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MARTINGALE VERSIONS OF YANO'S EXTRAPOLATION INEQUALITIES

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Abstract. In the paper we obtain some martingale versions of extrapolation theorems of Yano type by employing the technique of atomic decomposition in some Orlicz martingale spaces. Some variant extrapolation inequalities are also given.

1. Introduction and Preliminaries

The well-known extrapolation theorem of Yano (see [19] and [20, Theorem XII.4.41]) states that if for all p near 1, $p>1$, the sublinear operator T satisfies

$$\left(\int_\Omega |Tf(x)|^p d\mu\right)^{\frac{1}{p}} \leqslant \frac{c}{(p-1)^\alpha}\left(\int_\Omega |f(x)|^p d\mu\right)^{\frac{1}{p}},$$

where Ω is a finite measure space, c and α are both positive constants, then $T : L(\log L)(\Omega) \to L^1(\Omega)$ is bounded; later Sjölin [13] and Soria [15] gave weak extrapolation results, who can obtain endpoint estimate for any sublinear operator T satisfying restricted weak type estimate; in the 1990s, the theorem was put into the framework of abstract extrapolation theory (see [8], [9] and [14]); in recent years Carro [3] and [4] obtained new extrapolation estimates and discussed the extrapolation results for $p<1$, which improved the Yano theorem; a converse extrapolation theorem for translation-invariant operators was also obtained by Tao [16].

The purpose of this work is to investigate Yano's extrapolation theorem in the martingale setting (see Theorem 3.1, Theorem 3.2 and Theorem 3.3). Our main method is the atomic decompositions in Orlicz martingale spaces. In many areas of analysis, the idea of decomposition has turned out to be highly fruitful. It is often helpful to consider functions decomposed into suitable elementary pieces which are easier to handle. In connection with extrapolation procedures a lot of attention has always been paid to various expressions for the norm in Orlicz spaces $L(\log L)^\alpha$. The atomic decomposition in martingale theory, the idea of which is just derived from harmonic analysis, has

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been extensively studied. For instance, Weisz [17] gave some atomic decompositions on martingale Hardy spaces and proved many inequalities by atomic decompositions; Weisz [18] made a further study of atomic decomposition for weak Hardy spaces consisting of Vilenkin martingales, and proved a weak version of the Hardy-Littlewood inequality; Liu and Hou [11] investigated the atomic decomposition of martingales for the vector-valued case and the geometry properties of Banach spaces were characterized; Hou and Ren [6] considered the vector-valued weak atomic decompositions and weak martingale inequalities; Jiao, Liu and Peng [10] discussed the operator interpolation by atomic decompositions of weighted martingale Hardy spaces. In section 2 we respectively construct three atomic decomposition theorems for Orlicz martingale spaces $H^s(\log H)^\alpha$, $Q(\log Q)^\alpha$ and $\mathscr{D}(\log\mathscr{D})^\alpha$. In the section 3 and section 4 we give extremely simple proof of Yano type extrapolation and some variations in the target and domain spaces.

In the remainder of this section, we shall give some preliminaries necessary to the whole paper. Let (Ω,Σ,P) be a complete probability space and f a measurable function defined on Ω. The decreasing rearrangement of f is the function f^* defined by

$$f^*(t)=\inf\{s>0:P(|f|>s)\leqslant t\}.$$

We adopt the convention that $\inf\emptyset=\infty$. For any $\alpha>0$, $L(\log L)^\alpha=L(\log L)^\alpha(\Omega)$ is the Orlicz space generated by any Young function equivalent to $t\to t(\log t)^\alpha$ near infinity. It is well known that since the measure of Ω is finite, all such Young functions give the same space (up to equivalence of norms) and we can introduce a norm on $L(\log L)^\alpha$ by the formula

$$\|f\|_{L(\log L)^\alpha}=\int_0^1 f^*(t)\left(\log\frac{1}{t}\right)^\alpha dt,$$

which is in fact equivalent to the usual Luxemburg norm. For this we refer to see [1].

