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EXTENSIONS OF INEQUALITIES INVOLVING KANTOROVICH CONSTANT

MAREK NIEZGODA

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Abstract. In this paper, two methods of extending inequalities involving Kantorovich constant are presented. An inequality of Mićić et al. [Linear Algebra Appl., **318** (2000), 87–107] on positive linear maps and geometric mean of positive definite matrices is extended to arbitrary matrices having accretive transformation. A result of Dragomir [JIPAM 5 (3), Art.76, 2004] is applied to give new sufficient conditions for Greub-Reinboldt's inequality to hold.

1. Introduction and motivation

Throughout this paper, $\mathbf{C}^n$ is the complex space of column n-vectors with inner product $\langle x,y\rangle = \sum_{j=1}^{n} x_j\overline{y_j}$ and norm $\|x\| = \langle x,x\rangle^{1/2}$ for $x,y \in \mathbf{C}^n$. By $\mathbf{M}_n(\mathbf{C})$ we denote the C^*-algebra of all complex $n\times n$ matrices. The symbol $(\cdot)^*$ stands for the conjugate transpose of a matrix. As usual, I denotes the $n\times n$ identity matrix. For a matrix $X \in \mathbf{M}_n(\mathbf{C})$, we write $X \geqslant 0$ (resp. $X > 0$) if X is positive semidefinite (resp. positive definite). A linear map $\Phi : \mathbf{M}_n(\mathbf{C}) \to \mathbf{M}_k(\mathbf{C})$ is said to be *positive* if $\Phi(X) \geqslant 0$ for $X \geqslant 0$. Hereafter for a matrix $X \in \mathbf{M}_n(\mathbf{C})$ we denote

$$\operatorname{Re} X := (X+X^*)/2. \tag{1}$$

Let A and B be $n\times n$ positive definite matrices. The *geometric mean* of A and B is defined by

$$A\sharp B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2} \tag{2}$$

(see [1, 10]). The following result holds. If

$$0 < mI \leqslant A \leqslant MI \;\text{ and }\; 0 < mI \leqslant B \leqslant MI, \tag{3}$$

then for $x \in \mathbf{C}^n$

$$(x^*Ax)^{1/2}\cdot(x^*Bx)^{1/2} \leqslant \frac{M+m}{2\sqrt{mM}}\, x^*A\sharp Bx$$

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(see [8, 17]). The number $\kappa(m,M) = \frac{(M+m)^2}{4mM}$ is called *Kantorovich constant* [17, p. 688]. Note that $\sqrt{\kappa(m,M)} = \frac{M+m}{2\sqrt{mM}}$ is the ratio of the arithmetic to geometric mean of m and M. Furthermore, $\kappa(m,M) = \kappa(\sqrt{\frac{m}{M}}, \sqrt{\frac{M}{m}})$.

Mićić et al. [11, Corollary 3.7] showed that if

$$0 < m_1 I \leqslant A \leqslant M_1 I \quad \text{and} \quad 0 < m_2 I \leqslant B \leqslant M_2 I,$$

then

$$\Phi(A) \sharp \Phi(B) \leqslant \frac{\sqrt{M_1 M_2} + \sqrt{m_1 m_2}}{2\sqrt[4]{m_1 m_2 M_1 M_2}} \, \Phi(A \sharp B).$$

In particular, if (3) holds then

$$\Phi(A) \sharp \Phi(B) \leqslant \frac{M+m}{2\sqrt{mM}} \, \Phi(A \sharp B). \tag{4}$$

Recently Lee [10, Theorem 4] proved (4) under the assumption of the form $mA \leqslant B \leqslant MA$ with positive definite matrices A and B and positive scalars m,M. Niezgoda [15, Theorem 1.1] presented a proof of (4) by using the positive definite matrix $Z = |B^{1/2}A^{-1/2}| = (A^{-1/2}BA^{-1/2})^{1/2}$ such that $mI \leqslant Z \leqslant MI$ with positive scalars m,M.

