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NONLINEAR DIFFERENTIAL INEQUALITY

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Abstract. A nonlinear differential inequality is formulated in the paper. An estimate of the rate of growth/decay of solutions to this inequality is obtained. This inequality is of interest in a study of dynamical systems and nonlinear evolution equations in Banach spaces. It is applied to a study of global existence of solutions to nonlinear partial differential equations.

1. Introduction

In this paper the following nonlinear differential inequality

$$\dot{g}(t) \leqslant -\gamma(t)g(t) + \alpha(t,g(t)) + \beta(t), \qquad t \geqslant t_0, \quad \dot{g}(t) := \dot{g} = \frac{dg}{dt}, \quad g \geqslant 0, \tag{1}$$

is studied. In equation (1), $\beta(t)$ and $\gamma(t)$ are Lebesgue measurable functions, defined on $[t_0,\infty)$, where $t_0 \geqslant 0$ is a fixed number, and $\alpha(t,g)$ is defined on $[t_0,\infty)\times[0,\infty)$. The function $\alpha(t,g)$ is non-decreasing as a function of g for every $t \geqslant t_0$, and is $L^1_{loc}([t_0,\infty))$-function of t for every $g \geqslant 0$.

Inequality (1) was studied in [11] with $\alpha(t,y) = \tilde{\alpha}(t)y^2$, where $0 \leqslant \tilde{\alpha}(t)$ is a continuous function on $[t_0,\infty)$. This inequality arises in the study of the Dynamical Systems Method (DSM) for solving nonlinear operator equations. Sufficient conditions on β, α and γ which yields an estimate for the rate of growth/decay of $g(t)$ were given in [11]. A discrete analog of (1) was studied in [5]. An application to the study of a discrete version of the DSM for solving nonlinear equation was demonstrated in [5].

In [6] inequality (1) is studied in the case $\alpha(t,y) = \tilde{\alpha}(t)y^p$, where $p > 1$ and $0 \leqslant \tilde{\alpha}(t)$ is a continuous function on $[t_0,\infty)$. This equality allows one to study the DSM under weaker smoothness assumption on F than in the cited works. It allows one to study the convergence of the DSM under the assumption that F' is locally Hölder continuous. An application to the study of large time behavior of solutions to some partial differential equations was outlined in [6].

Assumption A): Assume that $0 \leqslant \alpha(t,g)$ *is a non-decreasing function of* g *on* $[0,\infty]$ *for every* $t \geqslant t_0$, *and is an* $L^1_{loc}([0,\infty))$ *function of* t *for every* $g \in [0,\infty)$, $\beta(t)$ *and* $\gamma(t)$ *are* $L^1_{loc}([0,\infty))$ *functions of* t.

Under this assumption, which holds throughout the paper, we give an estimate for the rate of growth/decay of $g(t)$ as $t \to \infty$ in Theorem 1.

A discrete version of (1) is also studied, and the result is stated in Theorem 3. In Section 3 possible applications of inequality (1) to the study of large time behavior of solutions to some partial equations are outlined.

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2. Main results

THEOREM 1. *Let Assumption A) hold. Suppose there exists a function* $\mu = \mu(t) > 0,\ \mu \in C^1[t_0,\infty)$*, such that*

$$\alpha\left(t,\frac{1}{\mu(t)}\right)+\beta(t)\leqslant \frac{1}{\mu(t)}\left[\gamma(t)-\frac{\dot\mu(t)}{\mu(t)}\right], \qquad t\geqslant t_0. \tag{2}$$

Let $g(t)\geqslant 0$ *be a solution to inequality* (1) *such that*

$$\mu(t_0)g(t_0)<1. \tag{3}$$

Then $g(t)$ *exists globally and the following estimate holds:*

$$0\leqslant g(t)<\frac{1}{\mu(t)}, \qquad \forall t\geqslant t_0. \tag{4}$$

Consequently, if $\lim_{t\to\infty}\mu(t)=\infty$*, then*

$$\lim_{t\to\infty} g(t)=0. \tag{5}$$

If inequality (3) *is replaced by*

$$\mu(t_0)g(t_0)\leqslant 1, \tag{6}$$

then inequality (4) *is replaced by*

$$0\leqslant g(t)\leqslant\frac{1}{\mu(t)}, \qquad \forall t\geqslant t_0. \tag{7}$$

