Mathematical
Inequalities
& Applications
Volume 15, Number 1 (2012), 27–35

GLOBAL ESTIMATES FOR THE MAXIMAL OPERATOR AND HOMOTOPY OPERATOR

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(*Communicated by I. Perić*)

Abstract. In this paper, we develop some estimates for the Hardy-Littlewood maximal operator and the sharp maximal operator. We also establish L^s-norm inequalities related to the composite operators.

1. Introduction

For a locally L^s-integrable form $u(y)$, the Hardy-Littlewood maximal operator $\mathbb{M}_s$ is defined by

$$\mathbb{M}_s(u) = \mathbb{M}_s u(x) = \sup_{r>0} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |u(y)|^s dy \right)^{1/s}, \tag{1.1}$$

where $B(x,r)$ is the ball of radius r, centered at x, $1 \leqslant s < \infty$. We write $\mathbb{M}(u) = \mathbb{M}_1(u)$ if $s = 1$. Similarly, for a locally L^s-integrable form $u(y)$, we define the sharp maximal operator $\mathbb{M}_s^\sharp$ by

$$\mathbb{M}_s^\sharp(u) = \mathbb{M}_s^\sharp u(x) = \sup_{r>0} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |u(y) - u_{B(x,r)}|^s dy \right)^{1/s}. \tag{1.2}$$

The maximal operators and homotopy operator are effective tools in analysis which have found many applications in different areas of mathematics, including topology, differential geometry and potential theory, etc. Some estimates for these operators have been established in recent years, see [1, 7, 8, 10].

We always assume that Ω is a domain in $\mathbb{R}^n$, $n \geqslant 2$, and $\wedge^l = \wedge^l(\mathbb{R}^n)$ is the set of all l-forms $\omega(x) = \sum_I \omega_I(x) dx_I = \sum \omega_{i_1 i_2 \cdots i_l}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}$ in $\mathbb{R}^n$, where $\omega_{i_1 i_2 \cdots i_l}(x)$ are functions in $\mathbb{R}^n$. Let $D'(\Omega, \wedge^l)$ be the space of all differential l-forms on Ω and $L^p(\Omega, \wedge^l)$ be the space of all l-forms on Ω satisfying $\int_\Omega |\omega_I|^p < \infty$ for all ordered l-tuples I. Let B be a ball and σB be the ball with the same center as B and with $\operatorname{diam}(\sigma B) = \sigma \operatorname{diam}(B)$, $\sigma > 0$. We call w a weight if $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $w > 0$

Mathematics subject classification (2010): Primary 26B10; Secondary 31B10, 46E35.
Keywords and phrases: Differential forms, harmonic equations, homotopy operator, maximal operator.
The first author was supported by Development Program for Outstanding Young Teachers in HIT and by Science Research Foundation in Harbin Institute of Technology (HITC200709).

Paper MIA-15-03

a.e.. We write $\|u\|_{s,\Omega} = (\int_\Omega |u|^s)^{1/s}$ and $||u||_{s,\Omega,w} = (\int_\Omega |u|^s w(x)dx)^{1/s}$, where $w(x)$ is a weight.

In this paper, we will develop some norm inequalities for the composition of maximal operator and homotopy operator applied to the solutions of the nonhomogeneous A-harmonic equation for differential forms

$$d^\star A(x,d\omega) = B(x,d\omega), \tag{1.3}$$

where $A:\Omega\times\wedge^l(\mathbb{R}^n)\to\wedge^l(\mathbb{R}^n)$ and $B:\Omega\times\wedge^l(\mathbb{R}^n)\to\wedge^{l-1}(\mathbb{R}^n)$ satisfy the conditions:

$$|A(x,\xi)|\leqslant a|\xi|^{p-1},\ A(x,\xi)\cdot\xi\geqslant|\xi|^p,\ |B(x,\xi)|\leqslant b|\xi|^{p-1} \tag{1.4}$$

for almost every $x\in\Omega$ and all $\xi\in\wedge^l(\mathbb{R}^n)$. Here $a,b>0$ are constants and $1<p<\infty$ is a fixed exponent associated with (1.3). If $B=0$ in (1.3), then equation (1.3) is called the A-harmonic equation. Some results about different versions of the A-harmonic equation have been obtained in recent years, see [1, 5, 9, 10].