Let $\{\Sigma_n\}_{n\geqslant 0}$ be a non-decreasing sequence of sub-σ-fields of Σ such that $\Sigma=\bigvee\Sigma_n$. We denote the expectation operator and the conditional expectation operator relative to Σ_n by E and E_n, respectively. For a martingale $f=(f_n)_{n\geqslant 0}$, we define $\triangle_n f=f_n-f_{n-1}$, $n\geqslant 0$ (with the convention that $f_{-1}=0,\Sigma_{-1}=\{\Omega,\emptyset\}$) and adopt the notions of its maximal function, square function and conditional square function as follows, respectively:

$$M_n(f)=\sup_{0\leqslant i\leqslant n}|f_i|,\quad M(f)=\sup_{n\geqslant 0}|f_n|,$$

$$S_n(f)=\Big(\sum_{i=0}^n|\triangle_i f|^2\Big)^{1/2},\quad S(f)=\Big(\sum_{n=0}^\infty|\triangle_n f|^2\Big)^{1/2},$$

$$s_n(f)=\Big(\sum_{i=0}^n E_{i-1}|\triangle_i f|^2\Big)^{1/2},\quad s(f)=\Big(\sum_{n=0}^\infty E_{n-1}|\triangle_n f|^2\Big)^{1/2}.$$

Denote by Λ the set of all non-decreasing, non-negative and adapted r.v. sequences $\rho=(\rho_n)_{n\geqslant 0}$ with $\rho_\infty=\lim_{n\to\infty}\rho_n$. We shall say a martingale $f=(f_n)_{n\geqslant 0}$ has predictable control in $L(\log L)^\alpha$ if there is a sequence $\rho = (\rho_n)_{n\geqslant 0} \in \Lambda$ such that

$$|f_n| \leqslant \rho_{n-1}, \quad \rho_\infty \in L(\log L)^\alpha.$$

Now we define some new Orlicz martingale spaces as follows,

$$H^s(\log H)^\alpha = \Big\{ f = (f_n)_{n\geqslant 0} : \|f\|_{H^s(\log H)^\alpha} = \|s(f)\|_{L(\log L)^\alpha} < \infty \Big\},$$

$$Q(\log Q)^\alpha = \Big\{ f = (f)_{n\geqslant 0} : \exists (\rho_n)_{n\geqslant 0} \in \Lambda, s.t.\, S_n(f) \leqslant \rho_{n-1}, \rho_\infty \in L(\log L)^\alpha \Big\},$$

$$\|f\|_{Q(\log Q)^\alpha} = \inf_\rho \|\rho_\infty\|_{L(\log L)^\alpha}$$

$$\mathscr{D}(\log \mathscr{D})^\alpha = \Big\{ f = (f)_{n\geqslant 0} : \exists (\rho_n)_{n\geqslant 0} \in \Lambda, s.t.\, |f_n| \leqslant \rho_{n-1}, \rho_\infty \in L(\log L)^\alpha \Big\},$$

$$\|f\|_{\mathscr{D}(\log \mathscr{D})^\alpha} = \inf_\rho \|\rho_\infty\|_{L(\log L)^\alpha}.$$

REMARK. If change the $L(\log L)^\alpha$-norms in definitions above by L_p-norms respectively, we get the usual Hardy martingale spaces H^s_p, Q_p and $\mathscr{D}_p$, respectively (see [12] and [17]).

It turns out that Orlicz martingale spaces, as many other quasi-Banach spaces, admit some sort of atomic decompositions. We will begin with the definition of atom.

DEFINITION 1.1. ([11,17]) A measurable function a is called a $(1,p,\infty)$-atom $\Big($or $(2,p,\infty)$-atom, $(3,p,\infty)$-atom, respectively$\Big)$ if there exists a stopping time τ such that

(i) $a_n = E_n a = 0,\ \forall n \leqslant \tau$,

(ii) $\|s(a)\|_\infty \leqslant P(\tau<\infty)^{-\frac{1}{p}}$ $\Big($or (ii) $\|S(a)\|_\infty \leqslant P(\tau<\infty)^{-\frac{1}{p}}$, (ii) $\|M(a)\|_\infty \leqslant P(\tau<\infty)^{-\frac{1}{p}}$, respectively$\Big)$.

But the atoms above are not available for Orlicz martingale spaces, we must define some new atoms as follows, which will be suitable for our purpose.

DEFINITION 1.2. A measurable function a is called a $(1,p,r)$-atom $\Big($or $(2,p,r)$-atom, $(3,p,r)$-atom, respectively$\Big)$ if there exists a stopping time τ such that

(i) $a_n = E_n a = 0,\ \forall n \leqslant \tau$,

(ii) $\|s(a)\|_\infty \leqslant P(\tau<\infty)^{-\frac{1}{p}}\left(\log P(\tau<\infty)^{-1}\right)^{-\frac{r}{p}}$ $\Big($or (ii) $\|S(a)\|_\infty \leqslant P(\tau<\infty)^{-\frac{1}{p}}\left(\log P(\tau<\infty)^{-1}\right)^{-\frac{r}{p}}$, (ii) $\|M(a)\|_\infty \leqslant P(\tau<\infty)^{-\frac{1}{p}}\left(\log P(\tau<\infty)^{-1}\right)^{-\frac{r}{p}}$, respectively$\Big)$.

