Mathematical
Inequalities
& Applications
Volume 15, Number 1 (2012), 1–14

THE MULTIDIMENSIONAL REVERSE HARDY INEQUALITIES

A. GOGATISHVILI AND R. CH. MUSTAFAYEV

(*Communicated by L.-E. Persson*)

Abstract. In this paper we characterize the validity of the multidimensional reverse Hardy inequalities

$$\|gw\|_{L_p(\mathbb{R}^n)} \leqslant C \left\| v(t) \int_{\complement B(0,t)} g(y)dy \right\|_{L_q(0,+\infty)}$$

and

$$\|gw\|_{L_p(\mathbb{R}^n)} \leqslant C \left\| v(t) \int_{B(0,t)} g(y)dy \right\|_{L_q(0,+\infty)}$$

for non-negative measurable functions on $\mathbb{R}^n$, where $B(0,t)$ is the closed ball in $\mathbb{R}^n$ centered at zero with radius t, $\complement B(0,t) = \mathbb{R}^n \setminus B(0,t)$, $0 < p \leqslant 1$, $0 < q \leqslant +\infty$, w and v are weight functions on $\mathbb{R}^n$ and $(0,+\infty)$, respectively. Obtained conditions are the natural extensions of one-dimensional conditions.

1. Introduction

The characterization of weights w, v for which the one-dimensional Hardy inequalities

$$\left(\int_a^b \left(w(x) \int_x^b f(t)\,dt \right)^q dx \right)^{\frac{1}{q}} \leqslant C \left(\int_a^b (v(x)f(x))^p\,dx \right)^{\frac{1}{p}} \tag{1.1}$$

and

$$\left(\int_a^b \left(w(x) \int_a^x f(t)\,dt \right)^q dx \right)^{\frac{1}{q}} \leqslant C \left(\int_a^b (v(x)f(x))^p\,dx \right)^{\frac{1}{p}} \tag{1.2}$$

hold for every non-negative Borel measurable functions f on the interval (a,b), $-\infty \leqslant a < b \leqslant +\infty$, $0 < q \leqslant +\infty$, $1 \leqslant p \leqslant +\infty$ have been extensively studied during the last two decades. A detailed account of the history of the topic can be found in the book [5] (see also [9] and [6]).

Mathematics subject classification (2010): 26D10, 26D15, 46E30.

Keywords and phrases: Multidimensional Hardy operator, Hardy inequality, reverse Hardy inequality, discretization.

The research of A. Gogatishvili was partially supported by the grant 201/08/0383 of the Grant Agency of the Czech Republic and by the Institutional Research Plan no. AV0Z10190503 of AS CR. The research of R.Ch. Mustafayev was partialy supported by a Post Doctoral Fellowship of INTAS (Grant 06-1000015-6385) and by the Institutional Research Plan no. AV0Z10190503 of AS CR.

In [2], W. D. Evans, A. Gogatishvili and B. Opic give a complete characterization of weights w and v on (a,b) for which so-called reverse Hardy inequalities

$$\left(\int_a^b \left(w(x)\int_x^b f(t)\,dt\right)^q dx\right)^{\frac{1}{q}} \geqslant C\left(\int_a^b (v(x)f(x))^p\,dx\right)^{\frac{1}{p}} \tag{1.3}$$

and

$$\left(\int_a^b \left(w(x)\int_a^x f(t)\,dt\right)^q dx\right)^{\frac{1}{q}} \geqslant C\left(\int_a^b (v(x)f(x))^p\,dx\right)^{\frac{1}{p}} \tag{1.4}$$

hold for every non-negative Borel measurable functions f on the interval (a,b), with $-\infty \leqslant a < b \leqslant +\infty$, $0 < q \leqslant +\infty$, $0 < p \leqslant 1$. In [2] the inequalities (1.3) and (1.4) were considered for non-negative Borel measures on (a,b) and above mentioned inequalities are a special case if we take an absolute continuous measures with respect to Lebesgue measure.

In [1], P. Drábek, H. P. Heinig, A. Kufner extended the inequalities (1.1) and (1.2) to n-dimensional case. In particular, in [1] it was shown that a necessary and sufficient conditions for validity of the inequalities

$$\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n\setminus B(0,|x|)} f(x)\right)^q u(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_{\mathbb{R}^n}(f(x))^p v(x)dx\right)^{\frac{1}{p}}$$

and

$$\left(\int_{\mathbb{R}^n}\left(\int_{B(0,|x|)} f(x)\right)^q u(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_{\mathbb{R}^n}(f(x))^p v(x)dx\right)^{\frac{1}{p}}$$

for every non-negative Borel measurable functions f on $\mathbb{R}^n$, with weights u,v on $\mathbb{R}^n$, $1 < p < +\infty$, $0 < q < +\infty$, are analogous to the corresponding conditions for the one-dimensional case.

In this paper, we will deal with multidimensional analogue of reverse Hardy inequalities (1.3) and (1.4). We make a comprehensive study of general inequalities of the form

$$\|gw\|_{L_p(\mathbb{R}^n)} \leqslant C\left\|v(t)\int_{\complement B(0,t)} g(y)dy\right\|_{L_q(0,+\infty)} \tag{1.5}$$

and

$$\|gw\|_{L_p(\mathbb{R}^n)} \leqslant C\left\|v(t)\int_{B(0,t)} g(y)dy\right\|_{L_q(0,+\infty)} \tag{1.6}$$

with complete proofs and estimates of the best constants C.

