

## A NOTE ON THE WEIGHTED HARMONIC–GEOMETRIC–ARITHMETIC MEANS INEQUALITIES

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(Communicated by Z. Páles)

*Abstract.* In this note, we derive non trivial sharp bounds related to the weighted harmonic-geometric-arithmetic means inequalities, when two out of the three terms are known. As application, we give an explicit bound for the trace of the inverse of a symmetric positive definite matrix and an inequality related to the coefficients of polynomials with positive roots.

### 1. Introduction and main results

The well known weighted harmonic-geometric-arithmetic means inequalities (HGA) can be stated as follows. Let  $\alpha_i > 0$  and  $x_i > 0$ ,  $i = 1, \dots, n$  with  $\sum_i \alpha_i = 1$ , and define  $h, g, a$  by

$$h = \left( \sum_{i=1}^n \frac{\alpha_i}{x_i} \right)^{-1}, \quad g = \prod_{i=1}^n x_i^{\alpha_i}, \quad a = \sum_{i=1}^n \alpha_i x_i.$$

Then the HGA inequalities state that

$$h \leq g \leq a. \tag{1}$$

One equality is reached if and only if all the  $x_i$  are equal, which then implies that both inequalities are in fact equalities. The terms of the previous inequalities are respectively called the harmonic, the geometric and the arithmetic mean of the  $x_i$  with weight  $\alpha_i$ . There exist several extensions of these inequalities, see for example [2, 4, 5, 6]. In this note we focus on the case where two of the means are known and non trivial bounds on the third have to be determined. Actually, Theorem 1.1 below gives a sharp lower bound and a sharp upper bound on the harmonic mean, when both the arithmetic and the geometric means are known. The dual bounds, i.e., an upper and a lower bound on the arithmetic mean when both the harmonic and the geometric means are known can easily be deduced with the change of variables  $y_i = x_i^{-1}$ . Theorem 1.2 gives a sharp lower bound and a sharp upper bound on the geometric mean, when both the harmonic and the arithmetic means are known, extending Inequalities (1) above when the two extreme values are in fact known.

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*Mathematics subject classification* (2010): Primary 26D15; Secondary 15A42.

*Keywords and phrases:* Classical means, weighted HGA inequalities, sharp inequalities.

The theory of complementary inequalities is a field, where upper bounds for the ratios  $a/g$ ,  $a/h$ ,  $g/h$  and for the differences  $a-g$ ,  $a-h$ ,  $g-h$  are obtained in terms of the upper and lower bounds for the variables  $x_i$ . For instance, Kantorovich's inequality, see e.g. [6], provides a well-known upper bound for  $a/h$  under these conditions. Despite our search for a similar result in the vast literature on the subject, we were not able to find the inequalities presented in this article in any published work. Related to our results, let us however mention [7], where the author considers the interplay of the three means  $h, g, a$ . In this paper, it is shown that the moment space of the triplets  $(h, g, a)$  is the set  $M = \{(u, v, w) \in \mathbb{R}^3 : 0 \leq u \leq v \leq w\}$ . This means that for any positive  $\varepsilon$ , there exists  $n \in \mathbb{N}$ ,  $x_1, \dots, x_n > 0$  such that

$$|h - u| \leq \varepsilon, |g - v| \leq \varepsilon, |a - w| \leq \varepsilon.$$

The meaning of the result is that if  $n$  is not fixed, then the only meaningful inequality for the three means is  $h \leq g \leq a$ . In the present paper,  $n$  is fixed and we will suppose that at least one  $x_i$  is different from the others, which insure that Inequalities (1) are strict. The main results are the following.

**THEOREM 1.1.** *With the above notations, if  $\alpha = \min_i \{\alpha_i\}$  and  $\xi_0 \in [0, 1]$ ,  $\xi_1 \in [1, 1/\alpha]$  are the solutions of the equation*

$$g = a \xi^\alpha \left( \frac{1 - \alpha \xi}{1 - \alpha} \right)^{1-\alpha},$$

then

$$a \frac{\xi_0(1 - \alpha \xi_0)}{\alpha - 2\alpha \xi_0 + \xi_0} \leq h \leq a \frac{\xi_1(1 - \alpha \xi_1)}{\alpha - 2\alpha \xi_1 + \xi_1}.$$

The first (resp. second) inequality reaches equality if and only if  $x_j = \xi_0$  (resp.  $x_j = \xi_1$ ) and  $x_l = x_k$ ,  $\forall l, k \neq j$  for some  $j$  with  $\alpha_j = \min_i \{\alpha_i\}$ .