Throughout the paper, we denote the set of integers and the set of non-negative integers by Z and N, respectively. We write $A \preceq B$ if $A \leqslant cB$ for some positive constant c independent of appropriate quantities involved in the expressions A and B and A $\sim$ B if $A \preceq B$ and $B \preceq A$.

2. Some Lemmas

Now we shall construct the atomic decomposition theorems.

LEMMA 2.1. *If the martingale $f \in H^s(\log H)^\alpha, \alpha > 0$, then there exist a sequence (a^k) of $(1,1,\alpha)$-atoms and a real number sequence $(\mu_k) \in \ell_1$ such that*

$$f_n = \sum_{k \in Z} \mu_k a_n^k, \quad \forall n \in N$$

and

$$\|(\mu_k)_{k \in Z}\|_{\ell_1} \preceq \|f\|_{H^s(\log H)^\alpha}.$$

Conversely, if the martingale f has the decomposition above, then $f \in H^s(\log H)^\alpha$ and

$$\|f\|_{H^s(\log H)^\alpha} \preceq \inf \|(\mu_k)_{k \in Z}\|_{\ell_1},$$

where the "inf" is taken over all the preceding decompositions of f.

Proof. Assume that $f \in H^s(\log H)^\alpha$, then $sf \in L_1$, so $sf < \infty$ a.e.. Let us consider the following stopping times for all $k \in Z$:

$$\tau_k = \inf\{n \in N : s_{n+1}(f) > e^k\} \, (\inf \emptyset = \infty).$$

The sequence of these stopping times is non-decreasing and $\tau_k \longrightarrow +\infty$ as $k \longrightarrow +\infty$; otherwise we would have $\tau_k \leqslant N$ on a set of positive measure for all k, which means that $s_{N+1}(f) > e^k$ for all k on the positive measure set. It is a contradiction with $sf < \infty$ *a.e.*. Let $f^{\tau_k} = (f_{(n \wedge \tau_k)})_{n \geqslant 0}$ be the stopping martingale. It is easy to see that

$$\begin{aligned}\sum_{k \in Z}(f_n^{\tau_{k+1}} - f_n^{\tau_k}) &= \sum_{k \in Z}\Big(\sum_{m=0}^{n} \chi_{\{m \leqslant \tau_{k+1}\}} \triangle_m f - \sum_{m=0}^{n} \chi_{\{m \leqslant \tau_k\}} \triangle_m f\Big) \\ &= \sum_{k \in Z}\Big(\sum_{m=0}^{n} \chi_{\{\tau_k < m \leqslant \tau_{k+1}\}} \triangle_m f\Big) = f_n\end{aligned}$$

where χ_A denotes the characteristic function of the set A. Since

$$\lim_{k \to +\infty} \tau_k = +\infty \quad \text{and} \quad \lim_{\mathrm{k} \to -\infty} \tau_{\mathrm{k}} = 0$$

for any martingale $f \in H^s(\log H)^\alpha$, there exists K_0 bigger enough and independent of f such that

$$f_n^{\tau_{k+1}} - f_n^{\tau_k} = 0, \quad \text{when } |\mathrm{k}| \geqslant \mathrm{K}_0.$$

Now let

$$\mu_k = \begin{cases} 3e^k P(\tau_k < \infty)\Big(\log P(\tau_k < \infty)^{-1}\Big)^\alpha, & |k| < K_0, \\ 0, & |k| \geqslant K_0. \end{cases}$$

Setting

$$a_n^k = \frac{f_n^{\tau_{k+1}} - f_n^{\tau_k}}{\mu_k}.$$

If $\mu_k = 0$ then let $a_n^k = 0$. Then for a fixed k, (a_n^k) is a martingale. Since $s(f_n^{\tau_k}) \leqslant e_k$, $s(f_n^{\tau_{k+1}}) \leqslant e^{k+1}$,

$$s(a_n^k) \leqslant \frac{s(f_n^{\tau_{k+1}}) + s(f_n^{\tau_k})}{\mu_k} \leqslant P(\tau_k < \infty)^{-1}\Big(\log P(\tau_k < \infty)^{-1}\Big)^{-\alpha}, \ \forall n \in N,$$

which implies that (a_n^k) is a L_2-bounded martingale, so there exists $a^k \in L_2$ such that $E_n a^k = a_n^k$. If $n \leqslant \tau_k$ then $a_n^k = 0$, so we get a^k is really a $(1,1,\alpha)$-atom.