In this note, our first aim is to extend (4) with the help of any $Z \in \mathbf{M}_n(\mathbf{C})$ such that $Z^*Z = A^{-1/2}BA^{-1/2}$ and

$$\operatorname{Re}(Z - mI)^*(MI - Z) \geqslant 0 \tag{5}$$

(see (1)). The condition (5), originated by Dragomir [5, 6], is known to be very useful in deriving matrix inequalities [3, 4, 5, 6, 13, 14, 15, 16]. It is related to the notion of *accretive operators*. In Section 2 we utilize such operators to generalized inequality (4). Related results are also given.

In Section 3 we study vectorial intervals to give another method of extending the range of applicability of some known inequalities. It is worth emphasising that some standard assumptions for many inequalities to hold are related to vectorial intervals induced by one (self-dual) cone, e.g. $\mathbf{R}^n_+$ in $\mathbf{R}^n$ or the Loewner cone $\mathbf{L}_n$ of positive semidefinite matrices in the (real) space $\mathbf{H}_n$ of $n \times n$ Hermitian matrices. In our approach, intervals are induced by an arbitrary pair of dual convex cones. This and special Dragomir's condition similar to (5) (see (18)) allow to establish some new sufficient conditions for such inequalities to be still valid. In Theorem 3.2 we reinterpret [3, Theorem 2.2] in the context of dual bases of the underlying linear space. In Corollary 3.5, we illustrate the above ideas by the classical inequality of Greub - Reinboldt (see [7]). In Corollary 3.6, we provide some new conditions implying G-R inequality.

2. Making use of accretive operators

A matrix $C \in \mathbf{M}_n(\mathbf{C})$ is said to be *accretive* if $\mathrm{Re}\langle Cx,x\rangle \geqslant 0$ for all $x \in \mathbf{C}^n$ [5, p. 2753].

For a matrix $Z \in \mathbf{M}_n(\mathbf{C})$ and scalars $m,M \in \mathbf{C}$, the symbol $C_{m,M}(Z)$ stands for the transform

$$C_{m,M}(Z) := (Z - mI)^*(MI - Z) \tag{6}$$

(see [5, p. 2752]).

It is not hard to verify that

$$C_{m,M}(Z) \text{ is accretive iff } \mathrm{Re}\, C_{m,M}(Z) \geqslant 0 \tag{7}$$

$$\text{iff } \mathrm{Re}\,\langle (Z-mI)x, (MI-Z)x\rangle \geqslant 0 \quad \text{for } x \in \mathbf{C}^n \tag{8}$$

(see (1)).

Let A and B be $n \times n$ positive definite matrices. It is well known that the geometric mean $X_0 = A\sharp B$ is the unique solution of the equation

$$XA^{-1}X = B \quad \text{with restriction} \quad X > 0$$

(see [10, p. 806]). Hence $Z_0 = A^{-1/2}X_0A^{-1/2}$ satisfies the equation

$$Z^2 = A^{-1/2}BA^{-1/2} \quad \text{with restriction} \quad Z > 0.$$

In the sequel, we are interested in any $Z \in \mathbf{M}_n(\mathbf{C})$ satisfying equation

$$Z^*Z = A^{-1/2}BA^{-1/2}. \tag{9}$$

The solution of (9) is not unique, since $Z = UZ_0$ with unitary U satisfies (9).

By analogy to (2), we denote

$$A\sharp_Z B := A^{1/2}ZA^{1/2},$$

where Z satisfies (9).

We are now in a position to give an extension of [10, Theorem 4] and [16, Theorem 1.1] (cf. [11, Corollary 3.7]).

THEOREM 2.1. *Let A and B be $n \times n$ positive definite matrices and let scalars $m,M \in \mathbf{C}$ with $\mathrm{Re}(\overline{m}M) > 0$. Let $Z \in \mathbf{M}_n(\mathbf{C})$ satisfy $Z^*Z = A^{-1/2}BA^{-1/2}$. Assume $\Phi : \mathbf{M}_n(\mathbf{C}) \to \mathbf{M}_k(\mathbf{C})$ is a strictly positive linear map.*

If $C_{m,M}(Z)$ is accretive, then

$$\Phi(A)\sharp\Phi(B) \leqslant \frac{1}{2\sqrt{\mathrm{Re}(\overline{m}M)}}\,\Phi(\mathrm{Re}((\overline{M+m})A\sharp_Z B)). \tag{10}$$

