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SOME LOWER BOUNDS FOR THE PERRON ROOT OF A NONNEGATIVE MATRIX

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Abstract. In this paper, we present some lower bounds for the Perron root of a symmetric nonnegative matrix, which are then applied to give the lower bounds of the Perron root of a general nonnegative matrix. These bounds improve the corresponding ones in [3] and [5]. Numerical examples are supplemented to illustrate the effectiveness of the presented bounds.

1. Introduction

Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a nonnegative matrix with entries a_{ij}, $i, j \in \langle n\rangle := \{1,2,\cdots,n\}$. Then A has a real eigenvalue equal to its spectral radius by Perron-Frobenius theory; see [1]. This eigenvalue is usually called the Perron root of A, and denoted by $\rho(A)$. The localization of the Perron root of a nonnegative matrix is a key problem in matrix theory and numerical analysis. A lot of bounds have been found by many authors; see, e.g., [1]–[12], [14]–[15].

Especially, the simplest lower bound

$$\rho(A) \geqslant \max_{1\leqslant i\leqslant n}\{a_{ii}\} \tag{1.1}$$

was given by Frobenius; see, e.g., [6]. Brauer and Gentry [3] showed that, for the irreducible matrix A,

$$\rho(A) > \frac{1}{2}\max_{i\neq j}\left\{a_{ii}+a_{jj}+\left[(a_{ii}-a_{jj})^2+4a_{ij}a_{ji}\right]^{\frac{1}{2}}\right\}, \tag{1.2}$$

which is sharper than the bound (1.1). Due to the monotonicity property of the Perron root, Kolotilina [5] further improved the lower bound (1.2) as

$$\rho(A) \geqslant \max_{1\leqslant i\leqslant n}\{\varsigma_i(A)\}, \tag{1.3}$$

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where

$$\varsigma_i(A) := \frac{1}{2}\left(a_{ii}+\mu+\left[(a_{ii}-\mu)^2+4\sum_{j\neq i}a_{ij}a_{ji}\right]^{\frac{1}{2}}\right),$$

and μ is the least diagonal element of A.

In this paper, we shall propose some lower bounds for the Perron root of a symmetric nonnegative matrix. These bounds are then used to derive the lower bounds of the Perron root of a general nonnegative matrix. The bounds obtained here improve the corresponding bounds (1.1)–(1.3).

The remainder of this paper is organized as follows. In Section 2, some lower bounds for the Perron roots of nonnegative matrices are established. In Section 3, some examples are given to illustrate the effectiveness of the presented bounds.

2. Lower bounds for the Perron root of a nonnegative matrix

Let $A=(a_{ij})\in\mathbb{R}^{n\times n}$ be a symmetric nonnegative matrix, and let S be a subset of $\langle n\rangle$. Then throughout the paper we define

$$r_i(A)=\sum_{k=1}^{n}a_{ik}^2, r_i^{(i,j)}(A)=\sum_{k\neq i,j}a_{ik}^2, \quad c_{ij}(A)=\sum_{k=1}^{n}a_{ik}a_{jk}, \quad c_{ij}^{(i,j)}(A)=\sum_{k\neq i,j}a_{ik}a_{jk}$$

and $A\{S\}=(\tilde{a}_{ij})$ (see, e.g., [6]) by

$$\tilde{a}_{ij}=\begin{cases}0 & \text{if } \quad i,j\in \mathrm{S}\\ a_{ij} & \text{otherwise}\end{cases}.$$

Let $B,C\in\mathbb{R}^{n\times n}$. Then we denote by I_n the $n\times n$ identity matrix; B^T the transpose of B; $\rho(B)$ the spectral radius of B; $||B||_2$ the spectral norm of B. We write $B\succ C$ ($B\succeq C$) if B,C are symmetric, and $B-C$ is symmetric positive definite (semidefinite).

LEMMA 1. ([13]) *Let $||\cdot||$ be a unitarily invariant norm, and let C' be any submatrix of an arbitrary matrix C. Then $||C'||\leqslant||C||$.*

LEMMA 2. ([1]) *Let $A=(a_{ij})\in\mathbb{R}^{n\times n}, B=(b_{ij})\in\mathbb{C}^{n\times n}$, and let $|B|\leqslant A$, i.e., $|b_{ij}|\leqslant a_{ij}, i,j\in\langle n\rangle$. Then $\rho(B)\leqslant\rho(A)$.*

Based on Lemma 1, we now give the first lower bound as follows, the proof of which is simple.