The uniqueness of the solutions  $\xi_0$  and  $\xi_1$  will be made clear in the sequel. Based on this result, we give explicit general lower and upper bounds for the harmonic and arithmetic means in Corollary 2.2. As application, we give an explicit bound for the trace of the inverse of a symmetric positive definite matrix in Example 5.1 and for the quotient of coefficients of polynomials with positive roots in Example 5.2.

**THEOREM 1.2.** *With the above notations, if  $\alpha = \min_i \{\alpha_i\}$  and  $\xi_0 \in [0, 1]$ ,  $\xi_1 \in [1, 1/\alpha]$  are the solutions of the equation*

$$h = a \frac{\xi(1 - \alpha \xi)}{\alpha - 2\alpha \xi + \xi},$$

then we have

$$a \xi_1^\alpha \left( \frac{1 - \alpha \xi_1}{1 - \alpha} \right)^{1-\alpha} \leq g \leq a \xi_0^\alpha \left( \frac{1 - \alpha \xi_0}{1 - \alpha} \right)^{1-\alpha}.$$

The first (resp. second) inequality reaches equality if and only if  $x_j = \xi_1$  (resp.  $x_j = \xi_0$ ) and  $x_l = x_k$ ,  $\forall l, k \neq j$  for some  $j$  with  $\alpha_j = \min_i \{\alpha_i\}$ .

Based on this result, we give explicit sharp lower and upper bounds for the geometric mean in Corollary 2.1 and simpler bounds in Corollary 2.2.

## 2. Explicit bounds

We postpone the proof of Theorem 1.1 and Theorem 1.2 and present explicit bounds for the different means. So let  $\alpha_i > 0$  with  $\sum \alpha_i = 1$  and  $x_i > 0$ ,  $i = 1, \dots, n$  be real numbers such that

$$h = \left( \sum_{i=1}^n \frac{\alpha_i}{x_i} \right)^{-1}, \quad g = \prod_{i=1}^n x_i^{\alpha_i}, \quad a = \sum_{i=1}^n \alpha_i x_i$$

and let  $\alpha = \min_i \{\alpha_i\}$ . Note that  $\sum_{i=1}^n \alpha_i = 1$  implies that  $\alpha \leq 1/n$ . In the case of Theorem 1.2, the equation for  $\xi$  is exactly solvable, and one readily verifies that a sharp bound in closed form can be computed as follows.

**COROLLARY 2.1.** *With the above notations, we have  $\alpha \leq 1/n$  and*

$$\begin{aligned} g &\leq \left( \frac{a - h(1 - 2\alpha) - \sqrt{(a-h)(a-h(1-2\alpha)^2)}}{2\alpha} \right)^\alpha \times \\ &\quad \times \left( \frac{a + h(1 - 2\alpha) + \sqrt{(a-h)(a-h(1-2\alpha)^2)}}{2(1-\alpha)} \right)^{1-\alpha}, \\ g &\geq \left( \frac{a - h(1 - 2\alpha) + \sqrt{(a-h)(a-h(1-2\alpha)^2)}}{2\alpha} \right)^\alpha \times \\ &\quad \times \left( \frac{a + h(1 - 2\alpha) - \sqrt{(a-h)(a-h(1-2\alpha)^2)}}{2(1-\alpha)} \right)^{1-\alpha}. \end{aligned}$$

The bounds of the next corollary are not sharp anymore but are both in closed form and simple.

**COROLLARY 2.2.** *With the above notations, we have*

$$a \cdot \left( \alpha e \left( \frac{a}{g} \right)^{1/\alpha} + 1 \right)^{-1} < h \leq g \leq a < h \cdot \left( \alpha e \left( \frac{g}{h} \right)^{1/\alpha} + 1 \right),$$

and

$$h \cdot \left( \frac{h}{a} \exp \left( \frac{h}{a} + \frac{n}{n-1} \right) \right)^{-\alpha} < g < a \cdot \left( \frac{h}{a} \exp \left( \frac{h}{a} + \frac{n}{n-1} \right) \right)^\alpha.$$

Asymptotically with  $n$ , the last two inequalities give an improvement of the usual HGA inequalities when  $h/a < t_0 = 0.278464\dots$ , where  $t_0 e^{t_0+1} = 1$ .