Proof. Denote

$$v(t):=g(t)e^{\int_{t_0}^t\gamma(s)ds}. \tag{8}$$

Then inequality (1) takes the form

$$\dot v(t)\leqslant a(t)\alpha\big(t,v(t)e^{-\int_{t_0}^t\gamma(s)ds}\big)+b(t), \qquad v(t_0)=g(t_0):=g_0, \tag{9}$$

where

$$a(t)=e^{\int_{t_0}^t\gamma(s)ds}, \qquad b(t):=\beta(t)e^{\int_{t_0}^t\gamma(s)ds}. \tag{10}$$

Denote

$$\eta(t)=\frac{e^{\int_{t_0}^t\gamma(s)ds}}{\mu(t)}. \tag{11}$$

From inequality (3) and relation (11) one gets

$$v(t_0)=g(t_0)<\frac{1}{\mu(t_0)}=\eta(t_0). \tag{12}$$

It follows from the inequalities (2), (9) and (12), and from the assumption that $\alpha(t,g)$ is non-decreasing with respect to g, that

$$\dot{v}(t_0) \leqslant \alpha(t_0, \frac{1}{\mu(t_0)}) + \beta(t_0) \leqslant \frac{1}{\mu(t_0)}\left[\gamma(t_0) - \frac{\dot{\mu}(t_0)}{\mu(t_0)}\right] = \frac{d}{dt}\frac{e^{\int_{t_0}^t \gamma(s)ds}}{\mu(t)}\Bigg|_{t=t_0} = \dot{\eta}(t_0). \tag{13}$$

From the inequalities (12) and (13) it follows that there exists $\delta > 0$ such that

$$v(t) < \eta(t), \qquad g(t) < \frac{1}{\mu(t)}, \qquad t_0 \leqslant t \leqslant t_0 + \delta. \tag{14}$$

To continue the proof we need two Claims.

Claim 1. If

$$v(t) \leqslant \eta(t), \qquad \forall t \in [t_0, T], \quad T > t_0, \tag{15}$$

then

$$\dot{v}(t) \leqslant \dot{\eta}(t), \qquad \forall t \in [t_0, T]. \tag{16}$$

Proof of Claim 1.

It follows from inequalities (2), (9), the non-decreasing of $\alpha(t,g)$ with respect to g, the inequality $g(t) < \frac{1}{\mu(t)}$ for $t \in [t_0, T)$, and the inequality $v(T) \leqslant \eta(T)$, that

$$\begin{aligned}
\dot{v}(t) &\leqslant e^{\int_{t_0}^t \gamma(s)ds}\alpha(t, \frac{1}{\mu(t)}) + \beta(t)e^{\int_{t_0}^t \gamma(s)ds} \\
&\leqslant \frac{e^{\int_{t_0}^t \gamma(s)ds}}{\mu(t)}\left[\gamma(t) - \frac{\dot{\mu}(t)}{\mu(t)}\right] \\
&= \frac{d}{dt}\frac{e^{\int_{t_0}^t \gamma(s)ds}}{\mu(t)}\Bigg|_{t=t} = \dot{\eta}(t), \qquad \forall t \in [t_0, T].
\end{aligned} \tag{17}$$

Claim 1 is proved. □

Denote

$$T := \sup\{\delta \in \mathbb{R}^+ : v(t) < \eta(t), \forall t \in [t_0, t_0 + \delta]\}. \tag{18}$$

Claim 2. One has $T = \infty$.

Claim 2 says that every non-negative solution $g(t)$ to inequality (1), satisfying assumption (3), is defined globally.

Proof of Claim 2.

