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EIGENVALUE ESTIMATES FOR STABLE MINIMAL HYPERSURFACES

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Abstract. In this article we provide estimates on the first eigenvalue of stable minimal hypersurfaces in a Riemannian manifold with sectional curvature bounded from below and above by negative constants. We also obtain a lower bound of the total scalar curvature of a stable minimal hypersurface if the scalar curvature of the ambient space is positive.

1. Introduction

As it is well known, the first Dirichlet eigenvalue of a Riemannian manifold Σ with boundary is characterized as

$$\lambda_1 = \inf_f \frac{\int_\Sigma |\nabla f|^2}{\int_\Sigma f^2},$$

where the infimum is taken over all piecewise smooth functions in Σ vanishing on the boundary $\partial\Sigma$. Recently Candel [1] gave an upper bound for the first eigenvalue of a stable minimal surface in $\mathbb{H}^3$. We say that a minimal hypersurface Σ in an $(n+1)$-dimensional Riemannian manifold M is *stable* if the second variation of its volume is always nonnegative for every compactly supported deformation of Σ in M^{n+1}. More precisely, an n-dimensional minimal hypersurface Σ in a Riemannian manifold M is called *stable* if for any compactly supported Lipschitz function f on M

$$\int_\Sigma |\nabla f|^2 - (\overline{\mathrm{Ric}}(e_{n+1}) + |A|^2) f^2 \geqslant 0 \tag{1.1}$$

holds, where $\overline{\mathrm{Ric}}(e_{n+1})$ is the Ricci curvature of M in the direction of e_{n+1}, e_{n+1} is the unit normal vector of Σ in M, and $|A|^2$ is the squared length of the second fundamental form of Σ.

Recall that the Yamabe invariant of the conformal class $[g]$ of an n-dimensional Riemannian manifold Σ is defined by

$$Y(g) = \inf\{E(\bar{g}) : \bar{g} = u(x)^{\frac{2n}{n-2}} g, u(x) > 0, u \in H^1(M)\},$$

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where $E(\bar{g})$ is defined by

$$E(\bar{g}) = \frac{\int_\Sigma (\frac{4(n-1)}{n-2}|\nabla u|^2 + R_g u^2) dv_g}{(\int_\Sigma u^{\frac{2n}{n-2}} dv_g)^{\frac{n-2}{n}}}.$$

Here R_g and dv_g denote the scalar curvature and the volume form of the metric g on Σ, respectively. Then the *Yamabe invariant* of Σ is defined by

$$\sigma(\Sigma) = \sup\{Y(g) : g \text{ is a smooth metric on } \Sigma\}.$$

Ho [4] generalized Candel's result to higher-dimensional cases. He gave estimates on the first eigenvalue of a stable minimal hypersurface Σ in hyperbolic space $\mathbb{H}^{n+1}$ for $n \geqslant 3$ under the assumption that the Yamabe invariant $\sigma(\Sigma)$ satisfies that $\sigma(\Sigma) < 0$. Actually he proved

THEOREM 1.1. *Let $\Sigma \subset \mathbb{H}^{n+1}$ be a compact stable minimal hypersurface with boundary. If $\sigma(\Sigma) < 0$, then the first eigenvalue of Σ satisfies*

$$\frac{1}{4}(n-1)^2 \leqslant \lambda_1(\Sigma) < \frac{n^2(n-2)}{7n-6}.$$

In this paper we extend Ho's result to stable minimal hypersurfaces in a Riemannian manifold of variable curvature. We provide the upper bound of the first eigenvalue when the ambient space has negative scalar curvature. More precisely, we prove

THEOREM 1.2. *Let M be an $(n+1)$-dimensional Riemannian manifold with scalar curvature S satisfying that $-S_0 \leqslant S < 0$ for some positive constant S_0, $n \geqslant 3$. Let $\Sigma \subset M$ be a compact stable minimal hypersurface with boundary. If $\sigma(\Sigma) < 0$, then the first eigenvalue of Σ satisfies*

$$\lambda_1(\Sigma) < \frac{S_0(n-2)}{2n}.$$

If the ambient space M has sectional curvature bounded from below and above by negative constants, we can extend Theorem 1.1 as follows.

