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AN INTEGRAL INEQUALITY FOR n–CONVEX FUNCTIONS

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Abstract. We extend Lupaş inequality for n-convex (n-concave) functions. As consequences some inequalities are derived.

1. Introduction

We say that a real function f is n-convex $(n \in \mathbb{N})$ on the segment $[a,b]$, if its divided differences involving $n+1$ points are nonnegative, i.e.

$$[x_0, x_1, \ldots, x_n; f] \geqslant 0, \tag{1}$$

for any $x_0, x_1, \ldots, x_n \in [a,b]$, where divided difference is defined by recurrence as (see [5, 9])

$$[x_0; f] := f(x_0),$$
$$[x_0, x_1, \ldots, x_n; f] := \frac{[x_1, \ldots, x_n; f] - [x_0, \ldots, x_{n-1}; f]}{x_n - x_0}.$$

Note that (1) can be written as follows

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_0^n & x_1^n & \cdots & x_n^n \\ f(x_0) & f(x_1) & \cdots & f(x_n) \end{vmatrix} \geqslant 0,$$

for any $x_0, x_1, \ldots, x_n \in [a,b]$ such that $x_0 < x_1 < \cdots < x_n$. The 2-convexity reduces to standard convexity.

Let $f, g : [a,b] \to \mathbb{R}$ be integrable functions, consider the Čebyšev functional

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f(x)\,dx \int_a^b g(x)\,dx, \tag{2}$$

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where the integrals involved exist.

In 1971, Atkinson [2] showed that if f,g are convex functions which are twice differentiable on $[a,b]$ and

$$\int_a^b \left(x-\frac{a+b}{2}\right) g(x)\,dx = 0,$$

then $T(f,g) \geqslant 0$. In 1972, Lupaş [6] proved the following inequality for convex functions

$$T(f,g) \geqslant \frac{12}{(b-a)^4}\int_a^b \left(x-\frac{a+b}{2}\right) f(x)\,dx \int_a^b \left(x-\frac{a+b}{2}\right) g(x)\,dx, \tag{3}$$

with equality when at least one of the functions f,g is an affine function on $[a,b]$.

In 2008, Ciobotariu-Boer [3] generalized (3) to 3-convex functions as

$$\begin{aligned} T(f,g) \geqslant\ & \frac{12}{(b-a)^4}\int_a^b \left(x-\frac{a+b}{2}\right) f(x)\,dx \int_a^b \left(x-\frac{a+b}{2}\right) g(x)\,dx \\ & +\frac{180}{(b-a)^6}\int_a^b \left(\left(x-\frac{a+b}{2}\right)^2-\frac{(b-a)^2}{12}\right) f(x)\,dx \\ & \times\int_a^b \left(\left(x-\frac{a+b}{2}\right)^2-\frac{(b-a)^2}{12}\right) g(x)\,dx, \end{aligned}$$

with equality when at least one of functions f,g is a polynomial of degree at most two on $[a,b]$.

Our goal is to generalize the above inequalities to n-convex functions $(n \geqslant 4)$. For reaching this aim, we need the following result (see [3, 9]).

LEMMA 1. *If f,g are n-convex (n-concave) functions on the interval $[a,b]$, then*

$$A_n(f,g) := \begin{vmatrix} 1 & F(e) & \cdots & F\left(e^{n-1}\right) & F(g) \\ F(e) & F\left(e^2\right) & \cdots & F\left(e^n\right) & F(eg) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F\left(e^{n-1}\right) & F\left(e^n\right) & \cdots & F\left(e^{2n-2}\right) & F\left(e^{n-1}g\right) \\ F(f) & F(ef) & \cdots & F\left(e^{n-1}f\right) & F(fg) \end{vmatrix} \geqslant 0, \tag{4}$$

where $e^i(x) = x^i$, $x \in [a,b]$ and F is one of the following functionals:

$$F(f) := \frac{1}{b-a}\int_a^b f(x)\,dx, \quad F(f) := \frac{\int_a^b p(x)f(x)\,dx}{\int_a^b p(x)\,dx},$$

$$F(f) := \sum_{i=1}^{n} p_i f(x_i) \quad \left(x_i \in [a,b]; i = 0,1,\cdots,n, \sum_{i=1}^{n} p_i = 1\right),$$

where $p : [a,b] \to \mathbb{R}^+$, *be an integrable function.*

2. Main results

Without loss of generality, we can work on $[-1,1]$, using the substitution

$$x = \frac{a+b}{2} + \left(\frac{b-a}{2}\right)t,$$

and (4) becomes

$$A_n(r,s) = \begin{vmatrix} 1 & 0 & \frac{1}{3} & \cdots & \frac{1}{2}\int_{-1}^{1} s(t)\,dt \\ 0 & \frac{1}{3} & 0 & \cdots & \frac{1}{2}\int_{-1}^{1} ts(t)\,dt \\ \frac{1}{3} & 0 & \frac{1}{5} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \frac{1}{2}\int_{-1}^{1} t^{n-1}s(t)\,dt \\ \frac{1}{2}\int_{-1}^{1} r(t)\,dt & \frac{1}{2}\int_{-1}^{1} tr(t)\,dt & \cdots & \frac{1}{2}\int_{-1}^{1} t^{n-1}r(t)\,dt & \frac{1}{2}\int_{-1}^{1} r(t)s(t)\,dt \end{vmatrix} \geqslant 0, \qquad (5)$$

where functions r and s are defined by

$$r(t) := f\left(\frac{a+b}{2} + \left(\frac{b-a}{2}\right)t\right), \; s(t) := g\left(\frac{a+b}{2} + \left(\frac{b-a}{2}\right)t\right). \qquad (6)$$

It is easy to see that functions f and g are n-convex on $[a,b]$ if and only if the functions r and s, define by (6) are n-convex on $[-1,1]$.

Let us consider $A_n(r,s)$ for $n = 2,3$ and 4. By simple computations, we obtain

$$A_2(r,s) = \frac{1}{3}\left[\frac{1}{2}\int_{-1}^{1} r(t)s(t)\,dt - \frac{1}{4}\int_{-1}^{1} r(t)\,dt\int_{-1}^{1} s(t)\,dt - \frac{3}{4}\int_{-1}^{1} tr(t)\,dt\int_{-1}^{1} ts(t)\,dt\right],$$

i.e.

$$A_2(r,s) = \frac{1}{3}A_1(r,s) - \frac{1}{4}\int_{-1}^{1} tr(t)\,dt \int_{-1}^{1} ts(t)\,dt.$$

$$A_3(r,s) = \frac{2}{135}\int_{-1}^{1} r(t)s(t)\,dt - \frac{1}{12}\int_{-1}^{1} t^2 r(t)\,dt \int_{-1}^{1} t^2 s(t)\,dt - \frac{1}{45}\int_{-1}^{1} tr(t)\,dt \int_{-1}^{1} tr(t)\,dt$$
$$+\frac{1}{36}\int_{-1}^{1} s(t)\,dt \int_{-1}^{1} t^2 r(t)\,dt + \frac{1}{36}\int_{-1}^{1} r(t)\,dt \int_{-1}^{1} t^2 s(t)\,dt - \frac{1}{60}\int_{-1}^{1} s(t)\,dt \int_{-1}^{1} r(t)\,dt,$$

$$A_3(r,s) = \frac{4}{135}\left[\frac{1}{2}\int_{-1}^{1} r(t)s(t)\,dt - \frac{1}{4}\int_{-1}^{1} r(t)\,dt \int_{-1}^{1} s(t)\,dt\right.$$
$$\left.-\frac{3}{4}\int_{-1}^{1} tr(t)\,dt \int_{-1}^{1} ts(t)\,dt - \frac{5}{4}\int_{-1}^{1}\left(\frac{3}{2}t^2 - \frac{1}{2}\right) r(t)\,dt \int_{-1}^{1}\left(\frac{3}{2}t^2 - \frac{1}{2}\right) s(t)\,dt\right],$$

