}$ be a minimum point for g. We claim that $|a_i - a_j| \leqslant 1$ for every $i,j \in \{1,2,\dots,n\}$. As above the proof of this claim will be by contradiction. We assume that there exists $i,j \in \{1,2,\dots,n\}$ such that $|a_i - a_j| \geqslant 2$. Without loss of generality we can assume that $a_1 \leqslant a_2 - 2$. Let $b_1 = a_1 + 1$ and $b_2 = a_2 - 1$, $Q : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $Q(x_1,x_2) = P(x_1,x_2,a_3,\dots,a_n)$. We have that $b_1 + b_2 = a_1 + a_2$, $b_1 \geqslant 0$, $b_2 \geqslant 0$, and $|b_1 - b_2| < |a_1 - a_2|$. As in the proof of *a*) we have that $(b_1,b_2,a_3,\dots,a_n) \in B$ and $P(a_1,a_2,\dots,a_n) \leqslant P(b_1,b_2,a_3,\dots,a_n)$ which implies that $(b_1,b_2,a_3,\dots,a_n) \in \tilde{B}$. We notice that $g((b_1,b_2,a_3,\dots,a_n) < g(a_1,a_2,\dots,a_n)$ which is a contradiction with the fact that $(a_1,a_2,\dots,a_n)$ is a minimum point for g and the claim is proved.

Next we claim that $a_i \in \{k,k+1\}$ for every $i \in \{1,2\dots,n\}$. Otherwise it would exists i such that either $a_i \leqslant k-1$ or $a_i \geqslant k+2$.

We assume that there exists i such that $a_i \leqslant k-1$ and we will reach a contradiction. Without loss of generality we can assume that $a_1 \leqslant k-1$. Since $\sum_{j=1}^n a_j = kn + p$ we have that $\sum_{j=2}^n a_j \geqslant k(n-1) + p + 1 \geqslant k(n-1) + 1$. At the same time $\max\{a_2,\dots,a_n\} \geqslant \frac{1}{n-1}\sum_{j=2}^n a_j \geqslant k + \frac{1}{n-1} > k$. We deduce that there exists $j \in \{2,\dots,n\}$ such that $a_j \geqslant k+1$. However we would have then that $|a_j - a_1| \geqslant 2$ which is a contradiction.

We assume that there exists i such that $a_i \geqslant k+2$ and again we will reach a contradiction. As before we assume that $a_1 \geqslant k+2$. We have that $\sum_{j=2}^n a_j \leqslant k(n-1) + p - 2 \leqslant k(n-1) + n - 3$ and hence $\min\{a_2,\dots,a_n\} \leqslant \frac{1}{n-1}\sum_{j=2}^n a_j \leqslant k + \frac{n-3}{n-1} < k+1$. We deduce that there exists $j \in \{2,\dots,n\}$ with $a_j \leqslant k$ which implies again the contradictory inequality $|a_j - a_1| \geqslant 2$.

It remains to notice that from $\sum_{j=1}^n a_j = kn + p$ it follows that the set $\{i : a_i = k\}$ has $n-p$ elements and the set $\{i : a_i = k+1\}$ has p elements. From the symmetry of P we may assume that $a_j = k$ for $1 \leqslant j \leqslant n-p$ and $a_j = k+1$ for $j \geqslant n$. $\square$

THEOREM 1. *If for $A \in \mathscr{M}_{2,n}(-1,1)$ we denote by n_A the number of non-zero 2×2 minors of A then* $\max\{n_A : A \in \mathscr{M}_n(-1,1)\} = \left\lfloor \frac{n^2}{4} \right\rfloor$ *and therefore* $a_{2,n} = 4\left\lfloor \frac{n^2}{4} \right\rfloor$.