From [4], we have the following result: Let $D\subset\mathbb{R}^n$ be a bounded, convex domain. To each $y\in D$ there is a linear operator $K_y: C^\infty(D,\wedge^l)\to C^\infty(D,\wedge^{l-1})$ defined by $(K_y\omega)(x;\xi_1,\cdots,\xi_{l-1})=\int_0^1 t^{l-1}\omega(tx+y-ty;x-y,\xi_1,\cdots,\xi_{l-1})dt$ and the decomposition $\omega=d(K_y\omega)+K_y(d\omega)$ holds at any point $y\in D$. A homotopy operator $T: C^\infty(D,\wedge^l)\to C^\infty(D,\wedge^{l-1})$ is defined by averaging K_y over all points y in D

$$T\omega=\int_D\varphi(y)K_y\omega dy\,, \tag{1.5}$$

where $\varphi\in C_0^\infty(D)$ is normalized by $\int_D\varphi(y)dy=1$. We define the l-form $\omega_D\in D'(D,\wedge^l)$ by

$$\omega_D=|D|^{-1}\int_D\omega(y)dy,\ \ l=0,\ \text{ and }\ \omega_D=d(T\omega),\ \ l=1,2,\cdots,n \tag{1.6}$$

for all $\omega\in L^p(D,\wedge^l)$, $1\leqslant p<\infty$, then $\omega_D=\omega-T(d\omega)$ and

$$\|Tu\|_{s,D}\leqslant C|D|\,\mathrm{diam}(D)\|u\|_{s,D}. \tag{1.7}$$

2. Local results and proofs

The following Lemma 2.1 appearing in [3] will be used in the proof of Theorem 2.3.

LEMMA 2.1. *Let $\mathbb{M}_s$, $s\geqslant1$, be the Hardy-Littlewood maximal operator defined in (1.1) and $u\in L^t(\Omega,\wedge^l)$, $l=1,2,\cdots,n$, $s<t<\infty$, be a differential form in a domain Ω. Then, $\mathbb{M}_s(u)\in L^t(\Omega)$ and*

$$\|\mathbb{M}_s(u)\|_{t,\Omega}\leqslant C\|u\|_{t,\Omega} \tag{2.1}$$

for some constant C, independent of u.

The following local Poincaré-type estimate for the homotopy operator T appears in [2].

LEMMA 2.2. *Assume that* $u \in L^s_{\mathrm{loc}}(\Omega, \wedge^l)$, $l = 1, 2, \cdots, n$, $1 < s < \infty$, *be a solution of the* A*-harmonic equation* (1.3) *in a convex, bounded domain* Ω *and* T *be the homotopy operator defined in* (1.5). *Then, there exists a constant* C, *independent of* u, *such that*

$$\|T(u) - (T(u))_B\|_{s,B} \leqslant C|B| \operatorname{diam}(B) \|u\|_{s,\rho B} \tag{2.2}$$

for all balls B *with* $\rho B \subset \Omega$, *where* $\rho > 1$ *is a constant.*

Note that (2.1) holds for any differential form in a domain Ω. Thus, if we replace u in (2.1) by $T(u)$, we obtain the following estimate for the composition of the Hardy-Littlewood maximal operator and the Homotopy operator.

THEOREM 2.3. *Assume that* $u \in L^t(\Omega, \wedge^l)$, $l = 1, 2, \cdots, n$, $s < t < \infty$, *is a differential form in a convex, bounded domain* Ω *and* $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ *is the homotopy operator defined in* (1.5). *Let* $\mathbb{M}_s$, $s \geqslant 1$, *be the Hardy-Littlewood maximal operator defined in* (1.1). *Then,* $\mathbb{M}_s(T(u)) \in L^t(\Omega)$ *and*

$$\|\mathbb{M}_s(T(u))\|_{t,\Omega} \leqslant C|\Omega| \operatorname{diam}(\Omega) \|u\|_{t,\Omega} \tag{2.3}$$

for some constant C, *independent of* u.