Assume the contrary, that is, $T < \infty$. The solution $v(t)$ to (9) is continuous at every point t at which it is bounded. From the definition of T and the continuity of v and η one gets

$$v(T) \leqslant \eta(T). \tag{19}$$

It follows from inequalities (18), (19), and *Claim 1* that

$$\dot{v}(t) \leqslant \dot{\eta}(t), \qquad \forall t \in [t_0, T]. \tag{20}$$

This implies

$$v(T) - v(t_0) = \int_{t_0}^{T} \dot{v}(s)ds \leqslant \int_{t_0}^{T} \dot{\eta}(s)ds = \eta(T) - \eta(t_0). \tag{21}$$

Since $v(t_0) < \eta(t_0)$ by assumption (3), it follows from inequality (21) that

$$v(T) < \eta(T). \tag{22}$$

Inequality (22) and inequality (20) with $t = T$ imply that there exists a $\delta > 0$ such that

$$v(t) < \eta(t), \qquad T \leqslant t \leqslant T + \delta. \tag{23}$$

This contradicts the definition of T in (18), and the contradiction proves the desired conclusion $T = \infty$.

Claim 2 is proved. □

It follows from the definitions of $\eta(t)$, T, $v(t)$, and from the relation $T = \infty$, that

$$g(t) = e^{-\int_{t_0}^{t} \gamma(s)ds} v(t) < e^{-\int_{t_0}^{t} \gamma(s)ds} \eta(t) = \frac{1}{\mu(t)}, \qquad \forall t > t_0. \tag{24}$$

The last statement of Theorem 1 is proved by the standard argument, and is left to the reader.

Theorem 1 is proved. □

THEOREM 2. *Let Assumption A) hold and $\alpha(t,g)$ is a continuous function of t for every $g \geqslant 0$ and satisfies Lipschitz condition with respect to g for every $t \geqslant t_0$, $|\alpha(t,g) - \alpha(t,h)| \leqslant L(t)|g-h|$, where $L(t) < \infty$ for all $t \geqslant t_0$. Let $0 \leqslant g(t)$ satisfy* (1), *$0 < \mu(t)$ satisfy* (2)*, and $\mu(t_0)g(t_0) \leqslant 1$. Then*

$$g(t) \leqslant \frac{1}{\mu(t)}, \qquad \forall t \geqslant t_0. \tag{25}$$

Proof. Let $v(t)$ be defined in (8). Then inequality (9) holds. Let $w_n(t)$ solve the following differential equation

$$\dot{w}_n(t) = a(t)\alpha\big(t, w_n(t)e^{-\int_{t_0}^{t} \gamma(s)ds}\big) + b(t), \qquad w_n(t_0) = g(t_0) - \frac{1}{n}, \qquad n \geqslant n_0, \tag{26}$$

where $a(t) := e^{\int_{t_0}^{t} \gamma(s)ds}$, n_0 is sufficiently large, and $g(t_0) > \frac{1}{n_0}$. Since $\alpha(t,y)$ is continuous with respect to t and locally Lipschitz-continuous with respect to y, there exists a unique local solution to (26).

From the proof of Theorem 1 one gets

$$w_n(t) < \frac{e^{\int_{t_0}^{t} \gamma(s)ds}}{\mu(t)}, \qquad \forall t \geqslant t_0, \forall n \geqslant n_0. \tag{27}$$

Let τ, $t_0 < \tau < \infty$, be an arbitrary number, and

$$w(t) = \lim_{n\to\infty} w_n(t), \qquad \forall t \in [t_0,\tau]. \tag{28}$$

This and the fact that $w_n(t)$ is uniformly continuous on $[0,\tau]$ imply that $w(t)$ solves the following equation:

$$\dot{w}(t) = a(t)\alpha\big(t, w(t)e^{-\int_{t_0}^t \gamma(s)ds}\big) + b(t), \qquad w(t_0) = g(t_0), \qquad \forall t \in [0,\tau]. \tag{29}$$

Note that the solution $w(t)$ to (29) is unique since $\alpha(t,y)$ is continuous with respect to t and locally Lipschitz-continuous with respect to y. From (9), (29), a comparison lemma (see, e.g., [11], p.99), the continuity of $w_n(t)$ with respect to $w_0(t_0)$ on $[0,\tau]$, and (27), one gets

$$v(t) \leqslant w(t) \leqslant \frac{e^{\int_{t_0}^t \gamma(s)ds}}{\mu(t)}, \qquad \forall t \in [t_0,\tau],\ \forall n \geqslant n_0. \tag{30}$$

Since $\tau > t_0$ is arbitrary, inequality (25) follows from (30).