Proof. Suppose that A is a matrix in $\mathscr{M}_{2,n}(-1,1)$. For every $j \in \{1,2,\ldots,n\}$ we let

$$\mathbf{a}^j = \begin{bmatrix} a_{1j} \\ a_{2j} \end{bmatrix}$$

Let also

$$\mathbf{v}^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \mathbf{v}^2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

We set, for $l \in \{1,2\}$, $C_l = \{j \in \{1,2,\ldots,n\} : \mathbf{a}^j = \mathbf{v}^l \text{ or } \mathbf{a}^j = -\mathbf{v}^l\}$ and we denote by x_l the number of elements of C_l. According to Lemma 1 the set of non-zero minors of A is in bijection with $C_1 \times C_2$. Therefore $n_A = x_1x_2$. As $x_1 + x_2 = n$, Lemma 2 implies that $\max\{n_A : A \in \mathscr{M}_n(-1,1)\} = k^2$ if $n = 2k$ and $\max\{n_A : A \in \mathscr{M}_n(-1,1)\} = k(k+1)$ if $n = 2k+1$. In both cases we get that $\max\{n_A : A \in \mathscr{M}_n(-1,1)\} = \left\lfloor \frac{n^2}{4} \right\rfloor$. $\square$

THEOREM 2. *If for $A \in \mathscr{M}_{3,n}(-1,1)$ we denote by n_A the number of non-zero 3×3 minors of A then* $\max\{n_A : A \in \mathscr{M}_n(-1,1)\} = \psi(n)$ *and therefore $a_{3,n} = 16\psi(n)$ where $\psi(n)$ is given by:*

$$\psi(n) = \begin{cases} 4k^3, & \text{if } n = 4k \\ 4k^3 + 3k^2, & \text{if } n = 4k+1 \\ 4k^3 + 6k^2 + 2k, & \text{if } n = 4k+2 \\ 4k^3 + 6k^2 + 6k + 1, & \text{if } n = 4k+3 \end{cases}.$$

Proof. Suppose that A is a matrix in $\mathscr{M}_{3,n}(-1,1)$. For every $j \in \{1,2,\ldots,n\}$ we let

$$\mathbf{a}^j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \end{bmatrix}$$

Let also

$$\mathbf{v}^1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ \mathbf{v}^2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},\ \mathbf{v}^3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},\ \mathbf{v}^4 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

We set, for $l \in \{1,2,3,4\}$, $C_l = \{j \in \{1,2,\ldots,n\} : \mathbf{a}^j = \mathbf{v}^l \text{ or } \mathbf{a}^j = -\mathbf{v}^l\}$ and we denote by x_l the number of elements of C_l. According to Lemma 1 the set of non-zero minors of A is in bijection to

$$(C_1 \times C_2 \times C_3) \cup (C_1 \times C_2 \times C_4) \cup (C_1 \times C_3 \times C_4) \cup (C_2 \times C_3 \times C_4).$$

Therefore $n_A = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4$. Note also that $x_1 + x_2 + x_3 + x_4 = n$. Hence we have to determine

$$\max\{P(x_1,x_2,x_3,x_4) : x_1,x_2,x_3,x_4 \in \mathbb{N},\ x_1 + x_2 + x_3 + x_4 = n\}$$

where $P(x_1,x_2,x_3,x_4) = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4$. Obviously P is a symmetric polynomial with positive coefficients and of degree 1 in each variable and therefore we can apply Lemma 2. It now remains to consider four cases: $n=4k$, $n=4k+1$, $n=4k+2$, $n=4k+3$ and to compute $P(k,k,k,k)$, $P(k+1,k,k,k)$, $P(k+1,k+1,k,k)$ and, respectively, $P(k+1,k+1,k+1,k)$. In each of these cases we will get the formula for $\psi(n)$. $\square$

COROLLARY 1. *For every $m \leqslant n$ positive integers $a_{m,n} \leqslant \chi(m,n)$ where $\chi(m,n)$ is defined as follows:*

$$\chi(m,n) = \begin{cases} (16\psi(n))^l, & \text{if } m = 3l \\ n(16\psi(n))^l, & \text{if } m = 3l+1 \\ 4\lfloor \frac{n^2}{4} \rfloor (16\psi(n))^l, & \text{if } m = 3l+2 \end{cases}.$$

