Mathematical
Inequalities
& Applications
Volume 15, Number 1 (2012), 49–59

VORONOVSKAJA'S THEOREM FOR SCHOENBERG OPERATOR

GANCHO TACHEV

(*Communicated by R. Mohapatra*)

Abstract. In this paper we represent new quantitative variants of Voronovskaja's Theorem for Schoenberg variation-diminishing spline operator. We estimate the rate of uniform convergence for $f \in C^2[0,1]$ and generalize the results obtained earlier by Goodman, Lee, Sharma, Gonska etc.

1. Introduction

We start with the definition of variation-diminishing operator, introduced by I. Schoenberg. For the case of equidistant knots we denote it by $S_{n,k}$. Consider the knot sequence $\Delta_n = \{x_i\}_{-k}^{n+k}$, $n \geqslant 1, k \geqslant 1$ with equidistant "interior knots", namely

$$\Delta_n : x_{-k} = \ldots = x_0 = 0 < x_1 < x_2 < \ldots < x_n = \ldots = x_{n+k} = 1$$

and $x_i = \frac{i}{n}$ for $0 \leqslant i \leqslant n$. For a bounded real-valued function f defined over the interval $[0,1]$ the variation-diminishing spline operator of degree k w.r.t. Δ_n is given by

$$S_{n,k}(f,x) = \sum_{j=-k}^{n-1} f(\xi_{j,k}) \dot{N}_{j,k}(x) \tag{1.1}$$

for $0 \leqslant x < 1$ and

$$S_{n,k}(f,1) = \lim_{y \to 1, y<1} S_{n,k}(f,y)$$

with the nodes (Greville abscissas)

$$\xi_{j,k} := \frac{x_{j+1} + \ldots + x_{j+k}}{k}, \; -k \leqslant j \leqslant n-1, \tag{1.2}$$

and the normalized $B-$ splines as fundamental functions

$$N_{j,k}(x) := (x_{j+k+1} - x_j)\dot{[}x_j, x_{j+1}, \ldots, x_{j+k+1}](\cdot - x)_+^k.$$

The first quantitative variant of Voronovskaja's Theorem for a broad class of linear positive operators L was obtained very recently by H. Gonska, P. Pitul and I. Rasa in [6] (see the proof of Theorem 6.2). We cite this result below.

Mathematics subject classification (2010): 41A10, 41A15, 41A17, 41A25, 41A17, 41A25, 41A36.
Keywords and phrases: Schoenberg-spline operator, Voronovskaja-type theorem, Degree of approximation.

THEOREM A. *Let $L: C[0,1] \to C[0,1]$ be a positive, linear operator reproducing linear functions. If $f \in C^2[0,1]$ and $x \in [0,1]$ then*

$$\left| L(f;x) - f(x) - \frac{1}{2} \cdot f''(x) \cdot L((e_1 - x)^2;x) \right|$$

$$\leqslant \frac{1}{2} \cdot L((e_1 - x)^2;x) \cdot \tilde{\omega}\left(f'', \frac{1}{3} \cdot \sqrt{\frac{L((e_1 - x)^4;x)}{L((e_1 - x)^2;x)}}\right). \tag{1.3}$$

Here $e_n : x \in [0,1] \to x^n$, $n = 0, 1, \dots$ are the monomial functions and $\tilde{\omega}(f, \cdot)$ denotes the least concave majorant of $\omega(f, \cdot)$ given by

$$\tilde{\omega}(f, \varepsilon) = \sup_{0 \leqslant x \leqslant \varepsilon \leqslant y \leqslant 1, x \neq y} \frac{(\varepsilon - x)\omega(f, y) + (y - \varepsilon)\omega(f, x)}{y - x},$$

for $0 \leqslant \varepsilon \leqslant 1$.