If in addition scalars m,M are positive, then (10) *gives*

$$\Phi(A)\sharp\Phi(B) \leqslant \frac{M+m}{2\sqrt{mM}}\,\Phi(\mathrm{Re}(A\sharp_Z B)). \tag{11}$$

Proof. Since $\sqrt{\operatorname{Re}(\overline{m}M)}\,\Phi(A)$ and $\frac{1}{\sqrt{\operatorname{Re}(\overline{m}M)}}\Phi(B)$ are positive definite matrices, it follows from the arithmetic-geometric inequality that

$$\Phi(A)\sharp\Phi(B) \leqslant \frac{1}{2}\left(\sqrt{\operatorname{Re}(\overline{m}M)}\Phi(A) + \frac{1}{\sqrt{\operatorname{Re}(\overline{m}M)}}\Phi(B)\right) \tag{12}$$

(see [17, p. 690]).

On the other hand, if $C_{m,M}(Z)$ is accretive, then (6) and (7) guarantee that

$$(\operatorname{Re}(\overline{m}M))I + Z^*Z \leqslant \operatorname{Re}(\overline{M+m}Z),$$

which implies

$$\sqrt{\operatorname{Re}(\overline{m}M)}I + \frac{1}{\sqrt{\operatorname{Re}(\overline{m}M)}}Z^*Z \leqslant \frac{1}{\sqrt{\operatorname{Re}(\overline{m}M)}}\operatorname{Re}(\overline{M+m}Z) \tag{13}$$

(see [15, Proposition 2.1]). Simultaneously, $A^{1/2}Z^*ZA^{1/2} = B$. So, by pre- and post-multiplying both sides of the inequality (13) by $A^{1/2}$, we obtain

$$\sqrt{\operatorname{Re}(\overline{m}M)}A + \frac{1}{\sqrt{\operatorname{Re}(\overline{m}M)}}B \leqslant \frac{1}{\sqrt{\operatorname{Re}(\overline{m}M)}}A^{1/2}(\operatorname{Re}(\overline{M+m}Z))A^{1/2}.$$

By using the positivity of Φ we get

$$\sqrt{\operatorname{Re}(\overline{m}M)}\Phi(A) + \frac{1}{\sqrt{\operatorname{Re}(\overline{m}M)}}\Phi(B) \leqslant \frac{1}{\sqrt{\operatorname{Re}(\overline{m}M)}}\Phi(A^{1/2}(\operatorname{Re}(\overline{M+m}Z))A^{1/2}). \tag{14}$$

Moreover,

$$A^{1/2}(\operatorname{Re}(\overline{M+m}Z))A^{1/2} = \operatorname{Re}((\overline{M+m})A\sharp_Z B). \tag{15}$$

Combining (12), (14) and (15) we deduce that (10) holds.

Inequality (11) is a direct consequence of (10). □

In Corollary 2.2, inequality (17) corresponds to a special case of [17, Theorem 2.2] with two factors (see also [8]).

COROLLARY 2.2. *Let A and B be $n\times n$ positive definite matrices and let scalars $m,M \in \mathbf{C}$ with* $\operatorname{Re}(\overline{m}M) > 0$. *Let $Z \in \mathbf{M}_n(\mathbf{C})$ satisfy $Z^*Z = A^{-1/2}BA^{-1/2}$.*

If $C_{m,M}(Z)$ is accretive, then for $h \in \mathbf{C}^n$ we have

$$\langle Ah,h\rangle^{1/2}\langle Bh,h\rangle^{1/2} \leqslant \frac{1}{2\sqrt{\operatorname{Re}(\overline{m}M)}}\langle(\operatorname{Re}((\overline{M+m})A\sharp_Z B))h,h\rangle. \tag{16}$$

If in addition scalars m,M are positive, then (16) *reduces to*

$$\langle Ah,h\rangle^{1/2}\langle Bh,h\rangle^{1/2} \leqslant \frac{M+m}{2\sqrt{mM}}\langle(\operatorname{Re}(A\sharp_Z B))h,h\rangle. \tag{17}$$

Proof. Straightforward application of Theorem 2.1 to the map $\Phi(X) = h^*Xh$ for $X \in \mathbf{M}_n(\mathbf{C})$. $\square$

By using the spectral decomposition of Hermitian matrices, one can prove Lemma 2.3 giving examples of the accretivity of the transform $C_{m,M}(Z)$. This lemma can be used to demonstrate inequalities (10) and (16) with some standard assumptions on A and B.