Now we shall estimate

$$\sum_{k\in Z} |\mu_k| = 3 \sum_{|k|<K_0} e^k P(\tau_k < \infty)\Big(\log P(\tau_k < \infty)^{-1}\Big)^{\alpha}.$$

It is a simple fact that

$$\|f\|_{H^s(\log L)^\alpha} = \int_0^1 s(f)^*(t)\left(\log\frac{1}{t}\right)^{\alpha} \sim \sum_{j=1}^{\infty} j^\alpha e^{-j} s(f)^*(e^{-j}).$$

For any fixed $k \in Z$, we can choose $j \in N$ such that $e^{-j} < P(\tau_k < \infty) \leqslant e^{-j+1}$ (for different k, the j may be identical, but such j's at most is finite). From the definition of stopping time, we get

$$e^{-j} < P\big(s(f) > e^k\big) \leqslant e^{-j+1},$$

thus $s(f)^*(e^{-j+1}) \leqslant e^k$, $s(f)^*(e^{-j}) > e^k$. Consequently,

$$\begin{aligned}\sum_{k\in Z}|\mu_k| &= 3\sum_{|k|<K_0} e^k P(\tau_k<\infty)\Big(\log P(\tau_k<\infty)^{-1}\Big)^{\alpha}\\ &= 3\sum_{|k|<K_0} e^k P\big(s(f)>e^k\big)\Big(\log P\big(s(f)>e^k\big)^{-1}\Big)^{\alpha}\\ &\preceq \sum_{j=1}^{\infty} e^{-j} s(f)^*(e^{-j}) j^\alpha\\ &\preceq \|f\|_{H^s(\log L)^\alpha}\end{aligned}$$

Conversely, suppose that f has the above decomposition. Since a^k is a $(1,1,\alpha)$-atom, it comes from its definition that

$$\chi_{(\tau_k \geqslant n)} E_{n-1}|\triangle_n a^k|^2 = E_{n-1}\chi_{(\tau_k\geqslant n)}|\triangle_n a^k|^2 = 0$$

thus $s(a^k) = 0$ on the set $\{\tau_k = \infty\}$. For any $k \in Z$, one can choose $j \in N$ such that $e^{-j} < P(\tau_k < \infty) \leqslant e^{-j+1}$. For the chosen j above, obviously $j \neq 0$, putting $p_j = 1 + \frac{1}{j}$ $(j \in N)$, we obtain

$$\begin{aligned}\|s(a^k)\|_{p_j} &\leqslant \|s(a^k)\|_\infty P(\tau_k<\infty)^{\frac{1}{p_j}}\\ &\leqslant P(\tau_k<\infty)^{-1}\Big(\log P(\tau_k<\infty)^{-1}\Big)^{-\alpha} P(\tau_k<\infty)^{\frac{1}{p_j}} \preceq j^{-a}.\end{aligned}$$

With the help of Stirling's formula it is not difficult to establish the following estimate for the norms in $H^s(\log H)^\alpha$ (see [5] for details),

$$\|f\|_{H^s(\log H)^\alpha} \leqslant c\left(\frac{p}{p-1}\right)^\alpha \|s(f)\|_p, \quad p \to 1_+,$$

where $c > 0$ independent of f and p. Since s is subliear, by the norm inequality we get

$$\begin{aligned} \|f\|_{H^s(\log H)^\alpha} &= \|\sum_{k\in Z} \mu_k a^k\|_{H^s(\log H)^\alpha} \leqslant \sum_{k\in Z} |\mu_k| \|s(a^k)\|_{L(\log L)^\alpha} \\ &\preceq \sum_{k\in Z} |\mu_k| \left(\frac{p_j}{p_j-1}\right)^\alpha \|s(a^k)\|_{p_j} \\ &\preceq \sum_{k\in Z} |\mu_k|, \end{aligned}$$

which gives the desired result. We complete the proof. □

LEMMA 2.2. *If the martingale $f \in Q(\log Q)^\alpha$, $\alpha > 0$, then there exist a sequence (a^k) of $(2,1,\alpha)$-atoms and a real number sequence $(\mu_k) \in \ell_1$ such that*

$$f_n = \sum_{k\in Z} \mu_k a_n^k, \quad \forall n \in N$$

and

$$\sum_{k\in Z} |\mu_k| \preceq \|f\|_{Q(\log Q)^\alpha}.$$

Conversely, if the martingale f has the decomposition above, then $f \in Q(\log Q)^\alpha$ and

$$\|f\|_{Q(\log Q)^\alpha} \preceq \inf \sum_{k\in Z} |\mu_k|$$

where the "inf" is taken all the proceding decompositions.