THEOREM 1.3. *Let M be an $(n+1)$-dimensional Riemannian manifold with sectional curvature K_M satisfying $-b \leqslant K_M \leqslant -a < 0$ for some positive constants a and b, $n \geqslant 3$. Let $\Sigma \subset M$ be a compact stable minimal hypersurface with boundary. Assume that $\sigma(\Sigma) < C_\sigma$ for some constant $C_\sigma \in [0, \frac{3n-2}{n-2}\frac{1}{C_s})$, where C_s is a Sobolev constant in* [5]. *Then the first eigenvalue of Σ satisfies*

$$\frac{(n-1)^2}{4}a \leqslant \lambda_1(\Sigma) < \frac{n(n+1)b - na}{\frac{3n-2}{n-2} - C_\sigma C_s}.$$

When the scalar curvature of the ambient space is positive, we estimate the total scalar curvature of a stable minimal hypersurface Σ and the Yamabe invariant $\sigma(\Sigma)$ of Σ.

THEOREM 1.4. *Let M be an $(n+1)$-dimensional compact Riemannian manifold with scalar curvature $S \geqslant n(n+1)k > 0$ for some positive constant k. Let $\Sigma \subset M$ be a compact stable minimal hypersurface without boundary. Then we have the following.*

(i) $\int_\Sigma R \geqslant n(n+1)k\mathrm{Vol}(\Sigma)$*, where R is the scalar curvature of* Σ.

(ii) $\sigma(\Sigma) > 0$*, when* $n \geqslant 3$.

We note that the Yamabe invariant $\sigma(\Sigma)$ is positive if and only if Σ admits a metric of positive scalar curvature. (See [6], [7] and [9].)

2. Proof of the theorems

Let M be an $(n+1)$-dimensional Riemannian manifold and let Σ be a stable minimal hypersurface in M. Choose an orthonormal frame $\{e_1, \cdots, e_n, e_{n+1}\}$ adapted to M, so that $e_1, \cdots, e_n$ are tangential and e_{n+1} is the unit normal vector. For $1 \leqslant i,j,k,l \leqslant n+1$, let R_{ijkl} denote the curvature tensor of M. For $1 \leqslant i,j,k,l \leqslant n$, let K_{ijkl} denote the curvature tensor of Σ with respect to the induced metric from M. For $1 \leqslant i,j \leqslant n$, let $h_{ij} = -\langle \bar{\nabla}_{e_i} e_{n+1}, e_j \rangle$ be the second fundamental form of Σ, where $\bar{\nabla}$ is the Riemannian connection on M. Then the Gauss curvature equation says

$$K_{ijij} = R_{ijij} + h_{ii}h_{jj} - h_{ij}^2 \tag{2.1}$$

for $1 \leqslant i,j \leqslant n$. Summing (2.1), we get

$$\sum_{i,j=1}^n K_{ijij} = \sum_{i,j=1}^n R_{ijij} + \Big(\sum_{i=1}^n h_{ii}\Big)^2 - \sum_{i,j=1}^n h_{ij}^2 .$$

Since $\sum_{i=1}^n h_{ii} = 0$ by the minimality of Σ, the scalar curvature S of M is

$$\begin{aligned} S = \sum_{i,j=1}^{n+1} R_{ijij} &= 2\sum_{i=1}^n R_{n+1,i,n+1,i} + \sum_{i,j=1}^n R_{ijij} \\ &= 2\overline{\mathrm{Ric}}(e_{n+1}) + R + \sum_{i,j=1}^n h_{ij}^2 \\ &= 2\overline{\mathrm{Ric}}(e_{n+1}) + R + |A|^2, \end{aligned} \tag{2.2}$$

where R is the scalar curvature of Σ. Putting this into the stability inequality (1.1), we therefore get

$$\frac{1}{2}\int_\Sigma Sf^2 - \frac{1}{2}\int_\Sigma Rf^2 + \frac{1}{2}\int_\Sigma |A|^2 f^2 \leqslant \int_\Sigma |\nabla f|^2 \tag{2.3}$$

for any compactly supported smooth function f defined on Σ.