LEMMA 2.3. **(i)** *Let Z be an $n \times n$ Hermitian matrix and let m, M be real scalars with $m \leqslant M$.*

Then the operator $C_{m,M}(Z)$ is accretive if and only if

$$mI \leqslant Z \leqslant MI.$$

(ii) *Let A and B be $n \times n$ positive definite matrices. Denote $Z = (A^{-1/2}BA^{-1/2})^{1/2}$.*

If

$$mA \leqslant B \leqslant MA \quad \text{with positive scalars } m, M,$$

then the operator $C_{\sqrt{m},\sqrt{M}}(Z)$ is accretive.

(iii) *Let A and B be $n \times n$ positive definite matrices. Denote $Z = (A^{-1/2}BA^{-1/2})^{1/2}$.*

If

$$m_1 I \leqslant A \leqslant M_1 I \quad \text{and} \quad m_2 I \leqslant B \leqslant M_2 I \quad \text{with positive scalars } m_1, M_1, m_2, M_2,$$

then the operator $C_{\sqrt{\frac{m_2}{M_1}},\sqrt{\frac{M_2}{m_1}}}(Z)$ is accretive.

As a result related to Theorem 2.1 and Corollary 2.2, we now quote a reverse to Cauchy-Schwarz inequality due to Dragomir [3, Theorem 2.2] (cf. [12, Proposition 3.4, part (c)]).

THEOREM 2.4. (Dragomir [3, Theorem 2.2]) *Let $x, y \in \mathbf{C}^n$ and $m, M \in \mathbf{C}$ with* $\operatorname{Re}(\overline{m}M) > 0$.

If

$$0 \leqslant \operatorname{Re}\langle x - my, My - x\rangle, \tag{18}$$

then

$$\|x\|\|y\| \leqslant \frac{\operatorname{Re}(\overline{M+m} \cdot \langle x, y\rangle)}{2\sqrt{\operatorname{Re}(\overline{m}M)}} \leqslant \frac{|M+m|}{2\sqrt{\operatorname{Re}(\overline{m}M)}} |\langle x, y\rangle|. \tag{19}$$

The singular values of a matrix $X \in \mathbf{M}_n(\mathbf{C})$ are denoted by $s_1(X) \geqslant \ldots \geqslant s_n(X)$ and arranged in decreasing order with repeated multiplicity. That is, for $j = 1, \ldots, n$, $s_j(X)$ is the jth largest eigenvalue of the positive semidefinite matrix $|X| = (X^*X)^{1/2}$.

The following result can be compared to [15, Corollary 2.6].

THEOREM 2.5. *Let $A, B \in \mathbf{M}_n(\mathbf{C})$ and let scalars $m, M \in \mathbf{C}$ satisfy $\mathrm{Re}\,(\overline{m}M) > 0$. If*

$$\mathrm{Re}\,(B - mA)^*(MA - B) \geqslant 0, \tag{20}$$

then

$$s_j(AB^*) \leqslant \frac{1}{2\sqrt{\mathrm{Re}\,(\overline{m}M)}} s_j(\mathrm{Re}\,(\overline{M + m}A^*B)) \quad \text{for } j = 1, \ldots, n. \tag{21}$$

If in addition A^*B is Hermitian, then (21) becomes

$$s_j(AB^*) \leqslant \frac{|\mathrm{Re}\,(M + m)|}{2\sqrt{\mathrm{Re}\,(\overline{m}M)}} s_j(A^*B) \quad \text{for } j = 1, \ldots, n. \tag{22}$$