Proof. Suppose that $f \in Q(\log Q)^\alpha$. For any non-decreasing adapted sequence $\beta = (\beta_n)_{n\geqslant 0} \in \Lambda$ such that $S_n(f) \leqslant \beta_{n-1}$, $\beta_\infty \in L(\log L)^\alpha$, the stopping time τ_k is defined in this case by

$$\tau_k = \inf\{n \in N : \beta^n > e^k\} \, (\inf \emptyset = \infty).$$

Then $\tau_k \uparrow \infty (k \uparrow \infty)$. Let a^k and $\mu_k \, (k \in Z)$ be defined as in the proof Lemma 2.1. Then for a fixed k, (a^k) is also a martingale. Since $S(f_n^{\tau_k}) = S_{\tau_k}(f) \leqslant \beta_{\tau_{k-1}} \leqslant e^k, S(f_n^{\tau_{k+1}}) \leqslant e^{k+1}$

$$S(a_n^k) \leqslant \frac{S(f_n^{\tau_{k+1}}) + S(f_n^{\tau_k})}{\mu_k} \leqslant P(\tau_k < \infty)^{-1}\Big(\log P(\tau_k < \infty)^{-1}\Big)^{-\alpha}, \forall n \in N.$$

Similarly to Lemma 2.1, we know (a^k) is really a $(2,1,\alpha)$-atom sequence. For $\forall k \in Z$, we can choose $j \in N$ such that $e^{-j} < P(\tau_k < \infty) \leqslant e^{-j+1}$, from the definition of stopping time τ_k, we get

$$e^{-j} < P(\beta_\infty > e^k) \leqslant e^{-j+1},$$

thus $(\beta_\infty)^*(e^{-j+1}) \leqslant e^k$, $(\beta_\infty)^*(e^{-j}) > e^k$. Consequently, by the fact that

$$\int_0^1 \beta_\infty^*(t)\left(\log\frac{1}{t}\right)^\alpha \sim \sum_{j=1}^\infty j^\alpha e^{-j}\beta_\infty^*(e^{-j}),$$

similarly to Lemma 2.1, we get

$$\begin{aligned}\sum_{k\in Z}|\mu_k| &= 3\sum_{k\in Z}2^k P(\tau_k<\infty)\Big(\log P(\tau_k<\infty)^{-1}\Big)^\alpha \\ &\preceq \sum_{j=1}^\infty j^\alpha e^{-j}\beta_\infty^*(e^{-j}) \preceq \|\beta_\infty\|_{L(\log L)^\alpha}.\end{aligned}$$

By taking infimum on $(\beta_n)_{n\geqslant 0}\in\Lambda$, we obtain $\sum_{k\in Z}|\mu_k| \preceq \|f\|_{Q(\log Q)^\alpha}$.

Conversely, suppose that f has the decomposition above. For any $k \in Z$, setting

$$\lambda_n^k = \chi_{\{\tau_k\leqslant n\}}\|S(a^k)\|_\infty \quad (n\in N),$$

where τ_k is the stopping time with respect to atom a^k. Then $(\lambda_n)_{n\geqslant 0}$ is a non-negative, non-decreasing and adapted sequence. It is clear that $S_{n+1}(a^k)=0$ on the set $\{\tau_k>n\}$. Hence we have

$$S_{n+1}(a^k) = \chi_{\{\tau_k\leqslant n\}}S_n(a^k) \leqslant \chi_{\{\tau_k\leqslant n\}}\|S(a^k)\|_\infty = \lambda_n^k$$

which shows that

$$\|a^k\|_{Q(\log Q)^\alpha} \leqslant \|\lambda_\infty^k\|_{L(\log L)^\alpha}$$

At the same time one can choose $j\in N$ such than $e^{-j-1} < P(\tau_k<\infty) \leqslant e^{-j}$. For the chosen j, putting $p_j = 1+\frac{1}{j}$ $(j\in N)$ and noting that $S(a^k)=0$ on the set $\{\tau_k=\infty\}$, we obtain

$$\begin{aligned}\|\lambda_\infty^k\|_{p_j} &\leqslant \|S(a^k)\|_\infty P(\tau_k<\infty)^{\frac{1}{p_j}} \\ &\leqslant P(\tau_k<\infty)^{-1}\Big(\log P(\tau_k<\infty)^{-1}\Big)^{-\alpha}P(\tau_k<\infty)^{\frac{1}{p_j}} \\ &\preceq j^{-\alpha}.\end{aligned}$$