The basic idea in [1] was to use polar coordinates in $\mathbb{R}^n$. Our method to study the inequalities (1.5) and (1.6) is different from the method used in [1]. Our approach for the inequality (1.5) based of the n-dimensional analogues of a discretization of function norms as in [2]. The discretization technique was investigated by K.-G. Grosse-Erdmann [4], where it is called the blocking technique. In [4] the discrete analogues of (1.3) and (1.4) were considered and it is also remarked that the techniques used in the proofs may be applicable to the continuous versions of the inequalities, namely to (1.3) and (1.4).

By changing variables we reduce the characterization of the inequality (1.6) to that of the inequality (1.5). But it should be noted that the discretization method works also for this inequality.

The paper is organized as follows. Main results, the necessary and sufficient conditions for the validity of the inequality (1.5) and (1.6), are formulated in Section 2. In Section 3, we give general discretization formulas of weighted function norms. In section 4, proofs of main results are given.

2. Main results

We start with some notations. Given a nonempty Borel subset E of $\mathbb{R}^n$ and f is a Lebesgue measurable function on E, then we put

$$\|f\|_{L_p(E)} := \begin{cases} (\int_E |f(y)|^p dy)^{\frac{1}{p}} & \text{if} \quad 0 < p < +\infty, \\ \operatorname{esssup}_{y\in E} |f(y)| & \text{if} \quad p = \infty. \end{cases}$$

If $I = (a,b)$ is a nonempty interval from $(0,+\infty)$ and f is a measurable function on I, then we define $\|f\|_{L_p(a,b)} = \|f\|_{L_p(I)}$ and $\|f\|_{L_\infty(a,b)} = \|f\|_{L_\infty(I)}$.

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r) := \{y \in \mathbb{R}^n : |x-y| \leqslant r\}$ be the closed ball centered at x of radius r and ${}^{\complement}B(x,r) := \mathbb{R}^n \backslash B(x,r)$.

Throughout the paper, u, v and w will denote weights, that is, locally integrable non-negative functions.

By $A \lesssim B$ we mean that $A \leqslant CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B *are equivalent*. We shall use throughout the paper the convention $1/(+\infty) = 0$, $0 \cdot (\pm\infty) = 0$, $0/0 = 0$, and $\infty/\infty = 0$. We put

$$p' := \begin{cases} \frac{p}{1-p} & \text{if} \quad 0 < p < 1, \\ +\infty & \text{if} \quad p = 1, \\ \frac{p}{p-1} & \text{if} \quad 1 < p < +\infty, \\ 1 & \text{if} \quad p = +\infty, \end{cases}$$

When $p < q$, we define r by

$$\frac{1}{r} = \frac{1}{p} - \frac{1}{q}. \tag{2.1}$$

Our first main result describes the necessary and sufficient condition for the validity of inequality (1.5) for all non-negative measurable g on $\mathbb{R}^n$ when $0 < q \leqslant p \leqslant 1$.

THEOREM 2.1. *Assume that* $0 < q \leqslant p \leqslant 1$. *Let* w *and* v *be weight functions on* $\mathbb{R}^n$ *and* $(0,\infty)$, *respectively. Let* $\|v\|_{L_q(0,t)} < +\infty$ *for all* $t \in (0,\infty)$. *Then the inequality* (1.5) *holds for all non-negative measurable* g *if and only if*

$$A_1 := \sup_{t\in(0,\infty)} \|w\|_{L_{p'}(B(0,t))} \|v\|_{L_q(0,t)}^{-1} < +\infty.$$

The best possible constant C *in* (1.5) *satisfies* $C \approx A_1$.

Our next result concerns the characterization of the inequality (1.5) when $0 < p \leqslant 1$, $p < q \leqslant +\infty$.

THEOREM 2.2. *Assume that* $0 < p \leqslant 1$, $p < q \leqslant +\infty$ *and* r *is given by* (2.1). *Let* w *and* v *be weight functions on* $\mathbb{R}^n$ *and* $(0,+\infty)$, *respectively. Let* v *satisfies* $\|v\|_{L_q(0,t)} < +\infty$ *for all* $t \in (0,+\infty)$ *and* $v \neq 0$ *a.e. on* $(0,+\infty)$. *Then the inequality* (1.5) *holds for all non-negative measurable* g *on* $\mathbb{R}^n$ *if and only if*

$$A_2 := \left(\int_{(0,+\infty)} \|w\|_{L_{p'}(B(0,t))}^r \, d\left(-\|v\|_{L_q(0,t)}^{-r} \right) \right)^{\frac{1}{r}} + \frac{\|w\|_{L_{p'}(\mathbb{R}^n)}}{\|v\|_{L_q(0,+\infty)}} < +\infty.$$

The best possible constant C *in* (1.5) *satisfies* $C \approx A_2$.

Our next assertion is a counterpart of Theorem 2.1 and concerns the characterization of the inequality (1.6) when $0 < q \leqslant p \leqslant 1$.

THEOREM 2.3. *Assume that* $0 < q \leqslant p \leqslant 1$. *Let* w *and* v *be weight functions on* $\mathbb{R}^n$ *and* $(0,+\infty)$, *respectively. Let* $\|v\|_{L_q(t,+\infty)} < +\infty$ *for all* $t \in (0,+\infty)$. *Then the inequality* (1.6) *holds for all non-negative measurable functions* g *on* $\mathbb{R}^n$ *if and only if*

$$B_1 := \sup_{t\in(0,\infty)} \|w\|_{L_{p'}({}^{\complement}B(0,t))} \|v\|_{L_q(t,+\infty)}^{-1} < +\infty. \tag{2.2}$$

The best possible constant C *in* (1.6) *satisfies* $C \approx B_1$.

Our last result concerns to the characterization of the inequality (1.6) when $0 < p \leqslant 1$, $p < q \leqslant +\infty$.