If in addition scalars m, M are positive, then (21) reduces to

$$s_j(AB^*) \leqslant \frac{M + m}{2\sqrt{mM}} s_j(\mathrm{Re}\,(A^*B)) \quad \text{for } j = 1, \ldots, n. \tag{23}$$

Proof. It is known that

$$s_j(CD^*) \leqslant \frac{1}{2} s_j(C^*C + D^*D) \quad \text{for } C, D \in \mathbf{M}_n(\mathbf{C}) \text{ and } j = 1, \ldots, n$$

(see [2, Theorem IX.4.2, p. 262]). Hence, by putting $C = \sqrt[4]{\mathrm{Re}\,(\overline{m}M)}A$ and $D = \frac{1}{\sqrt[4]{\mathrm{Re}\,(\overline{m}M)}}B$, we find that

$$s_j(AB^*) \leqslant \frac{1}{2} s_j(\sqrt{\mathrm{Re}\,(\overline{m}M)}A^*A + \frac{1}{\sqrt{\mathrm{Re}\,(\overline{m}M)}}B^*B)) \quad \text{for } j = 1, \ldots, n. \tag{24}$$

Similarly as in the proof of Theorem 2.1, (20) gives

$$\mathrm{Re}\,(\overline{m}M)\,A^*A + B^*B \leqslant \mathrm{Re}\,(MB^*A + \overline{m}A^*B).$$

Hence

$$\sqrt{\mathrm{Re}\,(\overline{m}M)}\,A^*A + \frac{1}{\sqrt{\mathrm{Re}\,(\overline{m}M)}} B^*B \leqslant \frac{1}{\sqrt{\mathrm{Re}\,(\overline{m}M)}} \mathrm{Re}\,(MB^*A + \overline{m}A^*B),$$

because $\mathrm{Re}\,(\overline{m}M) > 0$. Using the equality $\mathrm{Re}\,(MB^*A + \overline{m}A^*B) = \mathrm{Re}\,(\overline{M + m}A^*B)$, we get

$$\sqrt{\mathrm{Re}\,(\overline{m}M)}\,A^*A + \frac{1}{\sqrt{\mathrm{Re}\,(\overline{m}M)}} B^*B \leqslant \frac{1}{\sqrt{\mathrm{Re}\,(\overline{m}M)}} (\mathrm{Re}\,(\overline{M + m})A^*B). \tag{25}$$

Both the sides of (25) are positive semidefinite matrices. So, by Weyl's Monotonicity Theorem [2, Corollary III.2.3, p. 63],

$$s_j\left(\sqrt{\mathrm{Re}\,(\overline{m}M)}\,A^*A + \frac{1}{\sqrt{\mathrm{Re}\,(\overline{m}M)}} B^*B\right) \leqslant s_j\left(\frac{1}{\sqrt{\mathrm{Re}\,(\overline{m}M)}} (\mathrm{Re}\,(\overline{M + m})A^*B)\right) \tag{26}$$

for $j = 1, \ldots, n$. Combining (24) and (26) proves (21).

Inequalities (22) and (23) follows directly from (21). □

3. Dragomir's condition and vectorial intervals

As shown in Section 2, crucial assumptions in the previous results are conditions of type (8), (18) and (20). In this section we give a geometric interpretation and applications of Dragomir's condition (18) by using vectorial intervals.

Let V be a real linear space endowed with *real* inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}$.

If $\prec_1$ and $\prec_2$ are two preorders on V, then for given vectors $a, b \in V$, we define $\prec_1\prec_2$*-interval* as follows:

$$[a,b]_{\prec_1\prec_2} = \{v \in V : a \prec_1 v \prec_2 b\}$$

(cf. [9, pp. 120–121]).