We use Theorem A as an auxiliary tool to prove our main results – Theorems 2.1, 3.1 and 4.1. Here we point out that to estimate the argument of $\tilde{\omega}(f'', \cdot)$ we need a "good" upper bound for $L((e_1 - x)^4;x)$ and a "good" lower bound for the second moment $L((e_1 - x)^2;x)$. As far as we know, explicit presentation in compact form are still not available neither for the second moment, nor for the fourth moment of Schoenberg operator, except for the case $S_{n,1}$ – the piecewise linear interpolant at the knots $\frac{i}{n}$, $0 \leqslant i \leqslant n$ and when $n = 1, k > 1$ – the Bernstein polynomial of degree k, $S_{1,k} = B_k$. The problem to study the order of uniform convergence in Voronovskaja's Theorem for these two classical operators was very recently solved in [6,7]. For example for the case $k = 1, n > 1$ – linear interpolant at the knots x_i, H. Gonska established in [7] the following result (see Ex. 6.2):

THEOREM B. *If $z_n(x) = \{nx\} := nx - [nx]$ is the fractional part of nx then*

$$\left| \frac{n^2}{z_n(x)(1 - z_n(x))} [S_{n,1}(f,x) - f(x)] - \frac{1}{2} f''(x) \right| \leqslant \frac{1}{2} \tilde{\omega}\left(f'', \frac{1}{3n}\right). \tag{1.4}$$

This is obtained via representation of the second and fourth moments as given by A.Lupas in his Ph.D. thesis [8]

$$S_{n,1}((e_1 - x)^2;x) = \frac{1}{n^2} z_n(x)(1 - z_n(x))$$

$$S_{n,1}((e_1 - x)^4;x) = \frac{1}{n^4} z_n(x)(1 - z_n(x))[1 - 3z_n(x)(1 - z_n(x))].$$

In [7] it was proposed to find quantitative statements also for cases other than $S_{n,1}$. At the same time the degree changing of polynomials and the size of equidistance knots should be considered. Several authors – Schoenberg, Marsden, Riemenschneider, Lee, Goodman, Sharma [4, 9, 10, 11, 12] have established different types of Voronovskaja's Theorem for the Schoenberg operator. All these results are proved by the assumption that

$$\lim_{(n+k) \to \infty} \frac{n}{k} = \lambda, \tag{1.5}$$

where $\lambda \in [0,\infty)$. For example $\lambda = \infty$ is the case, considered by Schoenberg in [12] (see Theorem C). The case $0 \leqslant \lambda \leqslant \infty$ was studied by Marsden in [11]. Note that $n+k$ is the number of data points $f(\xi_{j,k})$, needed to specify $S_{n,k}(f,x)$, so that $n+k$ is a measure of the "complexity" of $S_{n,k}$. What is missing is to prove estimates, similar to Theorem A and thus showing the rate of uniform convergence in a quantitative form when $f \in C^2[0,1]$. Our second goal is to prove variants of Voronovskaja's Theorem – see Theorems 2.1 and 3.1, without the condition (1.5) and as a corollary to obtain the results mentioned above.

Also in Theorem 3.1 we establish for the first time a lower bound for the second moment of Schoenberg operator. In Corollary 3.4 we prove for $f \in C^2[0,1]$ uniform convergence by Bernstein operator, which is in a form stronger than that in original Theorem of E.Voronovskaja. We give several examples as applications of Theorems 2.1 and 3.1. As a result of Theorem A we obtain in Section 4 strong converse inequality for approximation by Bernstein operator in pointwise form, when f'' belongs to some classes $Lip_\alpha(L)$ of Hölder continuous functions.

To estimate the second and fourth moments in (1.3) we differ the cases $k \leqslant n$ and $k \geqslant n$. In section 2 we consider the so-called "spline" case $(k \leqslant n)$ and in section 3 we study the "polynomial" case $(k \geqslant n)$. In the last section we make concluding remarks and compare our results with other previous established versions of Voronovskaja's Theorem.