THEOREM 1. *Let $A=(a_{ij})\in\mathbb{R}^{n\times n}$ be a symmetric nonnegative matrix, let μ be the least diagonal element of A, and let $B=A-\mu I_n$. Then*

$$\rho(A)\geqslant\mu+\left[\max_{i\neq j}\{\tau_{ij}(B)\}\right]^{\frac{1}{2}}, \tag{2.1}$$

where

$$\tau_{ij}(B)=\frac{1}{2}\left\{r_i(B)+r_j(B)+\left[(r_i(B)-r_j(B))^2+4c_{ij}^2(B)\right]^{\frac{1}{2}}\right\}.$$

Proof. For any $i \neq j$, we set

$$F^{ij} = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{ii}-\mu & \cdots & a_{ij} & \cdots & a_{in} \\ a_{j1} & a_{j2} & \cdots & a_{ji} & \cdots & a_{jj}-\mu & \cdots & a_{jn} \end{bmatrix}.$$

Since $A-\mu I_n$ is symmetric nonnegative, from Lemma 1 we have, for any $i \neq j$,

$$\begin{aligned}(\rho(A)-\mu)^2 = ||B||_2^2 \geqslant \left\|F^{ij}\right\|_2^2 &= \rho\left(\begin{bmatrix} x^Tx & x^Ty \\ y^Tx & y^Ty \end{bmatrix}\right) \\ &= \frac{1}{2}\left(x^Tx+y^Ty+\left[(x^Tx-y^Ty)^2+4(x^Ty)^2\right]^{\frac{1}{2}}\right),\end{aligned}$$

where

$$x = (a_{i1}, a_{i2}, \cdots, a_{ii}-\mu, \cdots, a_{in})^T, \quad y = (a_{j1}, a_{j2}, \cdots, a_{jj}-\mu, \cdots, a_{jn})^T,$$

which yields the conclusion (2.1). □

REMARK 1. Since the spectral norm and the spectral radius of any normal matrix are equal, from the proof of Theorem 1, the conclusion of Theorem 1 still holds for any normal nonnegative matrix A.

Due to Lemma 2, we now present the second lower bound.

THEOREM 2. *Let $A=(a_{ij}) \in \mathbb{R}^{n\times n}$ be a symmetric nonnegative matrix, let μ be the least diagonal element of A, and let $\ell \geqslant \mu$ be any lower bound of the Perron root of A. Then*

$$\rho(A) \geqslant \mu + \left[\max_{i\neq j}\{\rho_{ij}^{\ell}(A)\}\right]^{\frac{1}{2}}, \tag{2.2}$$

where

$$\begin{aligned}\rho_{ij}^{\ell}(A) = \frac{1}{2}\Bigg\{&(\ell-\mu)(a_{ii}+a_{jj}-2\mu)+r_i^{(i,j)}(A)+r_j^{(i,j)}(A) \\ &+\left[\left((\ell-\mu)(a_{ii}-a_{jj})+r_i^{(i,j)}(A)-r_j^{(i,j)}(A)\right)^2+4\left((\ell-\mu)a_{ij}+c_{ij}^{(i,j)}(A)\right)^2\right]^{\frac{1}{2}}\Bigg\}.\end{aligned}$$

Proof. For any $i \neq j$, setting $S^{(i,j)} = \langle n\rangle - \{i,j\}$, we have

$$(A-\mu I_n)\{S^{(i,j)}\} = \begin{bmatrix} & & a_{1i} & & a_{1j} & & \\ & & \vdots & & \vdots & & \\ a_{i1} & \cdots & a_{ii}-\mu & \cdots & a_{ij} & \cdots & a_{in} \\ & & \vdots & & \vdots & & \\ a_{j1} & \cdots & a_{ji} & \cdots & a_{jj}-\mu & \cdots & a_{jn} \\ & & \vdots & & \vdots & & \\ & & a_{ni} & & a_{nj} & & \end{bmatrix} =: B^{ij}.$$

Obviously, B^{ij} is similar to the block matrix

$$C^{ij} = \begin{bmatrix} D_{11}^{ij} & D_{12}^{ij} \\ (D_{12}^{ij})^T & 0 \end{bmatrix},$$

where

$$D_{11}^{ij} = \begin{bmatrix} a_{ii}-\mu & a_{ij} \\ a_{ji} & a_{jj}-\mu \end{bmatrix}, \qquad D_{12}^{ij} = \begin{bmatrix} a_{i1} & \cdots & a_{ii-1} & a_{ii+1} & \cdots & a_{ij-1} & a_{ij+1} & \cdots & a_{in} \\ a_{j1} & \cdots & a_{ji-1} & a_{ji+1} & \cdots & a_{jj-1} & a_{jj+1} & \cdots & a_{jn} \end{bmatrix}.$$

It follows from Lemma 2 that

$$\rho(A)-\mu \geqslant \rho(B^{ij}) = \rho(C^{ij}).$$

We now consider the following two cases:

Case (i): $\rho(B^{ij})=0$. In this case, it is clear that $B^{ij}=0$ since B^{ij} is a symmetric matrix with all eigenvalues being zero. Hence, $\rho_{ij}^{\ell}(A)=0$.