Proof. Note that (2.1) holds for any differential form in a domain Ω. Thus, replacing u in (2.1) by $T(u)$, we obtain

$$\|\mathbb{M}_s(T(u))\|_{t,\Omega} \leqslant C_1 \|T(u)\|_{t,\Omega.}$$

Using inequality (1.7), it follows that

$$\begin{aligned} \|\mathbb{M}_s(T(u))\|_{t,\Omega} &\leqslant C_1 \|T(u)\|_{t,\Omega} \\ &\leqslant C_2 |\Omega| \operatorname{diam}(\Omega) \|u\|_{t,\Omega}. \end{aligned}$$

This ends the proof of Theorem 2.3. □

We now develop some estimates related to the sharp maximal operator $\mathbb{M}_s^\sharp$ and the homotopy operator. We also study the relationship between $\|\mathbb{M}_s^\sharp\|_{s,\Omega}$ and $\|\mathbb{M}_s\|_{s,\Omega}$.

THEOREM 2.4. *Assume that* $u \in L^s(\Omega, \wedge^l)$, $l = 1, 2, \cdots, n$, $1 < s < \infty$, *be a differential form satisfying* (1.3) *in a convex, bounded domain* Ω, *and* T *be the homotopy operator defined in* (1.5). $\mathbb{M}_s^\sharp$ *be the sharp maximal operator defined in* (1.2). *Then,*

$$\|\mathbb{M}_s^\sharp(T(u))\|_{s,\Omega} \leqslant C\|u\|_{s,\Omega} \tag{2.4}$$

for some constant C, *independent of* u.

Proof. Using Lemma 2.2 over the ball $B(x,r)$, we obtain

$$\begin{aligned} &\left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |T(u) - (T(u))_{B(x,r)}|^s dy \right)^{1/s} \\ &\qquad \leqslant C_1 |B(x,r)|^{1-1/s} \operatorname{diam}(B(x,r)) \left(\int_{\rho B(x,r)} |u|^s dy \right)^{1/s} \\ &\qquad \leqslant C_2 |\Omega|^{1-1/s} \operatorname{diam}(\Omega) \|u\|_{s,\Omega} \\ &\qquad \leqslant C_3 \|u\|_{s,\Omega} \end{aligned} \tag{2.5}$$

since $1-1/s>0$ and Ω is bounded. Thus, it follows that

$$\sup_{r>0}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}|T(u)-(T(u))_{B(x,r)}|^s dy\right)^{1/s}\leqslant C_3\|u\|_{s,\Omega}. \tag{2.6}$$

From (2.6), and using the definition of $\mathbb{M}_s^\sharp$, we have

$$\begin{aligned}\|\mathbb{M}_s^\sharp(T(u))\|_{s,\Omega}&=\left(\int_\Omega|\mathbb{M}_s^\sharp(T(u))|^s dx\right)^{1/s}\\&=\left(\int_\Omega\left|\sup_{r>0}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}|T(u)-(T(u))_{B(x,r)}|^s dy\right)^{1/s}\right|^s dx\right)^{1/s}\\&\leqslant\left(\int_\Omega\left|C_3\|u\|_{s,\Omega}\right|^s dx\right)^{1/s}\\&\leqslant C_4\|u\|_{s,\Omega}.\end{aligned}$$

We have completed the proof of Theorem 2.4. □

THEOREM 2.5. *Assume that $u\in L^s_{\text{loc}}(\Omega,\wedge^l)$, $l=1,2,\cdots,n$, $1<s<\infty$, is a solution of the A-harmonic equation (1.3) in a convex, bounded domain Ω and T is the homotopy operator defined in (1.5). Let $\mathbb{M}_s$ be the Hardy-Littlewood maximal operator defined in (1.1) and $\mathbb{M}_s^\sharp$ be the sharp maximal operator defined in (1.2). Then,*

$$\|\mathbb{M}_s^\sharp(T(u))\|_{s,\Omega}\leqslant C_3\|\mathbb{M}_s(u)\|_{s,\Omega}. \tag{2.7}$$

for some constant C, independent of u.