2. The case $k \leqslant n$

The main result in this section is

THEOREM 2.1. *For $f \in C^2[0,1]$ we have*

$$\left| S_{n,k}(f,x) - f(x) - \frac{1}{2} S_{n,k}((e_1-x)^2,x) f''(x) \right| \leqslant \frac{1}{2} S_{n,k}((e_1-x)^2,x) \cdot \tilde{\omega}\left(f'', \frac{1}{3}\cdot\frac{k+1}{2n}\right). \tag{2.1}$$

Proof. First we observe that in the definition of the fourth moment of Schoenberg operator given by

$$S_{n,k}((e_1-x)^4,x) = \sum_{j=-k}^{n-1} (\xi_{j,k}-x)^4 N_{j,k}(x) \tag{2.2}$$

only those summands are different from 0, for which $x \in [x_j, x_{j+k+1}]$ – the support of $N_{j,k}(x)$ by fixed x. Following the definition of $\xi_{j,k}$ in (1.2) it is easy to compute

$$|\xi_{j,k} - x| \leqslant \frac{k+1}{2n}. \tag{2.3}$$

From (2.2) and (2.3) we get

$$S_{n,k}((e_1-x)^4,x) \leqslant \left(\frac{k+1}{2n}\right)^2 \cdot S_{n,k}((e_1-x)^2,x).$$

Now using the estimate (1.3) in Theorem A we complete the proof. $\square$

COROLLARY 2.2. *If $k=1$ and using the representation of the second moment of piecewise linear interpolant we get the estimate* (1.4) *proved in [7].*

The following upper bound for the second moment of $S_{n,k}$ valid for all $n,k \geqslant 1$ and $x \in [0,1]$ was recently proved in [1]

$$S_{n,k}((e_1-x)^2,x) \leqslant 1 \cdot \frac{\min\left\{2x(1-x);\frac{k}{n}\right\}}{n+k-1}. \tag{2.4}$$

Suppose that $\frac{k}{n} \to 0$ (when $k+n \to \infty$) and that

$$\lim \frac{n^2}{k} \cdot S_{n,k}((e_1-x)^2,x) = g(x),$$

where the convergence is uniform w.r.t. $x \in [0,1]$. Now from (2.1) we get

COROLLARY 2.3. *If $\frac{k}{n} \to 0$ then*

$$\lim_{n+k\to\infty} \frac{n^2}{k} \cdot [S_{n,k}(f,x) - f(x)] = \frac{1}{2} f''(x) \cdot g(x).$$

The convergence is uniform w.r.t. $x \in [0,1]$.

The last statement is the same as in Theorem 3 in [4], where it was supposed that f is integrable, bounded and has a second derivative at the point x in $(0,1)$. Here we show the uniform convergence for $f \in C^2[0,1]$.

Here we recall that in the spline case $(\frac{k}{n} \to 0)$ it was I.Schoenberg, who proves the first known Voronovskaja – type theorem in [12] which we formulate as

THEOREM C. *Let f be bounded in $[0,1]$ and $k>2$. If $x \in (0,1)$ is such that $f''(x)$ exists then the following pointwise convergence holds*

$$\lim_{\frac{k}{n}\to 0} \frac{n^2}{k} \cdot [S_{n,k}(f,x) - f(x)] = \frac{f''(x)}{24}$$

3. The case $k \geqslant n$

For the polynomial case of Schoenberg operator $k \geqslant n$ we prove the following

THEOREM 3.1. *For $f \in C^2[0,1]$ we have*

$$\left| S_{n,k}(f,x) - f(x) - \frac{1}{2} S_{n,k}((e_1-x)^2,x) f''(x) \right|$$

$$\leqslant \frac{1}{2} S_{n,k}((e_1-x)^2,x) \cdot \tilde{\omega}\left(f'', \frac{1}{3} \cdot \sqrt{\Delta_{n,k}(x)}\right), \tag{3.1}$$

where $\Delta_{n,k}(x)$ *is defined as*

$$3\frac{n}{k}\cdot\left[3\left(1-\frac{2}{k}\right)x(1-x)+\frac{1}{k}\right]:=\Delta_{n,k}(x). \tag{3.2}$$