Theorem 2 is proved. □

REMARK 1. The results of Theorems 1,2 are closely related to the known comparison-type results in differential equations. The comparison lemmas, or lemmas about differential inequalities, are described in many books and papers, e.g., in [3], [7], [9], [13], [14], to mention a few.

In this remark we give an alternative proof of Theorem 1, which uses comparison results for differential inequalities. In this proof we have to assume that the Cauchy problem, corresponding to a differential equation, obtained from differential inequality (1), (see problem (31) below) has a unique solution. This assumption was not used in the proof of Theorem 1, given above. Uniqueness of the solution to this Cauchy problem holds, for example, if the function $a(t,g)$ satisfies Lipschitz condition with respect to g. Less restrictive conditions, e.g., one-sided inequalities, sufficient for the uniqueness of the solution to the Cauchy problem (31) are known (e.g., see [7]).

Let $\phi(t)$ solve the following Cauchy problem:

$$\dot{\phi}(t) = -\gamma(t)\phi(t) + \alpha(t,\phi(t)) + \beta(t), \quad t \geqslant t_0, \quad \phi(t_0) = \phi_0. \tag{31}$$

Inequality (2) can be written as

$$-\gamma(t)\mu^{-1}(t) + \alpha(t,\mu^{-1}(t)) + \beta(t) \leqslant \frac{d\mu^{-1}(t)}{dt}. \tag{32}$$

From the known comparison result (see, e.g., [3], Theorem III.4.1) it follows that

$$\phi(t) \leqslant \mu^{-1}(t) \quad \forall t \geqslant t_0, \tag{33}$$

provided that $\phi(t_0) \leqslant \mu^{-1}(t_0)$, where $\phi(t)$ is the *minimal* solution to problem (31).

Inequality (1) implies that

$$g(t) \leqslant \phi(t) \quad \forall t \geqslant t_0, \tag{34}$$

provided that $g(t_0) \leqslant \phi(t_0)$, where $\phi(t)$ is the *maximal* solution to problem (31).

Therefore, if problem (31) has at most one local solution, and

$$g(t_0) \leqslant \mu^{-1}(t_0), \tag{35}$$

then

$$g(t) \leqslant \mu^{-1}(t) \quad \forall t \geqslant t_0. \tag{36}$$

Since $\mu(t)$ is defined for all $t \geqslant t_0$, it follows that the solution to problem (31) with $\phi(t_0) = \mu^{-1}(t_0)$ is also defined for all $t \geqslant t_0$. Consequently, $g(t)$ is defined for all $t \geqslant t_0$. □

Let us consider a *discrete analog* of Theorem 1.

We obtain an upper bound for g_n as $n \to \infty$, sufficient conditions for the validity of the relation $\lim_{n\to\infty} g_n = 0$, and estimate the rate of growth/decay of g_n as $n \to \infty$. This result can be used in a study of evolution problems, in a study of iterative processes, and in a study of nonlinear PDE. Let us formulate the result.

THEOREM 3. *Let Assumption A) hold, g_n be a non-negative sequence of numbers, and β_n, γ_n be sequences of real numbers. Assume that*

$$\frac{g_{n+1} - g_n}{h_n} \leqslant -\gamma_n g_n + \alpha(n, g_n) + \beta_n, \qquad h_n > 0, \quad 0 < h_n\gamma_n < 1, \tag{37}$$

or, equivalently,

$$g_{n+1} \leqslant g_n(1 - h_n\gamma_n) + h_n\alpha(n, g_n) + h_n\beta_n, \qquad h_n > 0, \quad 0 < h_n\gamma_n < 1. \tag{38}$$

If there is a sequence of positive numbers $(\mu_n)_{n=1}^{\infty}$, such that the following conditions hold:

$$\alpha(n, \frac{1}{\mu_n}) + \beta_n \leqslant \frac{1}{\mu_n}\left(\gamma_n - \frac{\mu_{n+1} - \mu_n}{\mu_n h_n}\right), \tag{39}$$

$$g_0 \leqslant \frac{1}{\mu_0}, \tag{40}$$

then

$$0 \leqslant g_n \leqslant \frac{1}{\mu_n} \qquad \forall n \geqslant 0. \tag{41}$$

Therefore, if $\lim_{n\to\infty} \mu_n = \infty$, *then* $\lim_{n\to\infty} g_n = 0$.