The *dual cone* of a convex cone $K \subset V$ is defined by

$$\operatorname{dual} K = \{v \in V : \langle w, v\rangle \geqslant 0 \quad \text{for all } w \in K\}.$$

We write

$$y \prec_K x \quad \text{if} \quad x - y \in K,$$

and

$$y \prec_{\operatorname{dual} K} x \quad \text{if} \quad x - y \in \operatorname{dual} K.$$

For notational simplicity, the symbol $[a,b]_K$ stands for the vectorial interval $[a,b]_{\prec_K \prec_{\operatorname{dual} K}}$. Thus

$$x \in [a,b]_K \quad \text{iff} \quad a \prec_K x \prec_{\operatorname{dual} K} b.$$

It is well known that the distance between a number x in a real interval $[a,b] \subset \mathbf{R}$ and the center $(a+b)/2$ does not exceed $(b-a)/2$. A similar result is as follows. (The equivalence **(b)** $\Leftrightarrow$ **(c)** is due to Dragomir [4, Lemma 2.1].)

LEMMA 3.1. *For any vectors $a, b, x \in V$, the following statements are mutually equivalent.*

(a) *There exists a convex cone $K \subset V$ such that $x \in [a,b]_K$.*

(b) $\langle x - a, b - x\rangle \geqslant 0$.

(c) $\|x - \frac{a+b}{2}\| \leqslant \|\frac{b-a}{2}\|$.

Proof. **(a)** $\Leftrightarrow$ **(b)**. If there exists a convex cone $K \subset V$ such that $x \in [a,b]_K$, then $a \prec_K x \prec_{\operatorname{dual} K} b$, which means $x - a \in K$ and $b - x \in \operatorname{dual} K$. This gives $\langle x-a, b-x\rangle \geqslant 0$, completing the proof of **(b)**.

Conversely, assuming **(b)** and taking $K = \{t(x-a) : t \geqslant 0\}$, we obtain $b - x \in \operatorname{dual} K$. Clearly, $x - a \in K$. Therefore $x \in [a,b]_K$. This proves **(a)**.

(b) $\Leftrightarrow$ **(c)**. Apply [4, Lemma 2.1]. $\square$

Making use of Lemma 3.1, we deduce that Dragomir's condition (18) means

$$x \in [my, My]_K \quad \text{for some convex cone } K \subset V. \tag{27}$$

We now interpret [3, Theorem 2.2] (see (18) $\Rightarrow$ (19)) in the context of finitely generated cones.

THEOREM 3.2. *Let* $(V,\langle\cdot,\cdot\rangle)$ *be a real* n*-dimensional inner product space, and let* $\{e_i : i=1,\dots,n\}$ *and* $\{d_i : i=1,\dots,n\}$ *be two bases in* V*. Assume the bases are dual, that is* $\langle e_i,d_j\rangle=\delta_{ij}$*, the Kronecker delta,* $i,j=1,\dots,n$*.*

(A) *Let* $x,y\in V$ *and let* m,M *be real scalars with* $mM>0$*. Assume there exist index sets* I_1 *and* I_2 *with* $I_1\cup I_2=\{1,\dots,n\}$ *such that*

$$m\langle y,e_i\rangle \leqslant \langle x,e_i\rangle \quad \text{and} \quad \langle x,d_i\rangle \leqslant M\langle y,d_i\rangle \quad \text{for } i\in I_1\,, \tag{28}$$

and

$$m\langle y,e_i\rangle \geqslant \langle x,e_i\rangle \quad \text{and} \quad \langle x,d_i\rangle \geqslant M\langle y,d_i\rangle \quad \text{for } i\in I_2\,. \tag{29}$$

Then (19) *holds.*

(B) *Let* $x,y\in V$ *and let* m,M *be real scalars with* $mM>0$*. Assume that the following three conditions are satisfied:*

$$\langle y,e_i\rangle \neq 0 \quad \text{and} \quad \langle y,d_i\rangle \neq 0 \quad \text{for } i=1,\dots,n, \tag{30}$$

$$m\leqslant \frac{\langle x,e_i\rangle}{\langle y,e_i\rangle} \quad \text{and} \quad \frac{\langle x,d_i\rangle}{\langle y,d_i\rangle}\leqslant M \quad \text{for } i=1,\dots,n, \tag{31}$$

$$\{i\in\{1,\dots,n\} : \langle y,e_i\rangle>0\}=\{i\in\{1,\dots,n\} : \langle y,d_i\rangle>0\}. \tag{32}$$