Proof. If f is a convex function it is known that – see [3], p. 115 or Theorem 1 in [5]

$$S_{n,k}(f,x)\leqslant B_k(f,x),$$

where $B_k(f,x)$ is the Bernstein operator of degree k. Therefore from the well-known representation of the fourth moment of $B_k(f,x)$ we have

$$S_{n,k}((e_1-x)^4,x)\leqslant B_k((e_1-x)^4,x)=\frac{x(1-x)}{k^2}\cdot\left[3\left(1-\frac{2}{k}\right)x(1-x)+\frac{1}{k}\right] \tag{3.3}$$

Next we are going to prove a lower bound for the second moment of $S_{n,k}$. The following lower bound for the second moment in "spline case" , i.e. $2\leqslant k\leqslant n-1$ was established in Theorem 12 in [2]

$$c_k\cdot\frac{\min\left\{2x(1-x);\frac{k}{n}\right\}}{n(k-1)}\leqslant S_{n,k}((e_1-x)^2,x),$$

where $c_2=\frac{3}{124}\geqslant\frac{1}{42|}$ and $c_k=\frac{9}{88}\geqslant\frac{1}{10}$ for $k\geqslant 3$. We can not use the last estimate because it is valid only when $2\leqslant k\leqslant n-1$. Here we point out that the new lower bound for the second moment of $S_{n,k}$ is valid for all $k\geqslant 2, n\geqslant 2$. In Theorem 3 in [2] it was proved that

$$S_{n,k}((e_1-x)^2,x)=S_{n,k}(g_2,x), \tag{3.4}$$

where the function g_2 is given by

$$g_2(y)=\begin{cases}\left(-y^2+\frac{y}{3}\sqrt{\frac{8k}{n}\cdot y+\frac{1}{n^2}}\right)\cdot\frac{1}{k-1} & 0\leqslant y\leqslant\min\left\{\frac{k+1}{2n},\frac{n-1}{2k}\right\},\\ \frac{1}{k-1}\cdot\left(y-y^2-\frac{n^2-1}{6nk}\right), & \frac{n-1}{2k}\leqslant y\leqslant\frac{1}{2},\\ \frac{1}{k-1}\cdot\frac{(k+1)(k-1)}{12n^2}, & \frac{k+1}{2n}\leqslant y\leqslant\frac{1}{2},\\ g_2(1-y), & \frac{1}{2}\leqslant y\leqslant 1.\end{cases} \tag{3.5}$$

The function $g_2(y)$ is not concave and our goal is to bound it from below by an appropriate concave function $h_2(y)$. If this is possible it is easy to calculate

$$S_{n,k}(g_2,x)\geqslant S_{n,k}(h_2,x)\geqslant B_k(h_2,x) \tag{3.6}$$

and consequently

$$\frac{1}{S_{n,k}((e_1-x)^2,x)}=\frac{1}{S_{n,k}(g_2,x)}\leqslant\frac{1}{B_k(h_2,x)}. \tag{3.7}$$

To define the function h_2 we observe that

$$g_2'(0) = \frac{1}{3n(k-1)} \tag{3.8}$$

for all $n,k \geqslant 2$. If

$$h_2(y) = \frac{1}{3n(k-1)} y(1-y), y \in [0,1] \tag{3.9}$$

we verify that

$$g_2(y) \geqslant h_2(y). \tag{3.10}$$

Further we compute

$$B_k(h_2,x) = \frac{1}{3n(k-1)} \cdot \left[x - \left(x^2 + \frac{x(1-x)}{k}\right)\right] = \frac{x(1-x)\left(1-\frac{1}{k}\right)}{3n(k-1)}. \tag{3.11}$$