THEOREM 2.4. *Assume that* $0 < p \leqslant 1$, $p < q \leqslant +\infty$ *and* r *is given by* (2.1). *Let* w *and* v *be weight functions on* $\mathbb{R}^n$ *and* $(0,+\infty)$, *respectively. Let* v *satisfy* $\|v\|_{L_q(t,\infty)} < +\infty$ *for all* $t \in (0,+\infty)$ *and* $v \neq 0$ *a.e. on* $(0,+\infty)$. *Then the inequality* (1.6) *holds for all non-negative measurable functions* g *on* $\mathbb{R}^n$ *if and only if*

$$B_2 := \left(\int_{(0,+\infty)} \|w\|_{L_{p'}({}^{\complement}B(0,t))}^r \, d\left(\|v\|_{L_q(t,\infty)}^{-r} \right) \right)^{\frac{1}{r}} + \frac{\|w\|_{L_{p'}(\mathbb{R}^n)}}{\|v\|_{L_q(0,+\infty)}} < +\infty.$$

The best possible constant C *in* (1.6) *satisfies* $C \approx B_2$.

3. Discretization of function norms

We start with some basic definitions. We follow [2].
Let $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$.

DEFINITION 3.1. Let $N, M \in \overline{\mathbb{Z}}$, $N < M$. A positive non-increasing sequence $\{\tau_k\}_{k=N}^{M}$ is called *almost geometrically decreasing* if there are $\alpha \in (1, +\infty)$ and $L \in \mathbb{N}$ such that

$$\tau_k \leqslant \frac{1}{\alpha} \tau_{k-L} \quad \text{for all} \quad k \in \{N+L, \ldots, M\}.$$

A positive non-decreasing sequence $\{\sigma_k\}_{k=N}^{M}$ is called *almost geometrically increasing* if there are $\alpha \in (1, +\infty)$ and $L \in \mathbb{N}$ such that

$$\sigma_k \geqslant \alpha \sigma_{k-L} \quad \text{for all} \quad k \in \{N+L, \ldots, M\}.$$

REMARK 3.2. Definition 3.1 implies that if $0 < q < +\infty$, then the following three statements are equivalent:
(i) $\{\tau_k\}_{k=N}^{M}$ is an almost geometrically decreasing sequence;
(ii) $\{\tau_k^q\}_{k=N}^{M}$ is an almost geometrically decreasing sequence;
(iii) $\{\tau_k^{-q}\}_{k=N}^{M}$ is an almost geometrically increasing sequence.

Let $N, M \in \overline{\mathbb{Z}}$, $N \leqslant M$, $0 < q \leqslant +\infty$ and let $\{w_k\} = \{w_k\}_{k=N}^{M}$ be a sequence of positive numbers. We denote by $\ell^q(\{w_k\}, N, M)$ the following discrete analogue of a weighted Lebesgue space: if $0 < q < +\infty$, then

$$\ell^q(\{w_k\}, N, M) = \Big\{ \{a_k\}_{k=N}^{M} : \quad \|a_k\|_{\ell^q(\{w_k\}, N, M)} := \Big(\sum_{k=N}^{M} |a_k w_k|^q \Big)^{\frac{1}{q}} < +\infty \Big\}$$

and

$$\ell^\infty(\{w_k\}, N, M) = \Big\{ \{a_k\}_{k=N}^{M} : \quad \|a_k\|_{\ell^\infty(\{w_k\}, N, M)} := \sup_{N \leqslant k \leqslant M} |a_k w_k| < +\infty \Big\}.$$

If $w_k = 1$ for all $N \leqslant k \leqslant M$, we write simply $\ell^q(N, M)$ instead of $\ell^q(\{w_k\}, N, M)$.

We quote some known results. Proofs can be found in [7] and [8].

LEMMA 3.3. *Let $N, M \in \overline{\mathbb{Z}}$, $N \leqslant M$. Then, for any positive sequence $\{\tau_k\}_{k=N}^{M}$ and all $m \in \overline{\mathbb{Z}}$ satisfying $N < m < M$,*

$$\sum_{k=m}^{M} \tau_k \lesssim \tau_m \tag{3.1}$$

or

$$\sum_{k=N}^{m} \tau_k \lesssim \tau_m \tag{3.2}$$

if and only if the sequence $\{\tau_k\}_{k=N}^{M}$ is almost geometrically decreasing or increasing, respectively.

LEMMA 3.4. *Let $q \in (0,+\infty]$, $N,M \in \overline{\mathbb{Z}}$, $N \leqslant M$ and let $\{\tau_k\}_{k=N}^{M}$ be an almost geometrically decreasing sequence. Then*

$$\left\| \tau_k \sum_{m=N}^{k} a_m \right\|_{\ell^q(N,M)} \approx \|\tau_k a_k\|_{\ell^q(N,M)} \tag{3.3}$$

and

$$\|\tau_k \sup_{N \leqslant m \leqslant k} a_m\|_{\ell^q(N,M)} \approx \|\tau_k a_k\|_{\ell^q(N,M)} \tag{3.4}$$

for all non-negative sequences $\{a_k\}_{k=N}^{M}$.

LEMMA 3.5. *Let $q \in (0,+\infty]$, $N,M \in \overline{\mathbb{Z}}$, $N \leqslant M$ and let $\{\sigma_k\}_{k=N}^{M}$ be an almost geometrically increasing sequence. Then*

$$\left\| \sigma_k \sum_{m=k}^{M} a_m \right\|_{\ell^q(N,M)} \approx \|\sigma_k a_k\|_{\ell^q(N,M)} \tag{3.5}$$

and

$$\|\sigma_k \sup_{k \leqslant m \leqslant M} a_m\|_{\ell^q(N,M)} \approx \|\sigma_k a_k\|_{\ell^q(N,M)} \tag{3.6}$$

for all non-negative sequences $\{a_k\}_{k=N}^{M}$.