*Proof.* Let us start with the first set of inequalities. The root  $\xi = \xi_0 < 1$  of Theorem 1.1 satisfies the following inequalities,

$$(g/a)^{1/\alpha} = \xi \left( \frac{1 - \alpha\xi}{1 - \alpha} \right)^{\frac{1-\alpha}{\alpha}} = \xi \left( 1 + \frac{\alpha}{1 - \alpha}(1 - \xi) \right)^{\frac{1-\alpha}{\alpha}} < \xi e^{1-\xi} < \xi e$$

because  $\frac{\alpha}{1-\alpha} < 1$  and  $(1+u)^v < e^{uv}$ , as soon as  $v > 0$  and  $|u| < 1$ . Now, since  $\frac{1-\alpha\xi}{1-\alpha} > 1$  and  $1/\xi < e(a/g)^{1/\alpha}$ ,

$$\begin{aligned} \frac{1}{h} &= \sum_{i=1}^n \frac{\alpha_i}{x_i} = \frac{1}{a} \cdot \sum_{i=1}^n \frac{\alpha_i}{z_i} \leq \frac{1}{a} \cdot \left( \frac{\alpha - 2\alpha\xi + \xi}{\xi(1 - \alpha\xi)} \right) \\ &= \frac{1}{a} \cdot \left( \frac{\alpha}{\xi} + \frac{(1 - \alpha)^2}{1 - \alpha\xi} \right) < \frac{1}{a} \cdot (\alpha e(a/g)^{1/\alpha} + 1) \end{aligned}$$

which is equivalent to  $h > a \cdot (\alpha e(a/g)^{1/\alpha} + 1)^{-1}$ . By setting  $x'_i = 1/x_i$ , we have  $a' = 1/h, g' = 1/g, h' = 1/a$  and the inequality  $a < h \cdot (\alpha e(g/h)^{1/\alpha} + 1)$  is a direct consequence. Let us now prove the second set of inequalities. Since  $(1 - 2\alpha)^2 \leq 1$ , we have

$$(a - h)^2 \leq (a - h)(a - h(1 - 2\alpha)^2) \leq (a - h(1 - 2\alpha)^2)^2$$

and thus the upper bound of Corollary 2.1 gives

$$g \leq \left( \frac{a - h(1 - 2\alpha) - (a - h)}{2\alpha} \right)^\alpha \left( \frac{a + h(1 - 2\alpha) + (a - h(1 - 2\alpha)^2)}{2(1 - \alpha)} \right)^{1-\alpha}.$$

Since  $\alpha \leq 1/n$ ,  $\frac{1}{1-\alpha} \leq 1 + \frac{n}{n-1}\alpha$ , and after suitable simplifications, using once again the above exponential inequality, we obtain

$$\begin{aligned} g &\leq a \cdot \left( \frac{h}{a} \right)^\alpha \cdot \left( 1 + \frac{h}{a}(1 - 2\alpha)\alpha \right)^{1-\alpha} \cdot \left( 1 + \frac{n}{n-1}\alpha \right)^{1-\alpha} \\ &< a \cdot \left( \frac{h}{a} \exp \left( \frac{h}{a} + \frac{n}{n-1} \right) \right)^\alpha. \end{aligned}$$

The reverse inequality is once again obtained by setting  $z_i = 1/x_i$ . This finishes the proof of the lemma.  $\square$

### 3. The case $n = 2$

For the rest of the article, without loss of generality, we will assume that the  $x_i$  are normalized so that the arithmetic mean is equal to 1. This is simply obtained by the change of variable  $x'_i = x_i/a$ , leading to  $a' = 1, g' = g/a$  and  $h' = h/a$ .

Along the way of the proofs of the main results, we start with the case  $n = 2$ . This will turn out to be in fact the most important case, as the general case will be a consequence of it. The next two lemmas will be useful in the sequel.