The estimates (3.3) and (3.11) lead us to

$$\begin{aligned} \frac{S_{n,k}((e_1-x)^4,x)}{S_{n,k}((e_1-x)^2,x)} &\leqslant \frac{x(1-x)}{k^2} \cdot \frac{3n(k-1)}{x(1-x)(1-\frac{1}{k})} \cdot \left[3\left(1-\frac{2}{k}\right)x(1-x) + \frac{1}{k}\right] \\ &= 3\frac{n}{k} \cdot \left[3\left(1-\frac{2}{k}\right)x(1-x) + \frac{1}{k}\right] = \Delta_{n,k}(x). \end{aligned} \tag{3.12}$$

When $\frac{k}{n} \to \infty$ (the polynomial case) then $\lim_{\frac{k}{n}\to\infty} \Delta_{n,k}(x) = 0$. We apply Theorem A and the proof of Theorem 3.1 is completed. □

From (2.4) it follows that

$$k \cdot S_{n,k}((e_1-x)^2,x) \leqslant \frac{2x(1-x)k}{n+k-1} \leqslant 2x(1-x)$$

when $\frac{k}{n} \to \infty$. So if

$$\lim_{k\to\infty} kS_{n,k}((e_1-x)^2,x) = e(x)$$

then (3.11) yields

$$\begin{aligned} &\left| k \cdot (S_{n,k}(f,x) - f(x)) - \frac{1}{2} \cdot k \cdot S_{n,k}((e_1-x)^2,x) f''(x) \right| \\ &\leqslant x(1-x) \cdot \tilde{\omega}\left(f'', \frac{1}{3} \cdot \sqrt{\Delta_{n,k}(x)}\right), \end{aligned} \tag{3.13}$$

Hence we arrive at

COROLLARY 3.2. *If* $\frac{k}{n} \to \infty$ *and* $f \in C^2[0,1]$ *then*

$$\lim_{n+k\to\infty} k \cdot [S_{n,k}(f,x) - f(x)] = \frac{1}{2} e(x) f''(x).$$

The convergence is uniform w.r.t. $x \in [0,1]$.

The last statement is the same as in Theorem 1 in [4], where it was supposed, that f is integrable, bounded and has a second derivative at the point x in $(0,1)$. Here we show the uniform convergence, while the convergence in Theorem 1 in [4] is in a pointwise form.

COROLLARY 3.3. *If we set $n = 1$ in (3.2) we get exactly the result of Gonska, Rasa and Pitul in* [6] *(see Theorem 5.1 on p. 108), i.e.*

$$\left|k \cdot [S_{1,k}(f,x) - f(x)] - \frac{x(1-x)}{2} f''(x)\right| \leqslant \frac{x(1-x)}{2} \cdot \tilde{\omega}\left(f'', \sqrt{\frac{1}{k^2} + \frac{x(1-x)}{k}}\right).$$

COROLLARY 3.4. *If $f \in C^2[0,1]$ we have for $x \in [0,1]$*

$$\lim_{k\to\infty} \frac{k \cdot (B_k(f,x) - f(x))}{x(1-x)} = \frac{f''(x)}{2}. \tag{3.14}$$

The convergence is uniform w.r.t. $x \in [0,1]$.

Proof. From Corollary 3.3 we get

$$\left|\frac{k \cdot (B_k(f,x) - f(x))}{x(1-x)} - \frac{f''(x)}{2}\right| \leqslant \tilde{\omega}\left(f'', \sqrt{\frac{1}{k^2} + \frac{x(1-x)}{k}}\right).$$

The estimate (3.14) follows easily due to the fact that the argument of the modulus in the last inequality uniformly goes to 0 when $k \to \infty$. We recall the Theorem of Voronovskaja for Bernstein operator in its original form – see [13]

$$\lim_{k\to\infty} k \cdot (B_k(f,x) - f(x)) = \frac{x(1-x)}{2} \cdot f''(x), \tag{3.15}$$

where the convergence is uniform w.r.t. $x \in [0,1]$ for $f \in C^2[0,1]$. It is easy to verify that the statement of Corollary 3.4 is stronger than that in (3.15). $\square$