Case (ii): $\rho(B^{ij})\neq 0$. The matrix $\rho(B^{ij})I_n - C^{ij}$ can be decomposed into

$$\rho(B^{ij})I_n - C^{ij} = \begin{bmatrix} I_2 & \frac{-1}{\rho(B^{ij})}D_{12}^{ij} \\ & I_{n-2} \end{bmatrix} \begin{bmatrix} E & \\ & \rho(B^{ij})I_{n-2} \end{bmatrix} \begin{bmatrix} I_2 & \\ \frac{-1}{\rho(B^{ij})}(D_{12}^{ij})^T & I_{n-2} \end{bmatrix},$$

where

$$E = \rho(B^{ij})I_2 - D_{11}^{ij} - \frac{1}{\rho(B^{ij})}D_{12}^{ij}(D_{12}^{ij})^T.$$

Clearly,

$$\rho(B^{ij})I_n - C^{ij} \succeq 0,$$

and hence $E\succeq 0$. Since

$$\rho(A)-\mu \geqslant \rho(B^{ij}) \geqslant \rho(D_{11}^{ij}),$$

we obtain

$$\begin{aligned} (\rho(A) \; - \; \mu)^2 I_2 - (\rho(A)-\mu)D_{11}^{ij} - D_{12}^{ij}(D_{12}^{ij})^T \\ &= (\rho(A)-\mu)((\rho(A)-\mu)I_2 - D_{11}^{ij}) - D_{12}^{ij}(D_{12}^{ij})^T \\ &\succeq \rho(B^{ij})(\rho(B^{ij})I_2 - D_{11}^{ij}) - D_{12}^{ij}(D_{12}^{ij})^T = \rho(B^{ij})E \succeq 0, \end{aligned}$$

which implies

$$(\rho(A)-\mu)^2 I_2 - (\ell-\mu)D_{11}^{ij} - D_{12}^{ij}(D_{12}^{ij})^T \succeq 0.$$

Hence,

$$(\;\; \rho(A)-\mu)^2 \geqslant \rho((\ell-\mu)D_{11}^{ij} + D_{12}^{ij}(D_{12}^{ij})^T) = \rho_{ij}^{\ell}(A).$$

From Cases (i) and (ii), the proof is completed. □

REMARK 2. From the proof of Theorem 2, it should be noted that the lower bound in Theorem 2 increases as ℓ gets larger. Moreover, the bound in Theorem 2 can be better than that given by Theorem 1 when ℓ is near to $\rho(A)$, because of, for any $i \neq j$, $\rho(B^{ij}) = ||B^{ij}||_2 \geqslant ||F^{ij}||_2$.

Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a nonnegative matrix, and let $S(A) = (s_{ij})$ be the geometric symmetrization of A with $s_{ij} = (a_{ij}a_{ji})^{\frac{1}{2}}$. Schwenk [11] showed that

$$\rho(A) \geqslant \rho(S(A)),$$

which, together with Theorems 1 and 2, can be applied to give the lower bounds for the Perron root of a general nonnegative matrix.

COROLLARY 1. *Let $A \in \mathbb{R}^{n\times n}$ be a nonnegative matrix, let μ be the least diagonal element of A, and let $B = A - \mu I_n$. Then*

$$\rho(A) \geqslant \mu + \left[\max_{i\neq j}\left\{\tau_{ij}\left(S(B)\right)\right\}\right]^{\frac{1}{2}}.$$

COROLLARY 2. *Let $A \in \mathbb{R}^{n\times n}$ be a nonnegative matrix, let μ be the least diagonal element of A, and let $\ell \geqslant \mu$ be any lower bound of the Perron root of $S(A)$. Then*

$$\rho(A) \geqslant \mu + \left[\max_{i\neq j}\left\{\rho_{ij}^{\ell}\left(S(A)\right)\right\}\right]^{\frac{1}{2}}.$$

The following theorem shows that the bound in Corollary 1 is sharper than the bound (1.2).