LEMMA 3.1. *Let  $\alpha \in ]0, 1/2]$  and  $f$  and  $\varphi$  be the functions defined over  $[0, 1/\alpha]$  defined by*

$$f(x) = x^\alpha \left( \frac{1 - \alpha x}{1 - \alpha} \right)^{1-\alpha} \quad \text{and} \quad \varphi(x) = \sqrt{\frac{x(1 - \alpha x)}{(1 - 2\alpha)x + \alpha}}. \quad (2)$$

*Then  $\varphi(0) = \varphi(1/\alpha) = f(0) = f(1/\alpha) = 0$ ,  $f(1) = \varphi(1) = 1$ , they are strictly increasing over  $[0, 1]$  and strictly decreasing over  $[1, 1/\alpha]$ , and fulfill the property that  $f > \varphi$  over  $]0, 1[$ , and  $f < \varphi$  over  $]1, 1/\alpha[$ .*

*Proof.* Clearly,  $\varphi(0) = \varphi(1/\alpha) = f(0) = f(1/\alpha) = 0$ ,  $g(1) = f(1) = 1$ . A short analysis of  $f$  and of the radical of  $\varphi$  shows that they are strictly increasing over  $[0, 1]$  and strictly decreasing over  $[1, 1/\alpha]$ . The less obvious fact is that  $f > \varphi$  over  $]0, 1[$  and  $f < \varphi$  over  $]1, 1/\alpha[$ . In order to prove it, let us consider the quotient  $f/\varphi$ . Since  $(f/\varphi)(1) = 1$ , the statement would be proved if we can show that  $f/\varphi$  is strictly decreasing over  $[0, 1/\alpha]$ . Let us prove that it is the case by showing that  $(f/\varphi)' < 0$  over  $]0, 1[ \cup ]1, 1/\alpha[$ . First

$$(f/\varphi)(x) = \frac{x^\alpha \left( \frac{1 - \alpha x}{1 - \alpha} \right)^{1-\alpha}}{\frac{x^{1/2}(1 - \alpha x)^{1/2}}{((1 - 2\alpha)x + \alpha)^{1/2}}} = (1 - \alpha)^{\alpha-1} \cdot \left( \frac{1}{x} - \alpha \right)^{1/2-\alpha} \cdot ((1 - 2\alpha)x + \alpha)^{1/2}.$$

After suitable simplifications, we obtain

$$(f/\varphi)'(x) = -\alpha(1 - \alpha)^{\alpha-1} \cdot \frac{(x - 1)^2}{2x^2 \left( \frac{1}{x} - \alpha \right)^{\alpha+1/2} (\alpha + (1 - 2\alpha)x)^{1/2}}.$$

Note that  $\frac{1}{x} - \alpha > 0$  and  $1 - 2\alpha \geq 0$  so the condition  $(f/\varphi)' < 0$  is fulfilled.  $\square$

LEMMA 3.2. *If  $\alpha \in ]0, 1[$  and  $x \in [0, 1/\alpha]$ , then*

$$\frac{1 - x}{1 - \alpha x} + \ln \left( 1 - \frac{1 - x}{1 - \alpha x} \right) + \frac{(1 - x)^2}{(1 - \alpha x)(1 - (2\alpha - 1)x)} \begin{cases} \leq 0 & \text{if } x \in [0, 1], \\ \geq 0 & \text{if } x \in [1, 1/\alpha]. \end{cases} \quad (3)$$

*Proof.* If  $t = \frac{1-x}{1-\alpha x}$ , then  $-\infty < t \leq 0$  for  $x \in [1, 1/\alpha[$  and  $0 \leq t \leq 1$  for  $x \in [0, 1]$ . Since  $\frac{1-(2\alpha-1)x}{1-\alpha x} = 2 - t$ , the above expression is equal to  $s(t) = \frac{2t}{2-t} + \ln(1-t)$ . But since  $s(0) = 0$  and  $s$  has a non positive derivative  $s'(t) = \frac{-t^2}{(2-t)^2(1-t)} \leq 0$ , the function  $s$  is decreasing and  $s \leq 0$  for  $x \in [0, 1]$  and  $s \geq 0$  for  $x \in [1, 1/\alpha[$ .  $\square$

Returning to the original problem, let us focus on the case where both  $g$  and  $a$  are known and an upper and a lower bound on  $h$  is to be determined. If  $\alpha_1, \alpha_2 > 0$

and  $\alpha_1 + \alpha_2 = 1$ , up to a permutation of the indices, we can assume without loss of generality that  $\alpha_1 \leq 1/2$ . The two dimensional case can be stated as follows: given two real numbers  $0 < \alpha \leq 1/2$  and  $0 < g < 1$ , we want to find the minimal and the maximal value of

$$H(x, y) = (\alpha/x + (1 - \alpha)/y)^{-1},$$

where  $x$  and  $y$  fulfill the conditions

$$\alpha x + (1 - \alpha)y = 1 \quad \text{and} \quad x^\alpha y^{1-\alpha} = g.$$