If φ is a non-negative and monotone function on (a,b), then by $\varphi(a)$ and $\varphi(b)$ we mean the values $\varphi(a+) := \lim_{t \to a+} \varphi(t)$ and $\varphi(b-) := \lim_{t \to b-} \varphi(t)$, respectively.

LEMMA 3.6. ([2, Lemma 3.1]) *Let φ be a non-negative, non-decreasing, finite and right-continuous function on (a,b). There is a strictly increasing sequence $\{x_k\}_{k=N}^{M+1}$, $-\infty \leqslant N \leqslant M \leqslant +\infty$, with elements from the closure of the interval (a,b), such that:*

(i) *if $N > -\infty$, then $\varphi(x_N) > 0$ and $\varphi(x) = 0$ for every $x \in (a,x_N)$; if $M < +\infty$, then $x_{M+1} = b$;*

(ii) $\varphi(x_{k+1}-) \leqslant 2\varphi(x_k)$ *if* $N \leqslant k \leqslant M$*;*

(iii) $2\varphi(x_k-) \leqslant \varphi(x_{k+1})$ *if* $N < k < M$*.*

DEFINITION 3.7. ([2]) Let φ be a non-negative, non-decreasing, finite and right-continuous function on (a,b). A strictly increasing sequence $\{x_k\}_{k=N}^{M+1}$, $-\infty \leqslant N < M \leqslant +\infty$, is said to be *a discretizing sequence of the function* φ if it satisfies the conditions (i) – (iii) of Lemma 3.6.

REMARK 3.8. ([2]) We shall use the following *convention*: if $N = -\infty$, then we put $x_N = \lim_{k \to -\infty} x_k$. It is clear that if $N = -\infty$ and $x_N > a$, then $\varphi(x) = 0$ for all $x \in (a,x_N)$ (cf. condition (i) of Lemma 3.6).

THEOREM 3.9. ([2, Theorem 3.4]) *Let ν be a non-negative Borel measure on $I=(a,b)$ such that the function $\varphi(t)=\nu(a,t]$ is finite on I. If $\{x_k\}_{k=N}^{M+1}$ is a discretizing sequence of the function φ, then*

$$\int_{(a,b)} h(t)\,d\nu(t) \approx \sum_{k=N}^{M} h(x_k)\nu(a,x_k] \tag{3.7}$$

for all non-negative and non-increasing functions h on I.

THEOREM 3.10. ([2, Theorem 3.5]) *Let $I=(a,b)$ and u be a weight function on I such that the function $\|u\|_{\infty,(a,t]}<+\infty$ for all $t\in I$. If $\{x_k\}_{k=N}^{M+1}$ is a discretizing sequence of the function $\varphi(t)=\|u\|_{L_\infty((a,t+])}$, $t\in I$, then*

$$\|hu\|_{L_\infty((a,b))} \approx \sup_{N\leqslant k\leqslant M} h(x_k)\|u\|_{L_\infty((a,x_k+])} \tag{3.8}$$

for all non-negative, non-increasing and right-continuous functions h on I.

Let φ be a non-negative, non-decreasing, finite and right-continuous function on $(0,\infty)$. Using a discretizing sequence $\{x_k\}_{k=N}^{M+1}$ of φ, we define the sequence $\{J_k\}_{k=N}^{M}$ and $\{S_k\}_{k=N}^{M}$ as follows:

$$J_i=(x_i,x_{i+1}],\ \text{if } N\leqslant i<M,\ \text{and } J_M=(x_M,\infty)\ \text{if } M<+\infty. \tag{3.9}$$

$$\begin{aligned} S_i&=B(0,x_{i+1})\backslash B(0,x_i), \quad \text{if} \quad N\leqslant i<M, \quad \text{and}\\ S_M&={}^{\complement}B(0,x_M) \quad \text{if} \quad M<+\infty. \end{aligned} \tag{3.10}$$

THEOREM 3.11. *Let $0<q<+\infty$. Suppose that v be a weight function on $(0,\infty)$. Let v be such that the function $\varphi(t)=\|v\|_{L_q(0,t)}^q$ is finite on $(0,\infty)$. If $\{x_k\}_{k=N}^{M+1}$ is a discretizing sequence of φ, then*

$$\left\|v(t)\int_{{}^{\complement}B(0,t)} g(y)\,dy\right\|_{L_q(0,\infty)} \approx \left(\sum_{k=N}^{M}\left(\int_{S_k} g(y)\,dy\right)^q \|v\|_{L_q(0,x_k)}^q\right)^{\frac{1}{q}} \tag{3.11}$$

and

$$\left\|v(t)\|g\|_{L_\infty({}^{\complement}B(0,t))}\right\|_{L_q(0,\infty)} \approx \left(\sum_{k=N}^{M}\|g\|_{L_\infty(S_k)}^q\|v\|_{L_q(0,x_k)}^q\right)^{\frac{1}{q}} \tag{3.12}$$

for all non-negative measurable g on $\mathbb{R}^n$, where $\{S_k\}_{k=N}^{M}$ is defined by (3.10)*.*