THEOREM 3. *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a nonnegative matrix. Then the lower bound in Corollary* 1 *is sharper than the bound* (1.2).

Proof. Let $B = A - \mu I_n$ and $S(B) = (s_{ij})$. From the proof of Theorem 1 and Lemma 1 we obtain

$$\begin{aligned}
\mu + [\tau_{ij}(S(B))]^{\frac{1}{2}} &= \mu + \left\| \begin{bmatrix} s_{i1} & s_{i2} & \cdots & s_{in} \\ s_{j1} & s_{j2} & \cdots & s_{jn} \end{bmatrix} \right\|_2 \\
&\geqslant \mu + \left\| \begin{bmatrix} a_{ii}-\mu & (a_{ij}a_{ji})^{\frac{1}{2}} \\ (a_{ij}a_{ji})^{\frac{1}{2}} & a_{jj}-\mu \end{bmatrix} \right\|_2 = \left\| \begin{bmatrix} a_{ii} & (a_{ij}a_{ji})^{\frac{1}{2}} \\ (a_{ij}a_{ji})^{\frac{1}{2}} & a_{jj} \end{bmatrix} \right\|_2 \\
&= \frac{1}{2}\left(a_{ii}+a_{jj}+\left[(a_{ii}-a_{jj})^2+4a_{ij}a_{ji}\right]^{\frac{1}{2}}\right),
\end{aligned}$$

which implies that the lower bound in Corollary 1 is sharper than the bound (1.2). □

The following result shows that, the bound given by Corollary 2 is better than the bound (1.3) when ℓ is large enough.

THEOREM 4. *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a nonnegative matrix, let μ be the least diagonal element of A, and let ℓ be the lower bound of the Perron root of $S(A)$ satisfying*

$$\ell \geqslant \mu + \max_{i\neq j}\left\{\rho\left(S(B)\{S^{(i,j)}\}\right)\right\}, \tag{2.3}$$

where $B = A - \mu I_n$, and $S^{(i,j)}$ is defined as in the proof of Theorem 2. Then the lower bound in Corollary 2 is sharper than the bound (1.3).

Proof. We define, for any $i \in \langle n\rangle$,

$$S(B) = (s_{ij}), S^{(i)} = \langle n\rangle - \{i\}$$

and, for any $i \neq j$,

$$D_{11}^{ij} = \begin{bmatrix} s_{ii} & s_{ij} \\ s_{ji} & s_{jj} \end{bmatrix}, \qquad D_{12}^{ij} = \begin{bmatrix} s_{i1} & \cdots & s_{ii-1} & s_{ii+1} & \cdots & s_{ij-1} & s_{ij+1} & \cdots & s_{in} \\ s_{j1} & \cdots & s_{ji-1} & s_{ji+1} & \cdots & s_{jj-1} & s_{jj+1} & \cdots & s_{jn} \end{bmatrix}.$$

From the proof of Theorem 2 and (2.3) we can see that, for any $i \neq j$,

$$\begin{aligned} \sqrt{\rho_{ij}^{\ell}(S(A))} &= \sqrt{\rho\left((\ell-\mu)D_{11}^{ij} + D_{12}^{ij}(D_{12}^{ij})^T\right)} \\ &\geqslant \sqrt{\rho\left(\rho(S(B)\{S^{(i,j)}\})D_{11}^{ij} + D_{12}^{ij}(D_{12}^{ij})^T\right)} \\ &= \rho\left(S(B)\{S^{(i,j)}\}\right) \geqslant \max\left\{\rho\left(S(B)\{S^{(i)}\}\right), \rho\left(S(B)\{S^{(j)}\}\right)\right\} \\ &= \max\left\{\varsigma_i(A) - \mu, \varsigma_j(A) - \mu\right\}, \end{aligned}$$

which implies that the bound in Corollary 2 is sharper than the bound (1.3). □

REMARK 3. The bound in Corollary 2 may be sharper than the bound (1.3) even if the conditions of Theorem 4 are not satisfied; see Example 2 in the next section.

3. Numerical examples

In this section, the presented bounds are compared with some known ones by using two examples.