Clearly, these conditions imply that

$$f(x) = g \quad \text{where} \quad f(x) = x^\alpha \left( \frac{1 - \alpha x}{1 - \alpha} \right)^{1-\alpha}. \quad (4)$$

Note that the function  $f$  appears in Lemma 3.1. We call  $x_1$  and  $x_2$  the two unique solutions of Equation (4), with  $x_1 < 1 < x_2$ . Then, with  $\varphi$  being the function of Lemma 3.1,

$$H(x_i, y_i) = \left( \frac{\alpha}{x_i} + \frac{1 - \alpha}{y_i} \right)^{-1} = \frac{x_i(1 - \alpha x_i)}{x_i - 2x_i\alpha + \alpha} = \varphi^2(x_i)$$

and Lemma 3.1 implies that  $\varphi(x_2) > f(x_2) = f(x_1) > \varphi(x_1)$  because of the respective positions of  $f$  and  $\varphi$ . This directly gives the following lemma:

LEMMA 3.3. *Let  $0 < \alpha \leq 1/2$  and  $0 < g < 1$ . If  $x$  and  $y$  fulfill the conditions*

$$\alpha x + (1 - \alpha)y = 1 \quad \text{and} \quad x^\alpha y^{1-\alpha} = g,$$

then

$$\frac{x_1(1 - \alpha x_1)}{x_1 - 2x_1\alpha + \alpha} \leq \left( \frac{\alpha}{x} + \frac{1 - \alpha}{y} \right)^{-1} \leq \frac{x_2(1 - \alpha x_2)}{x_2 - 2x_2\alpha + \alpha}$$

where  $x_1$  and  $x_2$  are the unique solutions over  $[0, 1]$  and  $[1, 1/\alpha]$  respectively of the equation

$$g = x^\alpha \left( \frac{1 - \alpha x}{1 - \alpha} \right)^{1-\alpha}. \quad (5)$$

We would like now to prove that for a fixed  $g$ , and as a function of  $\alpha \in ]0, 1/2]$ , the minimum and the maximum values above  $H(x_1)$  and  $H(x_2)$  are increasing and decreasing functions respectively. This result will be useful in the sequel. More precisely, if we set

$$M(x, \alpha) = \left( \frac{\alpha}{x} + \frac{(1 - \alpha)^2}{1 - \alpha x} \right)^{-1}$$

and

$$\lambda_i(\alpha) = M(x_i(\alpha), \alpha)$$

where  $x_1(\alpha)$  and  $x_2(\alpha)$  are the unique roots of Equation (5) in  $[0, 1]$  and in  $[1, 1/\alpha]$  respectively, then we have the following lemma:

LEMMA 3.4. For a fixed  $g \in [0, 1]$ , the function  $\lambda_1$  is an increasing function and the function  $\lambda_2$  is an decreasing function over  $[0, \frac{1}{2}]$ .

*Proof.* First, let us note that the function  $f$  being strictly increasing over  $[0, 1]$  and strictly decreasing over  $[1, 1/\alpha]$ , the implicit function theorem can be used to define the implicit function  $\alpha \mapsto x_i(\alpha) \in [0, 1]$  given by the equation  $g = f(x) = x^\alpha \left(\frac{1-\alpha x}{1-\alpha}\right)^{1-\alpha}$ , where  $g$  is fixed. These functions are differentiable, and their derivative can be computed by implicitly differentiating the equation. In fact, taking the natural logarithm of Equation (5) and the derivative with respect to  $\alpha$ , after suitable simplifications, we obtain

$$x'(\alpha) = \frac{x}{\alpha} \cdot \left( -1 - \frac{1-\alpha x}{1-x} \cdot \ln \left( 1 - \frac{1-x}{1-\alpha x} \right) \right).$$

Using the chain rule, we have

$$\lambda'(\alpha) = \frac{\partial M}{\partial x}(x(\alpha), \alpha) \cdot x'(\alpha) + \frac{\partial M}{\partial \alpha}(x(\alpha), \alpha).$$