COROLLARY 3.5. *For $f \in C^2[0,1]$ it holds*

$$\lim_{k\to\infty} k \cdot \sup_{x\in[0,1]} \left|\frac{B_k(f,x) - f(x)}{x(1-x)}\right| = \frac{\|f''\|}{2}.$$

4. Concluding remarks

Next we suppose that

$$\lim_{(n+k)\to\infty} \frac{n}{k} = \lambda,$$

where $0 < \lambda < \infty$. The cases $\lambda = 0$ and $\lambda = \infty$ are considered in Theorems 3.1 and 2.1 respectively. Consequently $k \to \infty$. In this case it was proved by Goodman etc. in [4] (see Section 4 – p. 73) that the following uniform w.r.t. $x \in [0,1]$ convergence holds

$$\lim_{k\to\infty} S_{n,k}((e_1 - x)^2, x) = e(x),$$

where

$$e(x) = \begin{cases} ((2x)^{\frac{3}{2}} \dfrac{1}{3\sqrt{\lambda}} - x^2, & 0 \leqslant x \leqslant \frac{\lambda}{2} \\ x - x^2 - \dfrac{\lambda}{6}, & \frac{\lambda}{2} \leqslant x \leqslant 1 - \frac{\lambda}{2}, \end{cases} \tag{4.1}$$

if $0 < \lambda \leqslant 1$ and for $1 \leqslant \lambda < \infty$ the function is defined by

$$e(x) = \begin{cases} ((2x)^{\frac{3}{2}} \dfrac{1}{3\sqrt{\lambda}} - x^2, & 0 \leqslant x \leqslant \frac{1}{2\lambda} \\ \dfrac{1}{12\lambda^2}, & \frac{1}{2\lambda} \leqslant x \leqslant 1 - \frac{1}{2\lambda}, \end{cases}$$

and $e(x) = e(1-x)$. The same asymptotic uniform estimate was also established by M. Marsden in [11]. From Theorem 1a in [4] follows that

$$\lim_{k\to\infty} k^2 \cdot S_{n,k}((e_1 - x)^4, x) = \frac{4!}{2!} \cdot \left(\frac{e^2(x)}{2}\right)^2 = 3e^2(x). \tag{4.2}$$

THEOREM 4.1. *For $f \in C^2[0,1]$ and $\varepsilon > 0$ there exists $N > 0$ – integer number, such that for each $k \geqslant N$ we have*

$$\left| S_{n,k}(f,x) - f(x) - \frac{1}{2} S_{n,k}((e_1-x)^2,x) f''(x) \right|$$
$$\leqslant \frac{1}{2} S_{n,k}((e_1-x)^2,x) \cdot \tilde{\omega}\left(f'', \frac{1}{3} \cdot \sqrt{\frac{1}{k}(3e(x)+\varepsilon)}\right). \tag{4.3}$$

Proof. We present the ratio of the fourth and second moments of Schoenberg operator as

$$\frac{S_{n,k}((e_1-x)^4,x)}{S_{n,k}((e_1-x)^2,x)} = \frac{1}{k} \cdot \frac{k^2 S_{n,k}((e_1-x)^4,x)}{k S_{n,k}((e_1-x)^2,x)} \tag{4.4}$$

From the last formula,(4.1) and (4.2) using Theorem A we establish the proof of Theorem 4.1. □

Therefore the following holds true

COROLLARY 4.2. *If*

$$\lim_{(n+k)\to\infty} \frac{n}{k} = \lambda, \lambda \in (0,\infty)$$

then

$$\lim_{(n+k)\to\infty} \frac{S_{n,k}(f,x) - f(x)}{S_{n,k}((e_1-x)^2,x)} = \frac{f''(x)}{2}.$$

The convergence is uniform for $f \in C^2[0,1]$.