Proof. Let $A \in \mathscr{M}_{m,n}(-1,1)$ and let $\mathbf{a}^j$, $j=1,2,\dots,m$ be the row vectors of A. Suppose that $m=3l$. We let for $k=1,2,\dots l$ $A_k = \begin{bmatrix} \mathbf{a}^{3(k-1)+1} \\ \mathbf{a}^{3(k-1)+2} \\ \mathbf{a}^{3(k-1)+3} \end{bmatrix} \in \mathscr{M}_{3,n}(-1,1)$ and hence $A = \begin{bmatrix} A_1 \\ A_2 \\ \cdots \\ A_l \end{bmatrix}$. Using a well-known inequality (see for example [5]) and Theorem 2 we have:

$$\det(AA^T) \leqslant \det(A_1A_1^T)\det(A_2A_2^T)\cdots\det(A_lA_l^T) \leqslant (16\psi(n))^l.$$

If $m=3l+1$ we define A_k for $k=1,2,\dots l$ as before and $A_{l+1} = \mathbf{a}^m$. If $m=3l+2$ then we let $A_{l+1} = \begin{bmatrix} \mathbf{a}^{m-1} \\ \mathbf{a}^m \end{bmatrix}$. We use the same inequality as before the only difference is that in the case $m=3l+2$ we need to use Theorem 1 as well. In the case $m=3l+1$ we use the obvious fact that $\mathbf{a}^m(\mathbf{a}^m)^T = n$. $\square$

The following proposition gives a lower bound for $a_{m,n}$ in terms of $a_{m,m}$. For results regarding lower bounds for $a_{m,m}$ see [2].

PROPOSITION 1. *Let $n = km+p$ where $p \in \{0,1,\dots,m-1\}$. Then $a_{m,n} \geqslant a_{m,m}(k+1)^p k^{m-p}$.*

Proof. Let $B \in \mathscr{M}_m(-1,1)$ be such that $\det(B)^2 = \det(BB^T) = a_{m,m}$. Let $\mathbf{b}^1,\dots,\mathbf{b}^m$ be the column vectors of B. We define the matrix $A \in \mathscr{M}_{m,n}$ as follows: $A = [B\, B \cdots B\ \mathbf{b}^1 \cdots \mathbf{b}^p]$ if $p \geqslant 1$ and $A = [B\, B \cdots B]$ if $p=0$ where, in both cases the matrix B appears k times. Note now that that if C is $m \times m$ submatrix of A then either $\det(C)=0$ or C is obtained from B by a permutation of its columns. Therefore the absolute value of each non-zero $m \times m$ minor of A is $a_{m,m}$. On the other hand, since each of the columns $\mathbf{b}^1,\cdots,\mathbf{b}^p$ appear $k+1$ times among the columns of A and $\mathbf{b}^{p+1},\cdots,\mathbf{b}^m$ appear $m-k$ times, we get altogether $(k+1)^p k^{m-p}$ $m \times m$ non-zero minors. $\square$

REFERENCES

[1] J. BRENNER, L. CUMMINGS, *The Hadamard maximum determinant problem*, Amer. Math. Monthly, **79** (1972), 626–630.
[2] J. H. E. COHN, *On the value of determinants*, Proc. Amer. Math. Soc., **14** (1963), 581–588.
[3] H. EHLICH, *Determinantenabschätzungen für binäre Matrizen*, Math. Z., **83** (1964), 123–132.
[4] H. EHLICH, *Determinantenabschätzung für binäre Matrizen mit* $n \equiv 3 \bmod 4$, Math. Z., **84** (1964), 438–447.
[5] D. K. FADDEEV, I. S. SOMINSKII, *Problems in higher algebra*, Freeman and Co., San Francisco-London, 1965.
[6] M. G. NEUBAUER, A. J. RADCLIFFE, *The maximum determinant of* ± 1 *matrices*, Linear Algebra Appl., **257** (1997), 289–306.

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