After suitable simplifications, we obtain

$$\begin{aligned} \frac{\partial M}{\partial \alpha}(x, \alpha) &= -M^2(x, \alpha) \cdot \frac{(1-x)^2}{x(1-\alpha x)^2} \\ \frac{\partial M}{\partial x}(x, \alpha) &= -M^2(x, \alpha) \cdot \frac{-\alpha(1-x)(1-(2\alpha-1)x)}{x^2(1-\alpha x)^2} \end{aligned}$$

which leads to

$$\lambda'(\alpha) = -M^2(x, \alpha) \cdot \frac{1-(2\alpha-1)x}{x(1-\alpha x)} \left( \frac{1-x}{1-\alpha x} + \ln \left( 1 - \frac{1-x}{1-\alpha x} \right) + \frac{(1-x)^2}{(1-\alpha x)(1-(2\alpha-1)x)} \right).$$

Note that since  $\alpha \leq 1/2$ ,  $\frac{1-(2\alpha-1)x}{x(1-\alpha x)} \geq 0$ . An application of Lemma 3.2 shows that  $\lambda'_1 \geq 0$  and  $\lambda'_0 \leq 0$ . This finishes the proof of the lemma.  $\square$

Let us now focus on the case where both  $a$  and  $h$  are known, and an upper and a lower bound of  $g$  is to be found, when  $n = 2$ . The problem can now be formulated as follows. Given two real numbers  $0 < \alpha \leq 1/2$  and  $0 < h < 1$ , we want to find the minimal and the maximal value of

$$G(x, y) = x^\alpha y^{1-\alpha}$$

where  $x$  and  $y$  fulfill the conditions

$$\alpha x + (1-\alpha)y = 1 \quad \text{and} \quad \left( \frac{\alpha}{x} + \frac{1-\alpha}{y} \right)^{-1} = h.$$

These two conditions imply that

$$h = \left( \frac{\alpha}{x} + \frac{(1-\alpha)^2}{1-\alpha x} \right)^{-1} = \frac{x(1-\alpha x)}{\alpha - 2\alpha x + x} = \varphi^2(x) \quad (6)$$

and

$$G(x, y) = f(x) = x^\alpha \left( \frac{1-\alpha x}{1-\alpha} \right)^{1-\alpha}.$$

If  $x_1$  and  $x_2$  are the two unique solutions of Equation (6) with  $x_1 < 1 < x_2$ , Lemma 3.1 implies that  $f(x_1) > \varphi(x_1) = \varphi(x_2) > f(x_2)$  because of the respective positions of  $f$  and  $\varphi$ . This directly gives the following lemma:

LEMMA 3.5. *Let  $0 < \alpha \leq 1/2$  and  $0 < h < 1$ . If  $x$  and  $y$  fulfill the conditions*

$$\alpha x + (1-\alpha)y = 1 \quad \text{and} \quad \left( \frac{\alpha}{x} + \frac{1-\alpha}{y} \right)^{-1} = h,$$

then

$$x_2^\alpha \left( \frac{1-\alpha x_2}{1-\alpha} \right)^{1-\alpha} \leq x^\alpha y^{1-\alpha} \leq x_1^\alpha \left( \frac{1-\alpha x_1}{1-\alpha} \right)^{1-\alpha}$$

where  $x_1$  and  $x_2$  are the unique solutions over  $[0, 1]$  and  $[1, 1/\alpha]$  respectively of the equation

$$h = \frac{x(1-\alpha x)}{\alpha - 2\alpha x + x}. \quad (7)$$

As before, we would like now to prove that for a fixed  $h$ , and as a function of  $\alpha$ , the minimal and maximal values above  $G(x_2)$  and  $G(x_1)$  are increasing and decreasing functions respectively. More precisely, if we set

$$N(x, \alpha) = x^\alpha \left( \frac{1-\alpha x}{1-\alpha} \right)^{1-\alpha}$$

and

$$\gamma_i(\alpha) = N(x_i(\alpha), \alpha)$$

where  $x_1(\alpha)$  and  $x_2(\alpha)$  are the unique roots of Equation (7) in  $[0, 1]$  and  $[1, 1/\alpha]$  respectively, then we have the following lemma:

LEMMA 3.6. *For a fixed  $h \in [0, 1]$ , the function  $\gamma_1$  is a decreasing function and the function  $\gamma_2$  is an increasing function over  $[0, 1/2]$ .*

*Proof.* The same argument used in the proof of Lemma 3.4 (with  $\varphi$  instead of  $f$ ) shows that the function  $x(\alpha)$  is well defined, and after suitable simplifications, has the following derivative

$$x'(\alpha) = \frac{x}{\alpha} \cdot \frac{1-x}{(1-2\alpha)x+1}.$$



Using the chain rule, we have

$$\gamma'(\alpha) = \frac{\partial N}{\partial x}(x(\alpha), \alpha) \cdot x'(\alpha) + \frac{\partial N}{\partial \alpha}(x(\alpha), \alpha).$$