$$A_3(r,s) = \frac{4}{45}A_2(r,s) - \frac{1}{27}\int_{-1}^{1}\left(\frac{3}{2}t^2 - \frac{1}{2}\right) r(t)\,dt \int_{-1}^{1}\left(\frac{3}{2}t^2 - \frac{1}{2}\right) s(t)\,dt.$$

We continue in this fashion obtaining

$$A_4(r,s) = \frac{4}{175}A_3(r,s) - \frac{4}{3375}\int_{-1}^{1}\left(\frac{5}{2}t^3 - \frac{3}{2}t\right) r(t)\,dt \int_{-1}^{1}\left(\frac{5}{2}t^3 - \frac{3}{2}t\right) s(t)\,dt.$$

We are now able to guess the relation between A_{n+1} and A_n. The proof of this statement is in Lemma 4

$$A_{n+1}(r,s) = \frac{H_{n+1}}{H_n}A_n(r,s) - \frac{2n+1}{4}H_{n+1}\int_{-1}^{1} P_n(t)r(t)\,dt \int_{-1}^{1} P_n(t)s(t)\,dt, \qquad (7)$$

where $H_n := \det\left(\left(c_{i+j-2}\right)\right)_{1\leqslant i,j\leqslant n}$ with $c_i := \frac{1}{2}\int_{-1}^{1} t^i\,dt$, is the Hankel determinant and $P_n(t)$ is Legendre polynomial of degree n.

Rodrigues' formula gives

$$P_n(t) := \frac{1}{2^n n!}\frac{d^n}{dt^n}\left(t^2 - 1\right)^n, \;\; n \geqslant 0. \qquad (8)$$

The following well known properties, will be used [1, 10]

$$P_n(1) = 1, \tag{9}$$

$$P_n(-t) = (-1)^n P_n(t), \tag{10}$$

$$(2n+1)P_n(t) = P'_{n+1}(t) - P'_{n-1}(t), \tag{11}$$

$$\int_{-1}^{1} h(t)\, P_n(t)\, dt = \frac{(-1)^n}{2^n n!} \int_{-1}^{1} \left(t^2-1\right)^n \frac{d^n h(t)}{dt^n} dt, \tag{12}$$

where $h(t)$ is differentiable to the n^{th} order, on $-1 \leqslant t \leqslant 1$.

The first Legendre polynomials are

$P_0(t) = 1,$ $\qquad$ $P_1(t) = t,$

$P_2(t) = \frac{1}{2}\left(3t^2-1\right),$ $\qquad$ $P_3(x) = \frac{1}{2}\left(5t^3-3t\right),$

$P_4(t) = \frac{1}{8}\left(35t^4-30t^2+3\right),$ $\qquad$ $P_5(t) = \frac{1}{8}\left(63t^5-70t^3+15t\right).$

An important property of Legendre polynomials is that they are orthogonal with respect to classical inner product on the interval $[-1,1]$

$$\int_{-1}^{1} P_n(t)\, P_m(t)\, dt = \frac{2}{2n+1}\delta_{m,n}, \tag{13}$$

where $\delta_{m,n}$ denotes Kronecker symbol.

From the general theory of orthogonal polynomials, it follows that the polynomials $P_n(t)$ must have the determinant representation

$$P_n(t) = \frac{\binom{2n}{n}}{H_n 2^{2n}} \begin{vmatrix} \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ \mu_{n-1} & \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} \\ 1 & t & t^2 & \ldots & t^n \end{vmatrix}, \tag{14}$$

where $\mu_i := \int_{-1}^{1} t^i dt$.