Proof. Let us prove (41) by induction. Inequality (41) holds for $n = 0$ by assumption (40). Suppose that (41) holds for all $n \leqslant m$. From inequalities (37), (39), and from the induction hypothesis $g_n \leqslant \frac{1}{\mu_n}$, $n \leqslant m$, one gets

$$\begin{aligned} g_{m+1} &\leqslant g_m(1-h_m\gamma_m)+h_m\alpha(m,g_m)+h_m\beta_m \\ &\leqslant \frac{1}{\mu_m}(1-h_m\gamma_m)+h_m\alpha(m,\frac{1}{\mu_m})+h_m\beta_m \\ &\leqslant \frac{1}{\mu_m}(1-h_m\gamma_m)+\frac{h_m}{\mu_m}\left(\gamma_m-\frac{\mu_{m+1}-\mu_m}{\mu_m h_m}\right) \\ &= \frac{1}{\mu_{m+1}}-\frac{\mu_{m+1}^2-2\mu_{m+1}\mu_m+\mu_m^2}{\mu_m^2\mu_{m+1}} \leqslant \frac{1}{\mu_{m+1}}. \end{aligned} \tag{42}$$

Therefore, inequality (41) holds for $n = m+1$. Thus, inequality (41) holds for all $n \geqslant 0$ by induction. Theorem 3 is proved. □

Setting $h_n = 1$ in Theorem 3, one obtains the following result:

THEOREM 4. *Let the assumptions of Theorem 3 hold. and*

$$g_{n+1} \leqslant g_n(1-\gamma_n)+\alpha(n,g_n)+\beta_n, \qquad 0<\gamma_n<1. \tag{43}$$

If there is sequence $(\mu_n)_{n=1}^{\infty} > 0$ *such that the following conditions hold*

$$g_0 \leqslant \frac{1}{\mu_0}, \qquad \alpha(n,\frac{1}{\mu_n})+\beta_n \leqslant \frac{1}{\mu_n}\left(\gamma_n-\frac{\mu_{n+1}-\mu_n}{\mu_n}\right), \qquad \forall n \geqslant 0, \tag{44}$$

then

$$g_n \leqslant \frac{1}{\mu_n}, \qquad \forall n \geqslant 0. \tag{45}$$

3. Applications

Here we sketch an idea for possible applications of our inequalities in a study of dynamical systems in a Hilbert space H, see also [12].

In this Section we assume without loss of generality that $t_0 = 0$. Let

$$\dot{u}+Au = h(t,u)+f(t), \qquad u(0)=u_0, \quad \dot{u} := \frac{du}{dt}, \quad t \geqslant 0. \tag{46}$$

To explain the ideas, let us make simplifying assumptions: $A > 0$ is a selfadjoint time-independent operator in a real Hilbert space H, $h(t,u)$ is a nonlinear operator in H, locally Lipschitz with respect to u and continuous with respect to $t \in \mathbb{R}_+ := [0,\infty)$, and f is a continuous function on $\mathbb{R}_+$ with values in H, $\sup_{t\geqslant 0}\|f(t)\| < \infty$. The scalar product in H is denoted $\langle u,v\rangle$. Assume that

$$\langle Au,u\rangle \geqslant \gamma\langle u,u\rangle, \quad \gamma = const > 0, \quad \|h(t,u)\| \leqslant \alpha(t,\|u\|), \qquad \forall u \in D(A), \tag{47}$$

where $\alpha(t,y) \leqslant c|y|^p$, $p > 1$ and $c > 0$ are constants, and $\alpha(t,y)$ is a non-decreasing $C^1([0,\infty))$ function of y. Our approach works when $\gamma = \gamma(t)$ and $c = c(t)$, see Examples 1,2 below. The problem is to estimate the behavior of the solution to (46) as

$t \to \infty$ and to give sufficient conditions for a global existence of the unique solution to (46). Our approach consists of a reduction of this problem to the inequality (1) and an application of Theorem 1. A different approach, studied in the literature (see, e.g., [8], [10]), is based on the semigroup theory.