Then (19) *holds.*

Proof. **(A)**. On account of the validity of the implication (18) $\Rightarrow$ (19) (see [3, Theorem 2.2]), we only need to show (18). It is not hard to verify that

$$\langle x-my, My-x\rangle=\sum_{i=1}^{n}\langle x-my,e_i\rangle\langle My-x,d_i\rangle.$$

Observe that

$$\langle x-my,e_i\rangle=0=\langle My-x,d_i\rangle \quad \text{for } i\in I_1\cap I_2\,.$$

Therefore we have

$$\langle x-my, My-x\rangle=\sum_{i\in I_1}\langle x-my,e_i\rangle\langle My-x,d_i\rangle+\sum_{i\in I_2}\langle x-my,e_i\rangle\langle My-x,d_i\rangle. \tag{33}$$

Now, combining (28), (29) and (33), we get (18), as required.

(B). It is obvious that (30) and (32) give

$$\{i\in\{1,\dots,n\} : \langle y,e_i\rangle<0\}=\{i\in\{1,\dots,n\} : \langle y,d_i\rangle<0\}. \tag{34}$$

Denote the index sets of (32) by I_1 and of (34) by I_2, respectively. From (31), (32) and (34) it now follows that conditions (28)-(29) are true. Thus part (A) of Theorem 3.2 implies (19), as desired. □

REMARK 3.3. It can be proven that if $I_1 \cap I_2$ is the empty set, then the statements (28)-(29) have form (27) for the cone

$$K = \operatorname{cone}\{d_i : i \in I_1\} \cup \{-d_i : i \in I_2\} = \operatorname{dual}\operatorname{cone}\{e_i : i \in I_1\} \cup \{-e_i : i \in I_2\}.$$

Here the symbol $\operatorname{cone} V_0$ stands for the convex cone of all nonnegative linear combinations of vectors in a subset $V_0 \subset V$.

REMARK 3.4. If $\{e_i : i = 1,\dots,n\}$ is a self-dual basis, i.e., $d_i = e_i$ for $i = 1,\dots,n$, then (32) is satisfied automatically. Therefore (32) can be dropped from part (B) of Theorem 3.2. In this case, (31) can be rewritten as

$$m \leqslant \frac{\langle x, e_i\rangle}{\langle y, e_i\rangle} \leqslant M \quad \text{for } i = 1,\dots,n. \tag{35}$$

The following Greub-Reinboldt's inequality follows from (19). In particular, for $w_i = 1$, (37) reduces to Pólya-Szegö's inequality (see [7]).

COROLLARY 3.5. (See [7].) *Let $x = (x_1,\dots,x_n)$ and $y = (y_1,\dots,y_n)$ be two real n-tuples. Suppose that m_1, M_1, m_2, M_2 are constants such that*

$$0 < m_1 \leqslant x_i \leqslant M_1 \quad \textit{and} \quad 0 < m_2 \leqslant y_i \leqslant M_2, \quad i = 1,\dots,n. \tag{36}$$

Then, for $w_i > 0$,

$$\sum_{i=1}^{n} w_i x_i^2 \sum_{i=1}^{n} w_i y_i^2 \leqslant \frac{(M_1M_2 + m_1m_2)^2}{4m_1m_1M_1M_2}\left(\sum_{i=1}^{n} w_i x_i y_i\right)^2. \tag{37}$$

Proof. Inequality (37) is a special case of (19) for $m = \frac{m_1}{M_2}$ and $M = \frac{M_1}{m_2}$. To see this, set

$$V = \mathbf{R}^n \text{ and } \langle a, b\rangle = \sum_{i=1}^{n} w_i a_i b_i \quad \text{for } a, b \in \mathbf{R}^n. \tag{38}$$

Consider the standard orthonormal basis in $\mathbf{R}^n$, i.e.,

$$e_i = d_i = (0,\dots,0,1,0,\dots,0) \text{ with } 1 \text{ at } i\text{th position}, \quad i = 1,\dots,n.$$

Next, observe that (36) implies

$$0 < \frac{m_1}{M_2} \leqslant \frac{x_i}{y_i} \leqslant \frac{M_1}{m_2}, \quad i = 1,\dots,n,$$

which gives (35). Thus conditions (30), (31) and (32) are fulfilled (see Remark 3.4). As consequence of part (B) of Theorem 3.2, we get (19), which is equivalent to (37). □

In the next corollary, we present further sufficient conditions guaranteeing that Greub-Reinboldt's inequality (37) is satisfied.