Nothing that $\|\lambda_\infty\|_{L(\log L)^\alpha} \leqslant c(\frac{p_j}{p_j-1})^\alpha\|\lambda_\infty\|_{p_j}$, where $c>0$ independent of f and p_j, we get

$$\begin{aligned}\|f\|_{Q(\log Q)^\alpha} &\leqslant \sum_{k\in Z}|\mu_k|\|a^k\|_{Q(\log Q)^\alpha} \leqslant \sum_{k\in Z}|\mu_k|\|\lambda_\infty^k\|_{L(\log L)^\alpha}\\ &\preceq \sum_{k\in Z}|\mu_k|\Big(\frac{p_j}{p_j-1}\Big)^\alpha\|\lambda_\infty^k\|_{p_j}\\ &\preceq \sum_{k\in Z}|\mu_k|\Big(\frac{p_j}{p_j-1}\Big)^\alpha j^{-\alpha}\\ &\preceq \sum_{k\in Z}|\mu_k|,\end{aligned}$$

which gives the desired result. We complete the proof. □

LEMMA 2.3. *If the martingale* $f\in\mathscr{D}(\log\mathscr{D})^\alpha$*, then there exist a sequence* (a^k) *of* $(3,1,r)$*-atoms and a real number sequence* $(\mu_k)\in\ell_1$ *such that*

$$f_n=\sum_{k\in z}\mu_k a_n^k,\ \ \forall n\in N$$

and

$$\sum_{k\in z}|\mu_k|\preceq\|f\|_{\mathscr{D}(\log\mathscr{D})^\alpha}.$$

Conversely, if the martingale f *has the decomposition above, then* $f\in\mathscr{D}(\log\mathscr{D})^\alpha$ *and*

$$\|f\|_{\mathscr{D}(\log\mathscr{D})^\alpha}\preceq\inf\sum_{k\in z}|\mu_k|$$

where the "inf" *is taken all the proceding decompositions.*

The proof of Lemma 2.3 is similar to that of Lemma 2.2, so here omit it.

3. Extrapolation theorems

By the atomic decompositions we can give very short and simple proofs of extrapolation theorems of Yano type in the martingale setting. A map $T:X\to Y$, where X is a martingale space and Y is a measurable function space, is said to be sublinear if

$$|T(f+g)|\leqslant|Tf|+|Tg|,\ \ |T(\alpha f)|=|\alpha||Tf|\ \ a.e.,$$

where α is a scalar. For example, the Doob maximal operator, the square operator and the conditional square operator are all sublinear operators.

THEOREM 3.1. *Suppose that for all* p *near 1 with* $p>1$, $T:H_p^s\to L_p$ *is a bounded sublinear operator with* $\|T|H_p^s\to L_p\|\leqslant c(p-1)^{-\alpha}$ *for some* $\alpha>0$ *and* c *is independent of* p*, then* $T:H^s(\log H)^\alpha\to L_1$ *is bounded.*

Proof. By the decomposition from Lemma 2.1, for $f\in H^s(\log H)^\alpha$, we have $f=\sum_{k\in Z}|\mu_k|a^k$. For any $k\in Z$, one can choose $j\in N$ such that $e^{-j}<P(\tau_k<\infty)\leqslant e^{-j+1}$.

Because of p near 1 with $p>1$, for the chosen j, putting $p_j = 1+\frac{1}{j}$ $(j \in N)$, we obtain

$$\|s(a^k)\|_{p_j} \leqslant P(\tau_k<\infty)^{-1}\left(\log P(\tau_k<\infty)^{-1}\right)^{-\alpha} P(\tau_k<\infty)^{\frac{1}{p_j}} \preceq j^{-\alpha}.$$

By the Holder inequality, we get

$$\begin{aligned}
\|Tf\|_1 &\leqslant \Big\|\sum_{k\in Z}|\mu_k||Ta^k|\Big\|_1 \leqslant \sum_{k\in Z}|\mu_k|\|Ta^k\|_1 \\
&\leqslant \sum_{k\in Z}|\mu_k|\|Ta^k\|_{p_j} \preceq \sum_{k\in Z}|\mu_k|(p_j-1)^{-\alpha}\|s(a^k)\|_{p_j} \\
&\preceq \sum_{k\in Z}|\mu_k| j^{\alpha}\|s(a^k)\|_{p_j} \preceq \sum_{k\in Z}|\mu_k| \\
&\preceq \|f\|_{H^s(\log H)^\alpha}.
\end{aligned}$$

The proof is complete. □

THEOREM 3.2. *Suppose that for all p near 1 with $p>1$, $T: Q_p \to L_p$ is a bounded sublinear operator with $\|T|Q_p \to L_p\| \leqslant c(p-1)^{-\alpha}$ for some $\alpha>0$ and c is independent of p, then $T: Q(\log Q)^\alpha \to L_1$ is bounded.*