Proof. We prove (3.11) only (the proof of (3.12) is analogous). By Theorem 3.9,

$$\begin{aligned} \left\|v(t)\int_{{}^{\complement}B(0,t)} g(y)\,dy\right\|_{L_q(0,+\infty)} &\approx \left(\sum_{k=N}^{M}\left(\int_{{}^{\complement}B(0,x_k)} g(y)\,dy\right)^q \|v\|_{L_q(0,x_k)}^q\right)^{\frac{1}{q}}\\ &= \left(\sum_{k=N}^{M}\left(\sum_{i=k}^{M}\int_{S_k} g(y)\,dy\right)^q \|v\|_{L_q(0,x_k)}^q\right)^{\frac{1}{q}}. \end{aligned}$$

The condition (iii) of Lemma 3.6 implies that $\{\|v\|^q_{L_q(0,x_k)}\}_{k=N}^M$ is an almost geometrically increasing sequence. (We can take $\alpha = L = 2$ in Definition 3.1. Indeed, by the monotonicity of φ and the condition (iii) of Lemma 3.6, $2\varphi(x_{k-1}) \leqslant 2\varphi(x_k-) \leqslant \varphi(x_{k+1})$ if $N < k < M$, and, on putting $k-1 = m-2$, we arrive at $2\varphi(x_{m-2}) \leqslant \varphi(x_m)$ if $N+2 \leqslant m \leqslant M$.) Thus $\{\|v\|_{L_q(0,x_k)}\}_{k=N}^M$ is also an almost geometrically increasing sequence and (3.11) follows by applying Lemma 3.5. □

THEOREM 3.12. *Suppose that v be a weight function on $(0,\infty)$. Let v be such that the function $\varphi(t) = \|v\|_{L_\infty(0,t)}$ is finite on $(0,\infty)$. If $\{x_k\}_{k=N}^{M+1}$ is a discretizing sequence of the function $\varphi(t) = \|v\|_{L_\infty(a,t+)} := \lim_{s\to t+} \|v\|_{L_\infty(a,s)}$, $t \in (0,\infty)$, then*

$$\left\| v(t) \int_{\complement B(0,t)} g(y)\,dy \right\|_{L_\infty(0,\infty)} \approx \sup_{N \leqslant k \leqslant M} \left(\int_{S_k} g(y)\,dy \right) \|v\|_{L_\infty(0,x_k+)} \tag{3.13}$$

and

$$\left\| v(t) \|g\|_{L_\infty(\complement B(0,t))} \right\|_{L_\infty(0,\infty)} \approx \sup_{N \leqslant k \leqslant M} \|g\|_{L_\infty(S_k)} \|v\|_{L_\infty(0,x_k+)} \tag{3.14}$$

for all non-negative measurable g on $\mathbb{R}^n$, where $\{S_k\}_{k=N}^M$ is defined by (3.10).

Proof. This follows from Theorem 3.10 and Lemma 3.5. □

4. Proofs

In the proof of the necessity parts of our main theorems we will need the following two lemmas which are discrete versions of the classical Landau resonance theorems. Proofs can be found, for example, in [3].

LEMMA 4.1. *Let $0 < p \leqslant q \leqslant +\infty$, $N, M \in \overline{\mathbb{Z}}$, $N \leqslant M$ and let $\{v_k\}_{k=N}^M$ and $\{w_k\}_{k=N}^M$ be two sequences of positive numbers. Assume that, there is a constant $C > 0$ such that*

$$\|a_k\|_{\ell^q(\{w_k\},N,M)} \leqslant C \|a_k\|_{\ell^p(\{v_k\},N,M)} \tag{4.1}$$

for every sequence $\{a_k\}$. Then

$$\|\{w_k v_k^{-1}\}\|_{\ell^\infty(N,M)} \leqslant C, \tag{4.2}$$

LEMMA 4.2. *Let $0 < q < p \leqslant +\infty$, $N, M \in \overline{\mathbb{Z}}$, $N \leqslant M$ and let $\{v_k\}_{k=N}^M$ and $\{w_k\}_{k=N}^M$ be two sequences of positive numbers. Assume that* (4.1) *holds. Then*

$$\|\{w_k v_k^{-1}\}\|_{\ell^r(N,M)} \leqslant C, \tag{4.3}$$

where $1/r := 1/q - 1/p$

Proof of Theorem 2.1. Let $0 < q \leqslant 1$. By Theorem 3.11,

$$\left\| v(t) \int_{\complement B(0,t)} g(y)\,dy \right\|_{L_q(0,\infty)} \approx \left(\sum_{k=N}^{M} \left(\int_{S_k} g(y)\,dy \right)^q \|v\|^q_{L_q(0,x_k)} \right)^{\frac{1}{q}} \tag{4.4}$$

for all non-negative measurable g on $\mathbb{R}^n$, where $\{x_k\}_{k=N}^{M+1}$ is a discretizing sequence of the function $\varphi(t) = \|v\|^q_{L_q(0,t)}$, $t \in (0,\infty)$, and $\{S_k\}_{k=N}^{M}$ is defined by (3.10). By Lemma 3.6 (cf. also Remark 3.8),

$$\begin{aligned} &\text{if} \quad x_N > 0, \quad \text{then} \quad \|v\|_{L_q(0,x_N)} = 0; \\ &\text{if} \quad M < +\infty, \quad \text{then} \quad x_{M+1} = \infty; \end{aligned} \tag{4.5}$$

$$\|v\|^q_{L_q(0,x_{k+1})} \leqslant 2\|v\|^q_{L_q(0,x_k)} \quad \text{if} \quad N \leqslant k \leqslant M; \tag{4.6}$$

$$2\|v\|^q_{L_q(0,x_k)} \leqslant \|v\|^q_{L_q(0,x_{k+1})} \quad \text{if} \quad N < k < M. \tag{4.7}$$

Assume that $A_1 < +\infty$. This condition and (4.5) imply that

$$\|w\|_{L_{p'}(B(0,x_N))} = 0 \quad \text{if} \quad x_N > 0, \tag{4.8}$$

thus $w = 0$ a.e. in $B(0,x_N)$, consequently, $\|w\|_{L_p(B(0,x_N))} = 0$ if $x_N > 0$. Therefore,

$$\|gw\|_{L_p(\mathbb{R}^n)} = \left(\sum_{k=N}^{M} \|gw\|^p_{L_p(S_k)} \right)^{\frac{1}{p}} \tag{4.9}$$

for any non-negative measurable g on $\mathbb{R}^n$.