Proof. From Lemma 2.2, we know that

$$\left(\int_{B(x,r)}|T(u(y))-(T(u(y)))_{B(x,r)}|^s dy\right)^{1/s}\leqslant C_1|B(x,r)|^{1+1/n}\left(\int_{B(x,r)}|u|^s dy\right)^{1/s}$$

that is,

$$\begin{aligned}&\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}|T(u(y))-(T(u(y)))_{B(x,r)}|^s dy\right)^{1/s}\\&\qquad\leqslant C_1|B(x,r)|^{1+1/n}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}|u|^s dy\right)^{1/s}.\end{aligned}\tag{2.8}$$

Now, by definitions of the Hardy-Littlewood maximal operator $\mathbb{M}_s$ and the sharp maximal operator $\mathbb{M}_s^\sharp$, and (2.8), we obtain

$$\begin{aligned}
\|\mathbb{M}_s^\sharp(T(u))\|_{s,\Omega} &= \left(\int_\Omega |\mathbb{M}_s^\sharp(T(u))|^s dx\right)^{1/s} \\
&= \left(\int_\Omega \left|\sup_{r>0}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} |T(u(y)) - (T(u(y)))_{B(x,r)}|^s dy\right)^{1/s}\right|^s dx\right)^{1/s} \\
&\leqslant \left(\int_\Omega \left|\sup_{r>0} C_1|B(x,r)|^{1+1/n}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} |u(y)|^s dy\right)^{1/s}\right|^s dx\right)^{1/s} \\
&\leqslant \left(\int_\Omega \left|\sup_{r>0} C_1|\Omega|^{1+1/n}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} |u(y)|^s dy\right)^{1/s}\right|^s dx\right)^{1/s} \\
&\leqslant C_2\left(\int_\Omega \left|\sup_{r>0}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} |u(y)|^s dy\right)^{1/s}\right|^s dx\right)^{1/s} \\
&\leqslant C_2\left(\int_\Omega |\mathbb{M}_s(u)|^s dx\right)^{1/s} \\
&= C_3\|\mathbb{M}_s(u)\|_{s,\Omega}
\end{aligned}$$

which is equivalent to

$$\|\mathbb{M}_s^\sharp(T(u))\|_{s,\Omega} \leqslant C\|\mathbb{M}_s(u)\|_{s,\Omega}.$$

This ends the proof of Theorem 2.5. □

DEFINITION 2.6. We say that a weight $w(x)$ satisfies the $A_r(\Omega)$ condition, $r > 1$, write $w \in A_r(\Omega)$, if $w(x) > 0$ a.e., and

$$\sup_B \left(\frac{1}{|B|}\int_B w dx\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{(r-1)} < \infty \tag{2.9}$$

for any ball $B \subset \Omega$.

LEMMA 2.7. [8] *If $w \in A_r(\Omega)$, then there exist constants $\beta > 1$ and C, independent of w, such that*

$$\| w \|_{\beta,B} \leqslant C|B|^{(1-\beta)/\beta} \| w \|_{1,B} \tag{2.10}$$

for all balls $B \subset \Omega$.

LEMMA 2.8. [6] *Let u be a smooth differential form satisfying equation (1.3) in Ω, $\sigma > 1$ and $0 < s,t < \infty$. Then, there exists a constant C, independent of u, such that*

$$\|u\|_{s,B} \leqslant C|B|^{(t-s)/st}\|u\|_{t,\sigma B} \tag{2.11}$$

for all balls or cubes B with $\sigma B \subset \Omega$.

THEOREM 2.9. *Suppose that $u \in L^t_{\text{loc}}(\Omega, \wedge^l)$, $l = 1, 2, \cdots, n$, $s < t < \infty$, is a solution of the A-harmonic equation (1.3) in a convex, bounded domain Ω and T is the homotopy operator defined in (1.5). Let $\mathbb{M}^\sharp_s$ be the sharp maximal operator defined in (1.2). Assume that $w \in A_r(\Omega)$ for some $r > 1$. Then, there exists a constant C, independent of u, such that*

$$\|\mathbb{M}^\sharp_s(T(u))\|_{t,B,w} \leqslant C\|u\|_{t,\sigma B,w} \tag{2.12}$$

for all balls B with $\sigma B \subset \Omega$, where $\sigma > 1$ is a constant.