After suitable simplifications, we obtain

$$\begin{aligned} \frac{\partial N}{\partial \alpha}(x, \alpha) &= N(x, \alpha) \cdot \left( \ln \left( 1 - \frac{1-x}{1-\alpha x} \right) + \frac{1-x}{1-\alpha x} \right) \\ \frac{\partial N}{\partial x}(x, \alpha) &= N(x, \alpha) \cdot \frac{\alpha}{x} \cdot \frac{1-x}{1-\alpha x} \end{aligned}$$

which leads to

$$\gamma'(\alpha) = N(x, \alpha) \cdot \left( \frac{1-x}{1-\alpha x} + \ln \left( 1 - \frac{1-x}{1-\alpha x} \right) + \frac{(1-x)^2}{(1-\alpha x)(1-(2\alpha-1)x)} \right).$$

A straightforward application of Lemma 3.2 shows that  $\gamma'_1 \leq 0$  and  $\gamma'_2 \geq 0$  which finishes the proof of the lemma.  $\square$

#### 4. Proof of Theorems 1.1 and 1.2

The proofs of the two theorems are similar, so we treat them as a whole and make the differences precise when needed. Without loss of generality, we can suppose  $n \geq 3$ . In Theorem 1.1 (resp. Theorem 1.2), we suppose that the geometric mean  $g > 0$  (resp. harmonic mean  $h > 0$ ) and the arithmetic mean  $a > 0$  of a list of  $n$  strictly positive reals are given and we want to find sharp bounds on the harmonic mean (resp. geometric mean). Before going further, let us notice that the expression  $\left( \sum_i \frac{\alpha_i}{x_i} \right)^{-1}$ , defined for  $x_i > 0$ , can be continuously continued on  $[0, \infty[^n$  by setting its value to 0 as soon as  $x_i = 0$  for some  $i$ . Let  $\mathbb{R}_{\geq 0} = [0, \infty[$  and let us define the three sets  $S_h, S_g$  and  $S_a$  as follows:

$$S_a = \left\{ x \in \mathbb{R}^n \mid x_i \in [0, 1/\alpha_i], \sum \alpha_i x_i = 1 \right\},$$

and

$$S_g = \left\{ x \in \mathbb{R}_{\geq 0}^n \mid \prod x_i^{\alpha_i} = g \right\}, S_h = \left\{ x \in \mathbb{R}_{\geq 0}^n \mid \left( \sum \frac{\alpha_i}{x_i} \right)^{-1} = h \right\}.$$

The condition sets  $C_1$  and  $C_2$  on the  $x_i$  related to Theorems 1.1 and 1.2 respectively are given by  $C_1 = S_g \cap S_a$  and  $C_2 = S_h \cap S_a$ . Because they are defined through the preimage of closed sets via continuous maps, the sets  $S_g$  and  $S_h$  are closed and  $S_a$  is compact. Therefore  $C_1$  and  $C_2$  are compact in  $\mathbb{R}^n$  as the intersection between a compact and a closed set. Since the functions to optimize are well defined and continuous on these sets, their maximum and minimum are reached, and we will explicitly find them. The constraints being of class  $C^1$ , we use the Lagrange multipliers to find these optimums. When the expression to optimize is  $\sum_{i=1}^n \frac{\alpha_i}{x_i}$  and the geometric and the arithmetic means

are known, the Lagrange's conditions are

$$\frac{\partial}{\partial x_i} \left( \sum_{i=1}^n \frac{\alpha_i}{x_i} - A \cdot \left( \sum_{i=1}^n \alpha_i x_i - 1 \right) - B \cdot \left( \sum_{i=1}^n \alpha_i \ln(x_i) - \ln(g) \right) \right) = 0$$

which gives

$$-\frac{1}{x_i^2} - A - \frac{B}{x_i} = 0, \forall i = 1, \dots, n.$$

When the expression to optimize is  $\prod_{i=1}^n x_i^{\alpha_i}$  and the harmonic and the arithmetic means are known, the Lagrange's conditions applied to the natural logarithm of the product are

$$\frac{\partial}{\partial x_i} \left( \sum_{i=1}^n \alpha_i \ln(x_i) - A \cdot \left( \sum_{i=1}^n \alpha_i x_i - 1 \right) - B \cdot \left( \sum_{i=1}^n \frac{\alpha_i}{x_i} - h^{-1} \right) \right) = 0$$