We believe that it is possible to prove estimates similar to (4.3) for the case when the sequence $\left(\frac{n}{k}\right)$ is not convergent if $n+k \to \infty$. In this case we formulate the following

CONJECTURE 4.3. *For all* $k \geqslant 1, n \geqslant 1$

$$\frac{S_{n,k}((e_1-x)^4,x)}{S_{n,k}((e_1-x)^2,x)} \leqslant \frac{B_k((e_1-x)^4,x)}{B_k((e_1-x)^2,x)}. \tag{4.5}$$

This conjecture is confirmed at least when $k=n, k \to \infty$ by numerical experiments with MATLAB. If (4.5) is true then for $f \in C^2[0,1]$ we would be able to verify that

$$\left| S_{n,k}(f,x) - f(x) - \frac{1}{2} S_{n,k}((e_1-x)^2,x) f''(x) \right|$$

$$\leqslant \frac{1}{2} S_{n,k}((e_1-x)^2,x) \cdot \tilde{\omega}\left(f'', \frac{1}{3} \cdot \frac{1}{\sqrt{k}} \cdot \sqrt{3\left(1-\frac{1}{2}\right)x(1-x)+\frac{1}{k}}\right).$$

The last statement would lead us to

COROLLARY 4.4. *For* $f \in C^2[0,1]$ *the following convergence is uniform*

$$\lim_{(n+k)\to\infty} \frac{S_{n,k}(f,x)-f(x)}{S_{n,k}((e_1-x)^2,x)} = \frac{f''(x)}{2}.$$

As application of Theorem 2.1 we study the rate of approximation by Schoenberg operator for some concrete functions.

EXAMPLE 4.5. Let $f(x) = x^3, x \in [0,1], k=1, n>1$, i.e. $S_{n,1}$ – is a piecewise linear interpolant It is known the exact presentation of $\tilde{\omega}$ via K-functional

$$K\left(\frac{\varepsilon}{2}, f''; C[0,1], C^1[0,1]\right) = \frac{1}{2} \cdot \tilde{\omega}(f'', \varepsilon),\ \varepsilon \geqslant 0, \tag{4.6}$$

$$K\left(\frac{\varepsilon}{2}, f''; C[0,1], C^1[0,1]\right) := \inf_{g \in C^1[0,1]} \left\{ \|f''-g\|_C + \frac{\varepsilon}{2}\|g'\|_c \right\}. \tag{4.7}$$

Putting $g = f''$ in (4.7) from (2.1) we obtain

$$|S_{n,1}(f,x) - f(x)| \leqslant \frac{1}{n^2} \cdot z_n(x)(1-z_n(x)) \cdot \left(3x + \frac{1}{n}\right). \tag{4.8}$$

To obtain lower bound for the error of approximation we calculate from (2.1)

$$\begin{aligned} |S_{n,1}(f,x) - f(x)| &\geqslant \frac{1}{2} S_{n,1}((e_1-x)^2;x) \cdot \left[|f''(x)| - \tilde{\omega}\left(f'', \frac{1}{3n}\right) \right] \\ &\geqslant \frac{1}{2} S_{n,1}((e_1-x)^2;x) \cdot \left[6x - \frac{1}{3n} 6 \right] = S_{n,1}((e_1-x)^2;x)\left(3x - \frac{1}{n}\right). \end{aligned} \tag{4.9}$$

EXAMPLE 4.6. Let again $f(x) = x^3, k=2, n>1$. Here we use the upper bound (2.4) for the second moment. Here we point out that (2.4) is the first pointwise upper bound for the second moment of Schoenberg operator, which is a crucial step to establish various direct and inverse theorems. Hence

$$\begin{aligned} |S_{n,2}(f,x)-f(x)| &\leqslant \frac{1}{2}\cdot\frac{\min\left\{2x(1-x);\frac{2}{n}\right\}}{n+1}\cdot\left(6x+\frac{3}{2n}\right) \\ &< \frac{1}{n(n+1)}\cdot\left(6x+\frac{3}{2n}\right). \end{aligned} \tag{4.10}$$