Proof. Since $w \in A_r(\Omega)$, from Lemma 2.7, it follows that there exist constants $\beta > 1$ and $C_1 > 0$, such that

$$\| w \|_{\beta,B} \leqslant C_1 |B|^{(1-\beta)/\beta} \| w \|_{1,B} \tag{2.13}$$

for any ball $B \subset \Omega$. Let $k = t\beta/(\beta - 1)$, then $k > t$ and $\beta = k/(k-t)$. Using Hölder's inequality with $1/t = 1/k + (k-t)/kt$, we have

$$\begin{aligned} \|\mathbb{M}^\sharp_s(T(u))\|_{t,B,w} &= \left(\int_B \left(|\mathbb{M}^\sharp_s(T(u))| w^{1/t}\right)^t dx\right)^{1/t} \\ &\leqslant \left(\int_B |\mathbb{M}^\sharp_s(T(u))|^k dx\right)^{1/k} \left(\int_B \left(w^{1/t}\right)^{kt/(k-t)} dx\right)^{(k-t)/kt} \\ &\leqslant C_2 \|\mathbb{M}^\sharp_s(T(u))\|_{k,B} \cdot \|w\|^{1/t}_{\beta,B}. \end{aligned} \tag{2.14}$$

Since $k > t > s$, choose the domain Ω to be a ball B in Theorem 2.4, we obtain

$$\|\mathbb{M}^\sharp_s(T(u))\|_{k,B} \leqslant C_3 \|u\|_{k,B}. \tag{2.15}$$

Combining (2.13), (2.14) and (2.15) yields

$$\|\mathbb{M}^\sharp_s(T(u))\|_{t,B,w} \leqslant C_4 |B|^{(1-\beta)/\beta t} \|w\|^{1/t}_{1,B} \|u\|_{k,B}. \tag{2.16}$$

Now, set $m = t/r$, then $m < t$. By Lemma 2.8, we obtain

$$\|u\|_{k,B} \leqslant C_5 |B|^{(m-k)/mk} \|u\|_{m,\sigma B}. \tag{2.17}$$

Substituting (2.17) in (2.16), it follows that

$$\|\mathbb{M}^\sharp_s(T(u))\|_{t,B,w} \leqslant C_6 |B|^{\frac{1-\beta}{\beta t} + \frac{m-k}{mk}} \|w\|^{1/t}_{1,B} \cdot \|u\|_{m,\sigma B}. \tag{2.18}$$

Using Hölder's inequality with $1/m = 1/t + (t-m)/tm$ again, we have

$$\begin{aligned} \|u\|_{m,\sigma B} &= \left(\int_{\sigma B} \left(|u| w^{1/t} w^{-1/t}\right)^m dx\right)^{1/m} \\ &\leqslant \left(\int_{\sigma B} |u|^t w dx\right)^{1/t} \left(\int_{\sigma B} \left(\frac{1}{w}\right)^{m/(t-m)} dx\right)^{(t-m)/mt} \\ &\leqslant \|u\|_{t,\sigma B,w} \cdot \|1/w\|^{1/t}_{m/(t-m),\sigma B} \end{aligned} \tag{2.19}$$

for all balls B with $\sigma B \subset \Omega$. Substituting (2.19) in (2.18), we obtain

$$\|\mathbb{M}_s^\sharp(T(u))\|_{t,B,w} \leqslant C_7 |B|^{\frac{1-\beta}{\beta t}+\frac{m-k}{mk}} \|w\|_{1,B}^{1/t} \|1/w\|_{m/(t-m),\sigma B}^{1/t} \cdot \|u\|_{t,\sigma B,w}. \tag{2.20}$$