Let $g(t) := \|u(t)\|$. Problem (46) has a unique local solution under our assumptions. This local solution exists globally if $\sup_{t \geqslant 0} \|u(t)\| < \infty$. Multiply (46) by u and use (47) to get

$$\dot{g}g \leqslant -\gamma(t)g^2 + \alpha(t,g)g + \beta(t)g, \qquad \beta(t) := \|f(t)\|. \tag{48}$$

Since $g \geqslant 0$, one gets

$$\dot{g} \leqslant -\gamma(t)g + \alpha(t,g(t)) + \beta(t). \tag{49}$$

Now Theorem 1 is applicable and yields sufficient conditions (2) and (3) for the global existence of the solution to (46) and estimate (4) for the behavior of $\|u(t)\|$ as $t \to \infty$. The choice of $\mu(t)$ in Theorem 1 is often straightforward. For example, if $\alpha(t,g(t)) = \frac{c_0}{a(t)}g^2$, where $\lim_{t\to\infty} a(t) = 0$, $\dot{a}(t) < 0$, then one can often choose $\mu(t) = \frac{\lambda}{a(t)}$, $\lambda = const > 0$, see [11], p.116, and [4], p.487, for examples of applications of this approach.

The outlined approach is applicable to stability of the solutions to nonlinear differential equations, to semilinear parabolic problems, to hyperbolic problems, and other problems. There is a large literature on the stability of the solutions to differential equations (see, e.g., [1], [2], and references therein). Our approach yields some novel results. If the selfadjoint operator A depends on t, $A = A(t)$, and $\gamma = \gamma(t) > 0$, $\lim_{t\to\infty} \gamma(t) = 0$, one can treat problems with degenerate, as $t \to \infty$, elliptic operators A.

For instance, if the operator A is a second-order elliptic operator with matrix $a_{ij}(x,t)$, and the minimal eigenvalue $\lambda(x,t)$ of this matrix satisfies the condition

$$\min_x \lambda(x,t) := \gamma(t) \to 0, \quad \text{as} \quad t \to \infty,$$

then Theorem 1 is applicable under suitable assumptions on $\gamma(t)$, $h(t,u)$ and $f(t)$.

EXAMPLE 1. Consider

$$\dot{u} = -\gamma(t)u + a(t)u(t)|u(t)|^p + \frac{1}{(1+t)^q}, \qquad u(0) = 0, \tag{50}$$

where $\gamma(t) = \frac{c}{(1+t)^b}$, $a(t) = \frac{1}{(1+t)^m}$, p, q, b, c, and m are positive constants. Our goal is to give sufficient conditions for the solution to the above problem to converge to zero as $t \to \infty$. Multiply (50) by u, denote $g := u^2$, and get the following inequality

$$\dot{g} \leqslant -2\frac{c}{(1+t)^b}g + 2\frac{1}{(1+t)^m}g(t)^{1+0.5p} + 2\frac{1}{(1+t)^q}g^{0.5}, \qquad g = u^2. \tag{51}$$

Choose $\mu(t) = \lambda(1+t)^\nu$, where $\lambda > 0$ and $\nu > 0$ are constants.

Inequality (2) takes the form:

$$\frac{2}{(1+t)^m}[\lambda(1+t)^\nu]^{-1-0.5p} + \frac{2}{(1+t)^q}[\lambda(1+t)^\nu]^{-0.5} \leqslant [\lambda(1+t)^\nu]^{-1}\left(2\frac{c}{(1+t)^b} - \frac{\nu}{1+t}\right). \tag{52}$$