COROLLARY 3.6. *Let $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$ be two real n-tuples. Assume that for some positive scalars m,M, there exist index sets I_1 and I_2 with $I_1\cup I_2=\{1,\dots,n\}$ such that*

$$m\sum_{j=1}^{i} y_j \leqslant \sum_{j=1}^{i} x_j \quad \text{and} \quad x_i - x_{i+1} \leqslant M(y_i - y_{i+1}) \quad \text{for } i\in I_1, \tag{39}$$

and

$$m\sum_{j=1}^{i} y_j \geqslant \sum_{j=1}^{i} x_j \quad \text{and} \quad x_i - x_{i+1} \geqslant M(y_i - y_{i+1}) \quad \text{for } i\in I_2 \tag{40}$$

with the convention that $x_{n+1}=y_{n+1}=0$.

Then the following version of Greub-Reinboldt's inequality (37) *holds:*

$$\sum_{i=1}^{n} w_i x_i^2 \sum_{i=1}^{n} w_i y_i^2 \leqslant \frac{(M+m)^2}{4mM}\left(\sum_{i=1}^{n} w_i x_i y_i\right)^2. \tag{41}$$

Furthermore, if the following three conditions hold:

$$\sum_{j=1}^{i} y_j \neq 0 \quad \text{and} \quad y_i - y_{i+1} \neq 0 \quad \text{for } i=1,\dots n, \tag{42}$$

$$m \leqslant \frac{\sum_{j=1}^{i} x_i}{\sum_{j=1}^{i} y_i} \quad \text{and} \quad \frac{x_i - x_{i+1}}{y_i - y_{i+1}} \leqslant M \quad \text{for } i=1,\dots,n, \tag{43}$$

$$\{i\in\{1,\dots,n\} : \sum_{j=1}^{i} y_j > 0\} = \{i\in\{1,\dots,n\} : y_i - y_{i+1} > 0\}, \tag{44}$$

then (41) *holds.*

For instance, if

$$0 < m_1 \leqslant \sum_{j=1}^{i} x_j \quad \text{and} \quad 0 < x_i - x_{i+1} \leqslant M_1 \quad \text{for } i=1,\dots,n, \tag{45}$$

and

$$0 < m_2 \leqslant y_i - y_{i+1} \quad \text{and} \quad 0 < \sum_{j=1}^{i} y_j \leqslant M_2 \quad \text{for } i=1,\dots,n, \tag{46}$$

then (41) *holds.*

Proof. Consider $V=\mathbf{R}^n$ with the inner product given by (38). Take the basis

$$e_i = (\underbrace{1,\dots,1}_{i \text{ times}},0,\dots,0), \quad i=1,\dots,n.$$

The dual basis of $\{e_i : i = 1, \dots, n\}$ is given by

$$d_i = (\underbrace{0, \dots, 0}_{i-1 \text{ times}}, 1, -1, 0, \dots, 0), \quad i = 1, \dots, n-1, \quad \text{and} \quad d_n = (0, \dots, 0, 1).$$

According to Theorem 3.2, part (A), conditions (39)-(40) imply (41).

Likewise, by virtue of Theorem 3.2, part (B), we deduce that conditions (42)-(44) imply (41).

In particular, if (45)-(46) are met, then (42), (43) and (44) are achieved for $m = \frac{m_1}{M_2}$ and $M = \frac{M_1}{m_2}$. In other words, conditions (45)-(46) give (41). □

We conclude this section with the observation that G-R inequality (41) holds for *any* choice of pairs of dual bases e and d in $\mathbf{R}^n$ satisfying conditions (28)-(29) (or (30), (31) and (32)), as described in Theorem 3.2.

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Marek Niezgoda
Department of Applied Mathematics and Computer Science
University of Life Sciences in Lublin
Akademicka 13
20-950 Lublin
Poland
e-mail: marek.niezgoda@up.lublin.pl