Proof of Theorem 1.2. By the inequality (2.3), for any compactly supported function f, we have

$$\int_\Sigma Sf^2 \geqslant 2\int_\Sigma |\nabla f|^2 + \int_\Sigma Rf^2. \tag{2.4}$$

Since $\sigma(\Sigma) < 0$, $Y(g) < 0$ for the induced metric g on $\Sigma \subset M$. Thus there exists a smooth function f on Σ such that $E(f^{\frac{2n}{n-2}} g) < 0$, which implies

$$\int_\Sigma Rf^2 + \frac{4(n-1)}{n-2} \int_\Sigma |\nabla f|^2 < 0. \tag{2.5}$$

Combining (2.4) and (2.5), we have

$$\int_\Sigma Rf^2 < 2 \int_\Sigma |\nabla f|^2 - \frac{4(n-1)}{n-2} \int_\Sigma |\nabla f|^2.$$

Thus the curvature assumption on M gives

$$\frac{2n}{n-2} \int_\Sigma |\nabla f|^2 < -\int_\Sigma Sf^2 \leqslant S_0 \int_\Sigma f^2,$$

which implies

$$\lambda_1(\Sigma) \leqslant \frac{\int_\Sigma |\nabla f|^2}{\int_\Sigma f^2} < \frac{S_0(n-2)}{2n}. \quad \square$$

Before proving Theorem 1.3, we need the following Sobolev inequality.

LEMMA 2.1. ([5]) *Let Σ be an n-dimensional complete immersed minimal submanifold in a Riemannian manifold M with nonpositive sectional curvature, $n \geqslant 3$. Then for any $\phi \in W_0^{1,2}(M)$ we have*

$$\left(\int_\Sigma |\phi|^{\frac{2n}{n-2}} dv \right)^{\frac{n-2}{n}} \leqslant C_s \int_\Sigma |\nabla \phi|^2 dv,$$

where C_s depends only on n.

In [10], the author recently proved a lower bound part in Theorem 1.3. However, for completeness we shall give this part again here.

Proof of Theorem 1.3. First we estimate a lower bound of $\lambda_1(\Sigma)$. The Laplacian of the distance function r on $\Sigma \subset M$ satisfies

$$\Delta r \geqslant \sqrt{a}(n - |\nabla r|^2) \coth \sqrt{a} r \geqslant (n-1)\sqrt{a},$$

see [3]. Integrating both sides over a domain $\Omega \subset \Sigma$, we get

$$(n-1)\sqrt{a} \mathrm{Area}(\Omega) \leqslant \int_\Omega \Delta r dv = \int_{\partial\Omega} \frac{\partial r}{\partial \nu} ds \leqslant \mathrm{Length}(\partial\Omega). \tag{2.6}$$

Recall that for a Riemannian manifold Σ, the *Cheeger constant* $h(\Sigma)$ is defined by

$$h(\Sigma) := \inf_\Omega \frac{\mathrm{Length}(\partial\Omega)}{\mathrm{Area}(\Omega)},$$

where Ω ranges over all open submanifolds of Σ with compact closure in Σ. Then, applying Cheeger's inequality [2] and inequality (2.6), we obtain

$$\lambda_1(\Sigma) \geqslant \frac{1}{4}h(\Sigma)^2 = \frac{(n-1)^2}{4}a, \tag{2.7}$$

which gives the proof for the lower bound part.

Now we prove the upper bound part of the first eigenvalue $\lambda_1(\Sigma)$. Since Σ is stable, we have

$$\int_\Sigma (\overline{\mathrm{Ric}}(e_{n+1}) + |A|^2)f^2 \leqslant \int_\Sigma |\nabla f|^2$$

for any compactly supported smooth function f on Σ. Using the equation (2.2), we get

$$\int_\Sigma Sf^2 - \int_\Sigma Rf^2 - \int_\Sigma \overline{\mathrm{Ric}}(e_{n+1})f^2 \leqslant \int_\Sigma |\nabla f|^2.$$

From the curvature assumption on M, we have

$$-n(n+1)b \leqslant S \leqslant -n(n+1)a < 0 \text{ and}$$
$$-nb \leqslant \overline{\mathrm{Ric}}(e_{n+1}) = R_{n+1,1,n+1,1} + \cdots + R_{n+1,n,n+1,n} \leqslant -na < 0.$$

Hence

$$-n(n+1)b\int_\Sigma f^2 + na\int_\Sigma f^2 \leqslant \int_\Sigma Rf^2 + \int_\Sigma |\nabla f|^2. \tag{2.8}$$