Since $w \in A_r(\Omega)$, it follows that

$$\begin{aligned}
&\|w\|_{1,B}^{1/t} \cdot \|1/w\|_{m/(t-m),\sigma B}^{1/t} \\
&\quad = \left(\int_B w dx\right)^{1/t} \left(\int_{\sigma B} \left(\frac{1}{w}\right)^{m/(t-m)} dx\right)^{(t-m)/tm} \\
&\quad \leqslant \left(\left(\int_{\sigma B} w dx\right) \left(\int_{\sigma B} \left(\frac{1}{w}\right)^{1/(t/m-1)} dx\right)^{t/m-1}\right)^{1/t} \\
&\quad = \left(|\sigma B|^{\frac{t}{m}} \left(\frac{1}{|\sigma B|}\int_{\sigma B} w dx\right) \left(\frac{1}{|\sigma B|}\int_{\sigma B} (w)^{\frac{1}{1-r}} dx\right)^{r-1}\right)^{\frac{1}{t}} \\
&\quad \leqslant C_8 |B|^{1/m}.
\end{aligned} \tag{2.21}$$

Substituting (2.21) in (2.20), we obtain

$$\|\mathbb{M}_s^\sharp(T(u))\|_{t,B,w} \leqslant C\|u\|_{t,\sigma B,w} \tag{2.22}$$

for all balls B with $\sigma B \subset \Omega$. This ends the proof of Theorem 2.9. □

Using the similar method to the proof of Theorem 2.9, we can establish some other weighted inequalities, such as two weight cases.

THEOREM 2.10. *Suppose that $u \in L^t(\Omega, \wedge^l)$, $l = 1,2,\cdots,n$, $s < t < \infty$, is a solution of the A-harmonic equation (1.3) in a convex, bounded domain Ω and T is the homotopy operator defined in (1.5). Let $\mathbb{M}_s$, $s \geqslant 1$, be the Hardy-Littlewood maximal operator defined in (1.1). Assume that $w \in A_r(\Omega)$ for some $r > 1$. Then, there exists a constant C, independent of u, such that*

$$\|\mathbb{M}_s(T(u))\|_{t,B,w} \leqslant C|B| \operatorname{diam}(B) \|u\|_{t,\sigma B,w} \tag{2.23}$$

for all balls B with $\sigma B \subset \Omega$, where $\sigma > 1$ is a constant.

3. Global weighted estimates

We need the result about the Whitney cover appearing in [6] to prove the global results.

LEMMA 3.1. *Each domain Ω has a modified Whitney cover of cubes $\mathscr{V} = \{Q_i\}$ such that*

$$\cup_i Q_i = \Omega, \quad \sum_{Q\in\mathscr{V}} \chi_{\sqrt{\frac{5}{4}}Q} \leqslant N\chi_\Omega$$

for all $x \in \mathbb{R}^n$ *and some* $N > 1$ *, and if* $Q_i \cap Q_j \neq \phi$*, then there exists a cube* R *(this cube does not need be a member of* $\mathscr{V}$ *) in* $Q_i \cap Q_j$ *such that* $Q_i \cup Q_j \subset NR$*.*

We prove the following global weighted norm estimates related to the sharp maximal operator $\mathbb{M}_s^\sharp$ and the homotopy operator.

THEOREM 3.2. *Suppose that* $u \in L^t_{\text{loc}}(\Omega, \wedge^l)$*,* $l = 1, 2, \cdots, n$*,* $s < t < \infty$*, is a solution of the* A*-harmonic equation* (1.3) *in a convex, bounded domain* Ω *and* T *is the homotopy operator defined in* (1.5)*. Let* $\mathbb{M}_s^\sharp$ *be the sharp maximal operator defined in* (1.2)*. Assume that* $w \in A_r(\Omega)$ *for some* $r > 1$*. Then,*

$$\|\mathbb{M}_s^\sharp(T(u))\|_{t,\Omega,w} \leqslant C\|u\|_{t,\Omega,w} \tag{3.1}$$

holds for some constant C*, independent of* u*.*

Proof. Using the inequality $|\sum y_\alpha|^t \leqslant \sum |y_\alpha|^t, 0 \leqslant t \leqslant 1$, Theorem 2.9 and Lemma 3.1, we have