which gives

$$\frac{1}{x_i} - A - \frac{B}{x_i^2} = 0, \forall i = 1, \dots, n.$$

In both cases, each  $x_i$  is equal to one of the roots, say  $X, Y$ , of a second degree polynomial. Since we supposed that the  $x_i$ 's are not all equal, we have  $X \neq Y$ . Now, note that if  $\alpha = \sum_{i \in I} \alpha_i$  with  $I = \{i | x_i = X\}$ , then  $1 - \alpha = \sum_{j \in J} \alpha_j$  with  $J = \{j | x_j = Y\}$ , and we may suppose without loss of generality that  $\alpha \in [0, 1/2]$ . Moreover

$$\begin{aligned} \sum_{i=1}^n \alpha_i x_i &= \alpha X + (1 - \alpha)Y = 1 \\ \prod_{i=1}^n x_i^{\alpha_i} &= X^\alpha Y^{1-\alpha} = g \\ \left( \sum_{i=1}^n \frac{\alpha_i}{x_i} \right)^{-1} &= \left( \frac{\alpha}{X} + \frac{1-\alpha}{Y} \right)^{-1} = h. \end{aligned}$$

Suppose both the geometric and the arithmetic means are known and the minimum and the maximum of the harmonic mean have to be determined. Making use of Lemma 3.3 and 3.4 and the previous notations, since  $\alpha \leq 1/2$ , we have  $H(X) < h < H(Y)$  where  $X < 1 < Y$ . The functions  $H(X)$  and  $H(Y)$  being decreasing and increasing functions of  $\alpha$ , the minimum of  $H(X)$  and the maximum of  $H(Y)$  are reached when  $\alpha = \min_i \{\alpha_i\}$ ,  $\alpha = 0$  being impossible.

Similarly, suppose both the harmonic and the arithmetic means are known and the minimum and the maximum of the geometric mean have to be determined. Making use now of Lemma 3.5 and 3.6, since  $\alpha \leq 1/2$ , we have  $G(Y) < h < G(X)$  where  $X < 1 < Y$ . The functions  $G(Y)$  and  $G(X)$  being decreasing and increasing functions of  $\alpha$ , the minimum of  $G(Y)$  and the maximum of  $G(X)$  are reached as before when  $\alpha = \min_i \{\alpha_i\}$ .

The statement of each Theorem 1.1 and 1.2 follows then directly from the statements of Lemma 3.3 and 3.5.

## 5. Applications

EXAMPLE 5.1. The first application is a bound on the trace of the inverse of a matrix whose eigenvalues are all positive. This problem has been treated by several authors (see [1] and the reference therein). If  $\lambda_i$  are the eigenvalues of such an  $n \times n$  matrix  $A$ , then  $\det(A) = \prod_{i=1}^n \lambda_i$ ,  $\text{trace}(A) = \sum_{i=1}^n \lambda_i$ , and  $\text{trace}(A^{-1}) = \sum_{i=1}^n 1/\lambda_i$ . The connection with the arithmetic, geometric and harmonic means is clear, and Corollary 2.2 shows that

$$\text{trace}(A^{-1}) < e \cdot \left( \frac{\text{trace}(A)}{n} \right)^n \cdot \frac{1}{\det(A)} + \frac{n^2}{\text{trace}(A)}.$$

EXAMPLE 5.2. The second application is a bound on the quotient of some coefficients of polynomials with positive roots. It has been known since Fransén and Lohne [3] (see also [5]) that if the polynomial

$$a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$$

has positive roots, then

$$|a_{n-1}| \geq n^2 \left| \frac{a_0 a_n}{a_1} \right|.$$

An application of Corollary 2.2 shows that the following reverse inequality holds:

$$|a_{n-1}| \leq n^2 \left| \frac{a_0 a_n}{a_1} \right| + e |a_0| \left| \frac{a_1}{n a_0} \right|^n.$$

Indeed, if  $\lambda_i$  are the roots of the polynomial, then  $|a_n/a_0| = \prod_{i=1}^n \lambda_i$ ,  $|a_1/a_0| = \sum_{i=1}^n \lambda_i$ , and  $|a_{n-1}/a_n| = \sum_{i=1}^n 1/\lambda_i$ .

## Acknowledgments

The authors would like to thank the reviewer for several helpful suggestions.

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(Received September 30, 2009)

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