Proof. By the decomposition from Lemma 2.2, for $f \in Q(\log Q)^\alpha$, we have $f = \Sigma_{k\in Z}|\mu_k|a^k$. Let

$$\lambda_n^k = \chi_{\{\tau_k \leqslant n\}}\|S(a^k)\|_\infty \quad (n \in N),$$

where τ_k is the stopping time with respect to atom a^k. It is clear that $\|a^k\|_{Q_{p_j}} \leqslant \|\lambda_\infty^k\|_{p_j}$. Similarly to Theorem 3.1, we can show that $\|\lambda_\infty\|_{p_j} \preceq j^{-\alpha}$. Thus we get

$$\begin{aligned}
\|Tf\|_1 &\leqslant \Big\|\sum_{k\in Z}|\mu_k||Ta^k|\Big\|_{p_j} \preceq \sum_{k\in Z}|\mu_k|(p_j-1)^{-\alpha}\|a^k\|_{Q_{p_j}} \\
&\leqslant \sum_{k\in Z}|\mu_k| j^{\alpha}\|\lambda_\infty^k\|_{p_j} \preceq \sum_{k\in Z}|\mu_k| \\
&\preceq \|f\|_{Q(\log Q)^\alpha}.
\end{aligned}$$

The proof is finished. □

THEOREM 3.3. *Suppose that for all p near 1 with $p>1$, $T: \mathscr{D}_p \to L_p$ is a bounded operator with $\|T|\mathscr{D}_p \to L_p\| \leqslant c(p-1)^{-\alpha}$ for some $\alpha>0$ and c is independent of p. Then $T: \mathscr{D}(\log \mathscr{D})^\alpha \to L_1$ is bounded.*

Proof. By the decomposition of $f \in \mathscr{D}(\log \mathscr{D})^\alpha$, putting

$$\lambda_n^k = \chi(\tau_k \leqslant n)\|M(a^k)\|_\infty \quad (n \in N),$$

the proof is similar to that of Theorem 3.2. □

4. Variations on the target and domain spaces

The Lorentz space $L^{p,q}(\Omega) = L^{p,q}$, $0 < p < \infty$, $0 < q \leqslant \infty$ consists of all measurable functions with finite quasi-norm $\|f\|_{p,q}$ given by

$$\|f\|_{p,q} = \Big(\frac{q}{p}\int_0^\infty \big(t^{1/p}f^*(t)\big)^q\frac{dt}{t}\Big)^{1/q}, 0 < q < \infty,$$

$$\|f\|_{p,\infty} = \sup_{t>0} t^{1/p}f^*(t), q = \infty.$$

It is well known that if $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$, or $p = q = 1$, then $\|f\|_{p,q}$ is equivalent to a norm (see, for example [2] and [7]). If $p = q$, we get the usual $L_p(\Omega)$.

THEOREM 4.1. *Let* $1 \leqslant q \leqslant \infty$ *and suppose that for all* p *near 1 with* $p > 1$, $T : H_p^s \to L_{p,q}$ *is a bounded sublinear operator with* $\|T|H_p^s \to L_{p,q}\| \leqslant c(p-1)^{-\alpha}$ *for some* $\alpha > 0$ *and* c *is independent of* p, *then* $T : H^s(\log H)^{\alpha+\frac{1}{q'}} \to L_1$ *is bounded, where* $\frac{1}{q'} + \frac{1}{q} = 1$ ($1' = \infty$ and $\infty' = 1$).

Proof. First we shall consider the case $1 < q < \infty$. Let $f \in H^s(\log H)^{\alpha+\frac{1}{q'}}$, by the decomposition from Lemma 2.1,

$$f = \sum_{k\in Z}|\mu_k|a^k, \quad \sum_{k\in Z}|\mu_k| \preceq \|f\|_{H^s(\log H)^{\alpha+\frac{1}{q'}}}.$$

Let $p_j = 1 + \frac{1}{j} \leqslant 2$. From the preceding proof, we know $\|s(a^k)\|_{p_j} \preceq j^{-(\alpha+\frac{1}{q'})}$. Thus

$$\begin{aligned}
\|Tf\|_1 &= \int_0^1 t^{\frac{1}{p_j}}(Tf)^*(t)t^{1-\frac{1}{p_j}}\frac{dt}{t} \\
&\leqslant \Big(\int_0^1 \big(t^{\frac{1}{p_j}}(Tf)^*(t)\big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\Big(\int_0^1 (t^{1-\frac{1}{p_j}})q'\frac{dt}{t}\Big)^{\frac{1}{q'}} \\
&\preceq (p_j-1)^{-\frac{1}{q'}}\|Tf\|_{p_j,q} \\
&\preceq \sum_{k\in Z}|\mu_k|(p_j-1)^{-\frac{1}{q'}}\|Ta^k\|_{p_j,q} \\
&\preceq \sum_{k\in Z}|\mu_k|(p_j-1)^{-\alpha-\frac{1}{q'}}\|s(a^k)\|_{p_j} \\
&\preceq \sum_{k\in Z}|\mu_k| \preceq \|f\|_{H^s(\log H)^{\alpha+\frac{1}{q'}}}.
\end{aligned}$$

If $q = \infty$, we can make a standard rectification for the proof above; if $q = 1$, since $p > 1$ and $\|Tf\|_p \leqslant \|Tf\|_{p,1}$ (see [1, Proposition 4.4.2]), the claim follows immediately from Theorem 3.1.