If E is a measurable subset of $(0,+\infty)$ and g is a non-negative measurable function on $(0,+\infty)$, then by Hőlder's inequality (with the exponents $1/p$ and p'/p),

$$\|gw\|^p_{L_p(E)} \leqslant \|g\|^p_{L_1(E)} \|w\|^p_{Lp'(E)}. \tag{4.10}$$

Identity (4.9) and (4.10) (with $E = S_k$, $N \leqslant k \leqslant M$) give

$$\begin{aligned} \|gw\|_{L_p(\mathbb{R}^n)} &\leqslant \left(\sum_{k=N}^{M} \|g\|^p_{L_1(S_k)} \|w\|^p_{L_{p'}(S_k)} \right)^{\frac{1}{p}} \\ &\leqslant \left(\sup_{N \leqslant k \leqslant M} \|w\|_{L_{p'}(S_k)} \|v\|^{-1}_{L_q(0,x_k)} \right) \left(\sum_{k=N}^{M} \|g\|^p_{L_1(S_k)} \|v\|^p_{L_q(0,x_k)} \right)^{\frac{1}{p}}. \end{aligned}$$

Moreover, using the inequality $0 < q/p \leqslant 1$ and (4.4), we arrive at

$$\begin{aligned} \|gw\|_{L_p(\mathbb{R}^n)} &\leqslant \left(\sup_{N \leqslant k \leqslant M} \|w\|_{L_{p'}(S_k)} \|v\|^{-1}_{L_q(0,x_k)} \right) \left(\sum_{k=N}^{M} \|g\|^q_{L_1(S_k)} \|v\|^q_{L_q(0,x_k)} \right)^{\frac{1}{q}} \\ &\approx \left(\sup_{N \leqslant k \leqslant M} \|w\|_{L_{p'}(S_k)} \|v\|^{-1}_{L_q(0,x_k)} \right) \left\| v(x) \int_{\complement B(0,x)} g(y)\,dy \right\|_{L_q(0,\infty)}. \end{aligned} \tag{4.11}$$

Applying (4.6), we get

$$\sup_{N\leqslant k\leqslant M} \|w\|_{L_{p'}(S_k)} \|v\|^{-1}_{L_q(0,x_k)} \leqslant 2^{\frac{1}{q}} \sup_{N\leqslant k\leqslant M} \|w\|_{L_{p'}(B(0,x_{k+1}))} \|v\|^{-1}_{L_q(0,x_{k+1})} \leqslant 2^{\frac{1}{q}} A_1. \tag{4.12}$$

The inequality (1.5) (with $C \lesssim A_1$) follows from (4.11) and (4.12).

We now prove necessity. The validity of the inequality (1.5) and (4.4) imply that

$$\left(\sum_{k=N}^{M} \|gw\|^p_{L_p(S_k)}\right)^{\frac{1}{p}} \lesssim C \left(\sum_{k=N}^{M} \left(\int_{S_k} g(y)\,dy\right)^q \|v\|^q_{L_q(0,x_k)}\right)^{\frac{1}{q}} \tag{4.13}$$

for all non-negative measurable g on $\mathbb{R}^n$.

Let g_k, $N \leqslant k \leqslant M$, be non-negative measurable functions that saturate Hőlder's inequality (4.10) with $E = S_k$, $N \leqslant k \leqslant M$, that is, functions satisfying

$$\operatorname{supp} g_k \subset S_k, \quad \|g_k\|_{L_1(S_k)} = 1 \quad \text{and} \quad \|g_k w\|^p_{L_p(S_k)} \geqslant \frac{1}{2}\|w\|^p_{L_{p'}(S_k)}. \tag{4.14}$$

Then we define the test function g by

$$g = \sum_{k=N}^{M} a_k g_k, \tag{4.15}$$

where $\{a_k\}$ is a sequence of non-negative numbers. Consequently, (4.13) yields

$$\left(\sum_{k=N}^{M} a_k^p \|w\|^p_{L_{p'}(S_k)}\right)^{\frac{1}{p}} \lesssim C \left(\sum_{k=N}^{M} a_k^q \|v\|^q_{L_q(0,x_k)}\right)^{\frac{1}{q}}, \tag{4.16}$$

and, by Lemma 4.1,

$$\sup_{N\leqslant k\leqslant M} \|w\|_{L_{p'}(S_k)} \|v\|^{-1}_{L_q(0,x_k)} \lesssim C. \tag{4.17}$$

Assuming that $x_N > 0$, testing (1.5) with $g = \chi_{B(0,x_N)}$ and using (4.5), we arrive at $\|w\|_{L_p(B(0,x_N))} = 0$. This implies that $|B(0,x_N)| = 0$ or $w = 0$ a.e. in $B(0,x_N)$. Consequently, (4.8) holds.