Since $\sigma(\Sigma) < C_\sigma$, $Y(g) < C_\sigma$ for the induced metric g on Σ. Then there exists a smooth function f satisfying

$$\frac{\int_\Sigma Rf^2 + \frac{4(n-1)}{n-2}\int_\Sigma |\nabla f|^2}{\left(\int_\Sigma f^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}} < C_\sigma.$$

Applying Sobolev's inequality (Lemma 2.1), we have

$$\int_\Sigma Rf^2 + \frac{4(n-1)}{n-2}\int_\Sigma |\nabla f|^2 < C_\sigma C_s \int_\Sigma |\nabla f|^2. \tag{2.9}$$

Combining the inequalities (2.8) and (2.9), we obtain

$$\left(\frac{3n-2}{n-2} - C_\sigma C_s\right)\int_\Sigma |\nabla f|^2 < (n(n+1)b - na)\int_\Sigma f^2.$$

Note that the assumption on C_σ implies that $\frac{3n-2}{n-2} - C_\sigma C_s > 0$. Therefore it follows that

$$\lambda_1(\Sigma) \leqslant \frac{\int_\Sigma |\nabla f|^2}{\int_\Sigma f^2} < \frac{n(n+1)b - na}{\frac{3n-2}{n-2} - C_\sigma C_s},$$

which completes the proof of the upper bound part. $\square$

In particular, when $a = b = 1$, the ambient space M is isometric to the hyperbolic space $\mathbb{H}^{n+1}$. As a consequence of Theorem 1.3, we improve the upper bound of the Yamabe invariant in Theorem 1.1 as follows.

COROLLARY 2.2. *Let $\Sigma \subset \mathbb{H}^{n+1}$ be a compact stable minimal hypersurface. Assume that $\sigma(\Sigma) < C_\sigma$ for some constant $C_\sigma \in [0, \frac{3n-2}{n-2}\frac{1}{C_s})$, where C_s is a Sobolev constant as in Theorem* 1.3. *Then the first eigenvalue of Σ satisfies*

$$\frac{(n-1)^2}{4} \leqslant \lambda_1(\Sigma) < \frac{n^2}{\frac{3n-2}{n-2} - C_\sigma C_s}.$$

REMARK 2.3. If $C_\sigma = 0$, then this result is exactly the same as Theorem 1.1. Note that our scalar curvature is exactly twice the scalar curvature in Ho's paper [4].

Proof of Theorem 1.4. The inequality (2.3) says that

$$\frac{1}{2}\int_\Sigma Sf^2 \leqslant \int_\Sigma |\nabla f|^2 + \frac{1}{2}\int_\Sigma Rf^2$$

for any compactly supported smooth function f defined on Σ.

Since $S \geqslant n(n+1)k > 0$ by assumption, we have

$$\frac{n(n+1)k}{2}\int_\Sigma f^2 \leqslant \int_\Sigma |\nabla f|^2 + \frac{1}{2}\int_\Sigma Rf^2. \tag{2.10}$$

Choosing a test function $f \equiv 1$ on Σ gives

$$n(n+1)k\int_\Sigma S \leqslant \int_\Sigma R,$$

which completes the proof of (i).

For (ii), suppose that $\sigma(\Sigma) \leqslant 0$. Then there exists a smooth function $f > 0$ on Σ satisfying that

$$\int_\Sigma Rf^2 + \frac{4(n-1)}{n-2}\int_\Sigma |\nabla f|^2 \leqslant 0. \tag{2.11}$$

Then by (i) it is easy to see that f cannot be constant. Using the inequalities (2.10) and (2.11), we get

$$0 < n(n+1)k\int_\Sigma f^2 \leqslant 2\int_\Sigma |\nabla f|^2 - \frac{4(n-1)}{n-2}\int_\Sigma |\nabla f|^2 = -\frac{2n}{n-2}\int_\Sigma |\nabla f|^2 < 0,$$

which is a contradiction. Therefore we see that $\sigma(\Sigma) > 0$. $\square$

REMARK 2.4. In particular, when $n = 2$, it follows from the above theorem that $\int_\Sigma K_\Sigma > 0$, where K_Σ is the Gaussian curvature of Σ. By the Gauss-Bonnet theorem, Σ cannot have positive genus, which is a result of Schoen-Yau [8].

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