The following corollaries are easy, which are the variations in the assumed domain space. Since the proofs are similar to those already given, we omit them. $\square$

COROLLARY 4.2. *Suppose that for all p near 1 with $p>1$, $T: H^s_{p,\infty} \to L_1$ is a bounded sublinear operator with $\|T|H^s_{p,\infty} \to L_1\| \leqslant c(p-1)^{-\alpha}$ for some $\alpha > 0$ and c is independent of p, then $T: H^s(\log L)^{\alpha} \to L_1$ is bounded.*

COROLLARY 4.3. *Suppose that for some $1 \leqslant q \leqslant \infty$ and all p near 1 with $p>1$, $T: H^s_{p,\infty} \to L_{p,q}$ is a bounded sublinear with $\|T|H^s_{p,\infty} \to L_{p,q}\| \leqslant c(p-1)^{-\alpha}$ for some $\alpha > 0$ and c is independent of p. Then $T: H^s(\log L)^{\alpha+\frac{1}{q'}} \to L_1$ is bounded.*

It is well known that some of the theorems have been discussed to hold if the initial and target spaces are further logarithmically tuned (see, for example [9]). For example, if for all p near 1 with $p>1$, $T: L_p \to L_p$ is bounded with $\|T|L_p \to L_p\| \leqslant c(p-1)^{-\alpha}$ for some $\alpha > 0$, then for all $\beta > 0$, $T: L(\log L)^{\alpha+\beta} \to L(\log L)^{\beta}$ is bounded. The following result is a similar version in the martingale setting.

THEOREM 4.4. *Suppose that for all p near 1 with $p>1$, $T: H^s_p \to L_{p,\infty}$ is a bounded sublinear operator with $\|T|H^s_p \to L_{p,\infty}\| \leqslant c(p-1)^{-\alpha}$ for some $\alpha > 0$ and c is independent of p, then for all $\beta > 0$, $T: H^s(\log H)^{\alpha+\beta+1} \to L(\log L)^{\beta}$ is bounded.*

Proof. We use the same decomposition idea. As before we write $p_j = 1 + \frac{1}{j} (j \in N)$ and use the decomposition $f = \Sigma_{k \in Z} |\mu_k| a^k$ of $f \in H^s(\log L)^{\alpha+\beta+1}$. Then

$$\begin{aligned}
\|Tf\|_{L(\log L)^{\beta}} &\leqslant \sum_{k \in Z} |\mu_k| \int_0^1 t^{\frac{1}{p_j}} (Ta^k)^*(t) t^{-\frac{1}{p_j}} \left(\log \frac{1}{t}\right)^{\beta} dt \\
&\leqslant \sum_{k \in Z} |\mu_k| \|Ta^k\|_{p_j,\infty} \int_0^1 t^{-\frac{1}{p_j}} \left(\log \frac{1}{t}\right)^{\beta} dt \\
&\preceq \sum_{k \in Z} |\mu_k| (p_j - 1)^{-\alpha} \|s(a^k)\|_{p_j} \int_0^1 t^{-\frac{1}{p_j}} \left(\log \frac{1}{t}\right)^{\beta} dt \\
&\leqslant \sum_{k \in Z} |\mu_k| j^{\alpha} j^{-(\alpha+\beta+1)} \int_0^1 t^{-\frac{1}{p_j}} \left(\log \frac{1}{t}\right)^{\beta} dt
\end{aligned}$$

Change of variables gives

$$\int_0^1 t^{-\frac{1}{p_j}} \left(\log \frac{1}{t}\right)^{\beta} dt \leqslant j^{\beta+1} \Gamma(\beta+1),$$

where Γ is the Gamma function. Hence

$$\|Tf\|_{L(\log L)^{\beta}} \preceq \sum_{k \in Z} |\mu_k| \preceq \|f\|_{H^s(\log H)^{\alpha+\beta+1}},$$

since $\Gamma(\beta+1)$ is a finite positive constant. $\square$

REMARK. From Theorem 2.2 and 2.3, we can conclude that the familiar results respectively hold for $Q(\log Q)^{\alpha}$ and $\mathscr{D}(\log \mathscr{D})^{\alpha}$. We shall not state those explicitly.

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