EXAMPLE 1. Consider the positive matrix used in [3]–[5], [7], [12], [14]–[15]:

$$A_1 = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 5 \end{bmatrix}, \quad \rho(A_1) \approx 7.5311.$$

By some known lower bounds we get

Frobenius' bounds (Theorem II.1.1 in [10]): $\rho(A_1) \geqslant 4$;
Ledermann's bounds (Theorem II.1.3 in [10]): $\rho(A_1) \geqslant 4.2910$;
Ostrowski's bounds (Theorem II.1.4 in [10]): $\rho(A_1) \geqslant 4.7321$;
Brauer's bounds (Theorem II.1.5 in [10]): $\rho(A_1) \geqslant 6.1623$;
Brauer,Gentry's bounds(p.106 in [2]): $\rho(A_1) \geqslant 4.8730$;
(Theorem 2 in [3]): $\rho(A_1) \geqslant 6.6056$;
Deutsch, Wielandt's bounds (p. 251 in [4]): $\rho(A_1) \geqslant 7$;
Szulc's bounds (Theorem 1 in [14]): $\rho(A_1) \geqslant 4.9158$;
(Theorem 2 in [15]): $\rho(A_1) \geqslant 5.3589$;
Song's bounds (Theorem 2 in [12]): $\rho(A_1) \geqslant 6.5841$; see Example 2 in [12];
Kolotilina's bounds (Corollary 3 in [5]): $\rho(A_1) \geqslant 6.6095$;
(Theorem 5 in [5]): $\rho(A_1) \geqslant 7.1231$;
Lu's bounds (Theorem 4 in [7]): $\rho(A_1) \geqslant 6.8662$; see Example 2 in [7];
Merikoski, Virtanen's bounds (Corollary 5 in [9]): $\rho(A_1) \geqslant 4.9367$.

Applying Corollary 1 to A_1 yields $\rho(A_1) \geqslant 7.1111$. For A_1 the lower bound in Corollary 1 is better than the other listed known ones, except for the Kolotilina's bound in [5, Theorem 5]. Taking the lower bounds

$$\ell = 4, \quad \ell = 7, \quad \ell = 7.1111 \quad \text{and} \quad \ell = 7.1231$$

obtained above, by Corollary 2 we have, respectively,

$$\rho(A_1) \geqslant 6.0468, \quad \rho(A_1) \geqslant 7.3207, \quad \rho(A_1) \geqslant 7.3630 \quad \text{and} \quad \rho(A_1) \geqslant 7.3676.$$

EXAMPLE 2. Consider the matrix

$$A_2 = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, \qquad \rho(A_2) \approx 3.4551.$$

By simple computations, we derive

Frobenius' bounds (Theorem II. 1.1 in [10]): $\rho(A_2) \geqslant 2$;
Brauer, Gentry's bounds (p. 106 in [2]): $\rho(A_2) \geqslant 2.5616$;
(Theorem 2 in [3]): $\rho(A_2) \geqslant 2.6180$;
Szulc's bounds (Theorem 1 in [14]): $\rho(A_2) \geqslant 2.1794$;
Song's bounds (Theorem 2 in [12]): It is trivial;
Kolotilina's bounds (Corollary 3 in [5]): $\rho(A_2) \geqslant 3.2500$;
(Theorem 5 in [5]): $\rho(A_2) \geqslant 2.7321$;
Merikoski, Virtanen's bounds (Corollary 5 in [9]): $\rho(A_2) \geqslant 1.9315$.

Since A_2 is symmetric, using Theorem 1, we can get $\rho(A_2) \geqslant 3$, which is sharper than $\rho(A_2) \geqslant 2.7321$ by the Kolotilina's bound in [5, Theorem 5]. Taking the lower bounds

$$\ell = 2, \quad \ell = 3 \quad \text{and} \quad \ell = 3.25 \tag{3.1}$$

obtained above, by Theorem 2 we have, respectively,

$$\rho(A_2) \geqslant 2.8284, \quad \rho(A_2) \geqslant 3.1817 \quad \text{and} \quad \rho(A_2) \geqslant 3.2680,$$

which are all sharper than $\rho(A_2) \geqslant 2.7321$ by (1.3). But the condition (2.3) cannot be satisfied. In fact, by simple computations and Theorem 4, the values of ℓ given by (3.1) are strictly smaller than

$$\mu + \max_{i \neq j} \left\{ \rho \left(S(B)\{S^{(i,j)}\} \right) \right\} = 3.2774.$$

REMARK 4. From Examples 1-2, we remark that the lower bounds given in this paper are effective, and improve the lower bounds (1.2) and (1.3). It is worth noting that the presented bound in Corollary 1 is not always better than the existing lower bounds. The bound in Corollary 2 depends on the lower bound ℓ. If ℓ is chosen to be large enough, Examples 1-2 showed that the bound in Corollary 2 are always better than the existing lower bounds.

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