Therefore,

$$A_1 = \sup_{N\leqslant k\leqslant M} \sup_{x\in J_k} \|w\|_{L_{p'}(B(0,x))} \|v\|^{-1}_{L_q(0,x)}$$

and, on using (3.9), we obtain that

$$A_1 \leqslant \sup_{N\leqslant k\leqslant M} \|w\|_{L_{p'}(B(0,x_{k+1}))} \|v\|^{-1}_{L_q(0,x_k)}.$$

Applying (4.8) and (3.10) again, we arrive at

$$A_1 \leqslant \sup_{N\leqslant k\leqslant M} \left(\sum_{i=N}^{k} \|w\|^{p'}_{L_{p'}(S_i)}\right)^{\frac{1}{p'}} \|v\|^{-1}_{L_q(0,x_k)} \quad \text{if} \quad 0 < p < 1$$

and

$$A_1 \leqslant \sup_{N\leqslant k\leqslant M} \left(\sup_{N\leqslant i\leqslant k} \|w\|_{L_{p'}(S_i)} \right) \|v\|^{-1}_{L_q(0,x_k)} \quad \text{if} \quad p=1.$$

Now, using the fact that $\{\|v\|^{-1}_{L_q(0,x_k)}\}_{k=N}^M$ is almost geometrically decreasing (cf. (4.7)) and Lemma 3.4 we obtain

$$A_1 \lesssim \sup_{N\leqslant k\leqslant M} \|w\|_{L_{p'}(S_k)} \|v\|^{-1}_{L_q(0,x_k)},$$

which, together with (4.17), yields $A_1 \lesssim C$. □

REMARK 4.3. Let A_1 be the number defined in Theorem 2.1. If $p=1$, then

$$A_1 = \left\| w(x) \|v\|^{-1}_{L_q(0,|x|)} \right\|_{L_\infty(\mathbb{R}^n)}.$$

Indeed, exchanging essential suprema, we obtain

$$\begin{aligned} A_1 &= \left\| \|w\|_{L_\infty(B(0,t))} \|v\|^{-1}_{L_q(0,t)} \right\|_{L_\infty(0,\infty)} \\ &= \left\| \left\| w(x) \|v\|^{-1}_{L_q(0,t)} \right\|_{L_\infty(B(0,t))} \right\|_{L_\infty(0,\infty)} \\ &= \left\| \left\| w(x)\chi_{B(0,t)}(x) \|v\|^{-1}_{L_q(0,t)} \right\|_{L_\infty(\mathbb{R}^n)} \right\|_{L_\infty(0,+\infty)} \\ &= \left\| \left\| w(x) \|v\|^{-1}_{L_q(0,t)} \right\|_{L_\infty[|x|,\infty)} \right\|_{L_\infty(\mathbb{R}^n)} \\ &= \left\| w(x) \|v\|^{-1}_{L_q(0,|x|)} \right\|_{L_\infty(\mathbb{R}^n)}. \end{aligned}$$

Proof of Theorem 2.2. The proof of Theorem 2.2 is analogous to the proof of Theorem 4.4 from [2], the only difference is that the role of the interval $(x_i, x_{i+1}]$ is played by the set S_i defined by (3.10). □

In the proof of the Theorem 2.3 and Theorem 2.4 we are going to use the substitution $y = F(x) := \frac{x}{|x|^2}$, $x \neq 0$. Note that for the Jacobian of this mapping the equality

$$|\det J_F(x)| = |x|^{-2n}, \quad x \neq 0$$

holds. Indeed, observe at first that

$$F^{-1}(x) = F(x), \quad x \neq 0.$$

It is easy to calculate that

$$J_F(x) = \begin{pmatrix} \frac{|x|^2-2x_1^2}{|x|^4} & -\frac{2x_1x_2}{|x|^4} & \cdots & -\frac{2x_1x_n}{|x|^4} \\ -\frac{2x_2x_1}{|x|^4} & \frac{|x|^2-2x_2^2}{|x|^4} & \cdots & -\frac{2x_2x_n}{|x|^4} \\ \cdots & \cdots & \cdots & \cdots \\ -\frac{2x_nx_1}{|x|^4} & -\frac{2x_nx_2}{|x|^4} & \cdots & \frac{|x|^2-2x_n^2}{|x|^4}, \end{pmatrix}$$

and

$$J_{F^{-1}}(F(x)) = \begin{pmatrix} |x|^2-2x_1^2 & -2x_1x_2 & \cdots & -2x_1x_n \\ -2x_2x_1 & |x|^2-2x_2^2 & \cdots & -2x_2x_n \\ \cdots & \cdots & \cdots & \cdots \\ -2x_nx_1 & -2x_nx_2 & \cdots & |x|^2-2x_n^2 \end{pmatrix} = |x|^{4n}J_F(x).$$

Therefore

$$J_{F^{-1}}(F(x)) = |x|^{4n}J_F(x). \tag{4.18}$$

Since

$$J_{F^{-1}}(F(x))J_F(x) = I,$$

by (4.18) we get

$$|x|^{4n}J_F(x)J_F(x) = I.$$

Thus

$$|\det J_F(x)| = |x|^{-2n}.$$

Proof of Theorem 2.3. Writing the inequality (1.6) for $|y|^{-2n}g(\frac{y}{|y|^2})$ instead of g and using the substitutions $x = \frac{y}{|y|^2}$ on the left-hand side and $x = \frac{y}{|y|^2}$ and $\tau = \frac{1}{t}$ on the right-hand side, we obtain

$$\left(\int_{\mathbb{R}^n} g(x)^p \left(w\left(\frac{x}{|x|^2}\right)|x|^{-\frac{2n}{p'}}\right)^p dx\right)^{\frac{1}{p}} \leqslant C\left(\int_0^\infty v\left(\frac{1}{\tau}\right)^q \frac{1}{\tau^2}\left(\int_{\complement_{B(0,\tau)}} g(x)dx\right)^q d\tau\right)^{\frac{1}{q}}. \tag{4.19}$$