Choose p, q, m, c, λ and ν so that inequality (52) be valid and $\lambda u(0)^2 < 1$, so that condition (3) with $t_0 = 0$ holds. If this is done, then $u^2(t) \leqslant \frac{1}{\lambda(1+t)^\nu}$, so $\lim_{t\to\infty} u(t) = 0$. For example, choose $b = 1$, $\nu = 1$, $q = 1.5$, $m = 1$, $\lambda = 4$, $c = 4$, $p \geqslant 1$. Then inequality (52) is valid, and if $u(0)^2 < 1/4$, then (3) with $t_0 = 0$ holds, so $\lim_{t\to\infty} u(t) = 0$. The choice of the parameters can be varied. In particular, the nonlinearity growth, governed by p, can be arbitrary in power scale. If $b = 1$ then three inequalities $m + 0.5p\nu \geqslant 1$, $q - 0.5\nu \geqslant 1$, and $\lambda^{1/2} + \lambda^{-0.5p} \leqslant c - 0.5\nu$ together with $u(0)^2 < \lambda^{-1}$ are sufficient for (3) and (52) to hold, so they imply $\lim_{t\to\infty} u(t) = 0$.

In Example 1 one could use the argument, given in Remark 1.

EXAMPLE 2. Consider problem (46) with A, h and f satisfying (47) with $\gamma \equiv 0$. So one gets inequality (49) with $\gamma(t) \equiv 0$. Choose

$$\mu(t) := c + \lambda(1+t)^{-b}, \qquad c > 0, b > 0, \lambda > 0, \tag{53}$$

where c, λ, and b are constants. Inequality (2) takes the form:

$$\alpha(t, \frac{1}{\mu(t)}) + \beta(t) \leqslant \frac{1}{\mu(t)} \frac{b\lambda}{(1+t)[\lambda + c(1+t)^b]}. \tag{54}$$

Let $\theta \in (0,1)$, $p > 0$, and $C > 0$ be constants. Assume that

$$\alpha(t, |y|) \leqslant \theta C |y|^p \frac{b\lambda}{(\lambda+c)(1+t)^{1+b}}, \qquad \beta(t) \leqslant (1-\theta)\frac{b\lambda}{(c+\lambda)^2(1+t)^{1+b}}, \tag{55}$$

for all $t \geqslant 0$, and

$$C = \begin{cases} c^{p-1} & \text{if} \quad p > 1, \\ (\lambda + c)^{p-1} & \text{if} \quad p \leqslant 1. \end{cases} \tag{56}$$

Let us verify that inequality (54) holds given that (55) and (56) hold.

It follows from (53) that $c < \mu(t) \leqslant c + \lambda$, $\forall t \geqslant 0$. This and (55) imply

$$\beta(t) \leqslant (1-\theta)\frac{1}{(c+\lambda)^2(1+t)^{1+b}} \leqslant (1-\theta)\frac{1}{\mu(t)} \frac{1}{(1+t)(c+\lambda(1+t)^b)}. \tag{57}$$

From (56) and (53) one gets

$$\frac{C}{\mu^{p-1}(t)} \leqslant C \max(c^{1-p}, (c+\lambda)^{1-p}) \leqslant 1, \qquad \forall t \geqslant 0. \tag{58}$$

From (55) and (58) one obtains

$$\begin{aligned}\alpha(t,\frac{1}{\mu(t)}) &\leqslant \theta C\frac{1}{\mu(t)}\frac{1}{\mu^{p-1}(t)}\frac{b\lambda}{(1+t)[\lambda(1+t)^b+c(1+t)^b]}\\ &\leqslant \theta\frac{1}{\mu(t)}\frac{b\lambda}{(1+t)[\lambda+c(1+t)^b]}.\end{aligned} \tag{59}$$

Inequality (54) follows from (57) and (59). From (54) and Theorem 1 one obtains

$$g(t)\leqslant\frac{1}{\mu(t)}<\frac{1}{c},\qquad \forall t>0, \tag{60}$$

provided that $g(0)<(c+\lambda)^{-1}$. From (49) with $\gamma(t)=0$ and (55)–(60), one gets $\dot g(t)=O(\frac{1}{(1+t)^{1+b}})$. Thus, there exists finite limit $\lim_{t\to\infty}g(t)=g(\infty)\leqslant c^{-1}$.

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