$$\begin{aligned}
\|\mathbb{M}_s^\sharp(T(u))\|_{s,\Omega,w} &= \left(\int_\Omega |\mathbb{M}_s^\sharp(T(u))|^s w dx\right)^{1/s} \\
&\leqslant \left(\sum_{B \in \mathscr{V}} \int_B |\mathbb{M}_s^\sharp(T(u))|^s w dx\right)^{1/s} \\
&\leqslant \sum_{B \in \mathscr{V}} \left(\int_B |\mathbb{M}_s^\sharp(T(u))|^s w dx\right)^{1/s} \\
&\leqslant \sum_{B \in \mathscr{V}} \left(\int_B |\mathbb{M}_s^\sharp(T(u))|^s w \cdot \chi_{\sqrt{\frac{5}{4}}B} dx\right)^{1/s} \\
&\leqslant C_1 \sum_{B \in \mathscr{V}} \left(\int_{\sigma B} |u|^s w \cdot \chi_{\sqrt{\frac{5}{4}}B} dx\right)^{1/s} \\
&\leqslant C_1 \cdot N \left(\int_\Omega |u|^s w dx\right)^{1/s} \\
&\leqslant C_2 \left(\int_\Omega |u|^s w dx\right)^{1/s}
\end{aligned}$$

since Ω is bounded. The proof of Theorem 3.2 has been completed. □

Using the similar method to the proof of Theorem 3.2, we can establish the global weighted norm estimates related to the maximal operator $\mathbb{M}_s$ and the homotopy operator T.

THEOREM 3.3. *Suppose that* $u \in L^t(\Omega, \wedge^l)$*,* $l = 1, 2, \cdots, n$*,* $s < t < \infty$*, is a solution of the* A*-harmonic equation* (1.3) *in a convex, bounded domain* Ω *and* T *is the homotopy operator defined in* (1.5)*. Let* $\mathbb{M}_s$*,* $s \geqslant 1$*, be the Hardy-Littlewood maximal operator defined in* (1.1)*. Assume that* $w \in A_r(\Omega)$ *for some* $r > 1$*. Then,*

$$\|\mathbb{M}_s(T(u))\|_{t,\Omega,w} \leqslant C\|u\|_{t,\Omega,w} \tag{3.2}$$

holds for some constant C*, independent of* u*.*

REFERENCES

[1] R. P. AGARWAL AND S. DING, *Advances in differential forms and the A-harmonic equations*, Mathematical and Computer Modelling **37** (2003), 1393–1426.
[2] S. DING, *Lipschitz and BMO norm inequalities for operators,* Nonlinear Analysis **71** (2009), e2350–e2357.
[3] S. DING, *Norm estimates for the maximal operator and Green's operator*, The Proceeding of the 6th international conference on Differential Equations and Dynamical Systems, 2008.
[4] T. IWANIEC AND A. LUTOBORSKI, *Integral estimates for null Lagrangians*, Arch. Rational Mech. Anal. **125** (1993), 25–79.
[5] B. LIU, $A_r^{\lambda}(\Omega)$*-weighted imbedding inequalities for A-harmonic tensors*, J. Math. Anal. Appl. **273** (2002), 667–676.
[6] C. A. NOLDER, *Hardy-Littlewood theorems for A-harmonic tensors*, Illinois J. of Math. **43**(1999), 613–631.
[7] C. SCOTT, L^p*-theory of differntial forms on manifolds*, Trans. Amer. Soc **347** (1995), 2075–2096.
[8] [8] E. M. STEIN, *Harmonic analysis*, Princeton University Press, Princeton, 1993.
[9] Y. XING, *Two-weight imbedding inequalities for solutions to the A-harmonic equation*, J. Math. Anal. Appl. **307** (2005), 555–564.
[10] Y. XING AND C. WU, *Global weighted inequalities for operators and harmonic forms on manifolds*, J. Math. Anal. Appl. **294** (2004), 294–309.

(Received September 16, 2009)

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