Consequently, the inequality (1.6) holds for all non-negative measurable g on $\mathbb{R}^n$ if and only if the inequality (4.19) holds for all non-negative measurable g on $\mathbb{R}^n$. We deduce from Theorem 2.1 that the inequality (1.6) holds for all non-negative measurable g on $\mathbb{R}^n$ if and only if

$$\sup_{t\in(0,+\infty)}\left(\int_{B(0,t)}\left(w\left(\frac{x}{|x|^2}\right)|x|^{-\frac{2n}{p'}}\right)^{p'}dx\right)^{\frac{1}{p'}}\left(\int_0^t v\left(\frac{1}{\tau}\right)^q\frac{1}{\tau^2}d\tau\right)^{-\frac{1}{q}} < +\infty,$$

that is,

$$\sup_{t\in(0,+\infty)} \|w\|_{L_{p'}(\complement_{B(0,t)})}\|v\|^{-1}_{L_q(t,+\infty)} < +\infty. \qquad \square \tag{4.20}$$

REMARK 4.4. Let B_1 be the number defined in Theorem 2.3. If $p=1$, then

$$B_1 = \left\| w(x) \|v\|^{-1}_{L_q(|x|,+\infty)} \right\|_{L_\infty(\mathbb{R}^n)}.$$

Indeed, using the idea of the proof of Theorem 2.3, we obtain the result from Remark 4.3.

Proof of Theorem 2.4. Suppose first that $q<+\infty$. As in the proof of Theorem 2.3, one can show that the inequality (1.6) holds if and only if the inequality (4.19) is satisfied for all non-negative measurable g on $\mathbb{R}^n$. Thus, by Theorem 2.2, the inequality (1.6) holds if and only if

$$\left(\int_{(0,+\infty)} \left(\int_{B(0,t)} \left(w\left(\frac{x}{|x|^2}\right) |x|^{-\frac{2n}{p'}} \right)^{p'} dx \right)^{\frac{r}{p'}} d\left(-\int_{(0,t)} v\left(\frac{1}{\tau}\right)^q \frac{1}{\tau^2} d\tau \right)^{-\frac{r}{q}} \right)^{\frac{1}{r}}$$

$$+ \frac{\left(\int_{\mathbb{R}^n} \left(w\left(\frac{x}{|x|^2}\right) |x|^{-\frac{2n}{p'}} \right)^{p'} dx \right)^{\frac{1}{p'}}}{\left(\int_0^{+\infty} v\left(\frac{1}{\tau}\right)^q \frac{1}{\tau^2} d\tau \right)^{\frac{1}{q}}} < +\infty,$$

that is,

$$\left(\int_{(0,+\infty)} \|w\|^r_{L_{p'}(\complement B(0,t))} d\left(\|v\|^{-r}_{L_q(t-,+\infty)} \right) \right)^{\frac{1}{r}} + \frac{\|w\|_{L_{p'}(\mathbb{R}^n)}}{\|v\|_{L_q(0,+\infty)}} < +\infty.$$

If $q=+\infty$, the statement can be proved analogously. □

REFERENCES

[1] P. DRÁBEK, H.P. HEINIG AND A.KUFNER, *Higher-dimensional Hardy inequality*, In: General Inequalities 7 (Oberwolfach, 1995), Internat. Ser. Numer. Math. **123** (1997), Birkhäuser, Basel, 1997, 3–16.

[2] W. D. EVANS, A. GOGATISHVILI AND B. OPIC, *The reverse Hardy inequality with measures*, Math. Ineq. and Appl., bf 11 (2008), 43–74.

[3] A. GOGATISHVILI AND L. PICK, *Discretization and anti-discretization of rearrangement-invariant norms*, Publ. Mat., **47** (2003), 311–358.

[4] K.-G. GROSSE-ERDMANN, *The Blocking Technique. Weighted Mean Operators and Hardy's Inequality*, Lect. Notes Math. 1679, Springer, Berlin, 1998.

[5] A. KUFNER, L. MALIGRANDA AND L. E. PERSSON, *The Hardy Inequality. About its History and Some Related Results*, Vydavatelský servis Publishing House, Pilsen, 2007.

[6] A. KUFNER AND L.-E. PERSSON, *Weighted inequalities of Hardy type*, World Scientific Publishing Co, Singapore, 2003.

[7] L. LEINDLER, *Inequalities of Hardy and Littlewood type*, Acta Sci. Math. (Szeged), bf 2 (1976), 117–123.

[8] L. LEINDLER, *On the converses of inequalities of Hardy and Littlewood*, Acta Sci. Math. (Szeged), **58** (1993), 191–196.

[9] B. OPIC AND A. KUFNER, *Hardy–type inequalities*, Pitman Research Notes in Mathematics Series, 219, Longman Scientific & Technical, Harlow, 1990.

(Received October 16, 2009)

Amiran Gogatishvili
Institute of Mathematics of the
Academy of Sciences of the Czech Republic
Žitna 25, 115 67 Prague 1
Czech Republic
e-mail: gogatish@math.cas.cz

Rza Mustafayev
Institute of Mathematics and Mechanics
Academy of Sciences of Azerbaijan
F. Agayev St. 9
Baku, AZ 1141
Azerbaijan
e-mail: rzamustafayev@gmail.com
or
Institute of Mathematics of the
Academy of Sciences of the Czech Republic
Žitna 25, 115 67 Prague 1
Czech Republic
e-mail: mustafa@math.cas.cz