Our next goal is to establish strong converse inequality for approximation by Bernstein operator $B_k(f,x)$ in pointwise form for some subspace of $f\in C^2[0,1]$. We consider the class $Lip_\alpha(L)$ of Hölder continuous functions with exponent α for some $0<\alpha\leqslant 1$ and constant L,i.e. which obey

$$|f(x)-f(y)|\leqslant L\cdot|x-y|^\alpha,\ x,y\in[0,1].$$

THEOREM 4.7. *Let $f''\in Lip_\alpha(L)$, $0<\alpha\leqslant 1$ and $f''(x)\neq 0$ at some fixed point $x\in[0,1]$. Then there exists a natural number $k_0=k_0(x,L,\alpha,f)$ such that for every $k\geqslant k_0$ the following holds true*

$$\frac{1}{4}\cdot\frac{x(1-x)}{k}\cdot|f''(x)| < |B_k(f,x)-f(x)| < \frac{3}{4}\cdot\frac{x(1-x)}{k}\cdot|f''(x)|. \tag{4.11}$$

Proof. It is known that

$$\tilde{\omega}(f,\varepsilon)\leqslant 2\omega(f,\varepsilon),\ \varepsilon>0,$$

where $\omega(f,\varepsilon)$ is the usual modulus of continuity. Hence

$$\tilde{\omega}(f,\varepsilon)\leqslant 2L\varepsilon^\alpha.$$

This observation and Corollary 3.2 lead us to

$$\begin{aligned} \frac{k|B_k(f,x)-f(x)|}{x(1-x)} &\leqslant \frac{|f''(x)|}{2}+\frac{1}{2}\tilde{\omega}\left(f'',\sqrt{\frac{1}{k^2}+\frac{x(1-x)}{k}}\right) \\ &\leqslant \frac{|f''(x)|}{2}+L\cdot\left(\frac{1}{k^2}+\frac{x(1-x)}{k}\right)^{\frac{\alpha}{2}}. \end{aligned} \tag{4.12}$$

If $f''(x)\neq 0$ then there exists a natural number $k_0=k_0(x,L,\alpha,f)$ such that for $k\geqslant k_0$

$$L\cdot\left(\frac{1}{k^2}+\frac{x(1-x)}{k}\right)^{\frac{\alpha}{2}} < \frac{|f''(x)|}{4}. \tag{4.13}$$

Obviously

$$\frac{k|B_k(f,x)-f(x)|}{x(1-x)} \geqslant \frac{|f''(x)|}{2}-L\cdot\left(\frac{1}{k^2}+\frac{x(1-x)}{k}\right)^{\frac{\alpha}{2}}. \tag{4.14}$$

The last three inequalities complete the proof of Theorem 4.7. □

As a straightforward corollary from Theorem 4.7 we get

COROLLARY 4.8. *Let* $f'' \in Lip_\alpha(L),\ 0 < \alpha \leqslant 1$ *and* $|f''(x)| > m > 0$ *for all* $x \in [0,1]$. *Then there exists a natural number* $k_0 = k_0(m,L,\alpha)$, *independent on the position of* x, *such that for* $k \geqslant k_0$ *and for all* $x \in [0,1]$ *the strong converse pointwise inequality* (4.11) *holds true.*

It is easy to observe that the constants $\frac{3}{4}$ and $\frac{1}{4}$ in Theorem 4.7 and Corollary 4.8 could be replaced by $\frac{1}{2}+c$ and $\frac{1}{2}-c$ for $c > 0$-arbitrary small and appropriate choice of k_0.

Acknowledgements. This work was done during my stay in January – February 2008 as DAAD fellow by prof. H. Gonska at the University of Duisburg-Essen. I am sincerely grateful to the anonymous referee for his/her thorough reading of the paper and helpful remarks.

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(Received April 9, 2009)

Gancho T. Tachev
Dept. of Mathematics
University of Architecture
Civil Engineering and Geodesy
BG-1046, Sofia, Bulgaria
e-mail: gtt_fte@uacg.bg