The following lemma states some properties of the Hankel determinant H_n.

LEMMA 2. *For $n \geqslant 1$, we have*

$$H_n = \frac{2^{n(n-1)}}{n^n} \prod_{i=1}^{n-1} \frac{(i!)^2}{\left(n^2-i^2\right)^{n-i}}, \tag{15}$$

$$H_{n+1} = \frac{2^{2n}}{(2n+1)\binom{2n}{n}^2} H_n, \tag{16}$$

$$\frac{2n+1}{4}\left(\int_{-1}^{1} t^n P_n(t)\,dt\right)^2 = \frac{H_{n+1}}{H_n}. \tag{17}$$

Proof. It is immediate that $H_n = 2^{n(n-1)}\det\left(\frac{1}{i+j-1}\right)_{1\leqslant i,j\leqslant n}$, using Cauchy determinant formula we get (15). We obtain (16) as consequence of (15). For (17), let us denote

$$I_n := \int_{-1}^{1} t^n P_n(t)\,dt,$$

applying (11), we have

$$I_n = \frac{1}{2n+1}\left[\int_{-1}^{1} t^n P'_{n+1}(t)\,dt - \int_{-1}^{1} t^n P'_{n-1}(t)\,dt\right],$$

using integration by parts, and from (9) and (10) we find

$$I_n = \frac{n}{2n+1} I_{n-1},$$

i.e.

$$I_n = \frac{2^{n+1}}{(2n+1)\binom{2n}{n}},$$

it is obvious that this implies (17). □

The notation

$$\phi_i := \frac{1}{2}\int_{-1}^{1} t^i r(t)\,dt,$$

$$\tau_i := \frac{1}{2}\int_{-1}^{1} t^i s(t)\,dt,$$

is used to simplify the following Lemmas. The next Lemma is needed to prove the next result.

LEMMA 3. *We have*

$$\frac{2^{n-1}}{\binom{2n}{n}} H_n \int_{-1}^{1} P_n(t)\, r(t)\,dt = \begin{vmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_1 & c_2 & c_3 & \cdots & c_{n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{n-1} & c_n & c_{n+1} & \cdots & c_{2n-1} \\ \phi_0 & \phi_1 & \phi_2 & \ldots & \phi_n \end{vmatrix}.$$

Proof. It follows from (14) that

$$\frac{H_n 2^{2n}}{\binom{2n}{n}} P_n(t) = 2^{n+1} \begin{vmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_1 & c_2 & c_3 & \cdots & c_{n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{n-1} & c_n & c_{n+1} & \cdots & c_{2n-1} \\ \frac{1}{2} & \frac{1}{2}t & \frac{1}{2}t^2 & \ldots & \frac{1}{2}t^n \end{vmatrix}.$$

Thus, upon integration

$$\frac{H_n 2^{n-1}}{\binom{2n}{n}} \int_{-1}^{1} P_n(t) r(t)\, dt = \begin{vmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_1 & c_2 & c_3 & \cdots & c_{n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{n-1} & c_n & c_{n+1} & \cdots & c_{2n-1} \\ \frac{1}{2}\int_{-1}^{1} r(t)\, dt & \frac{1}{2}\int_{-1}^{1} t r(t)\, dt & \frac{1}{2}\int_{-1}^{1} t^2 r(t)\, dt & \ldots & \frac{1}{2}\int_{-1}^{1} t^n r(t)\, dt \end{vmatrix}.$$

The proof is now complete. □

LEMMA 4. *For $n \geqslant 1$, the determinant A_n satisfies the recurrence relation*

$$A_{n+1}(r,s) = \frac{H_{n+1}}{H_n} A_n(r,s) - \frac{2n+1}{4} H_{n+1} \int_{-1}^{1} P_n(t) r(t)\, dt \int_{-1}^{1} P_n(t) s(t)\, dt.$$

and is given explicitly by

$$A_n(r,s) = H_n \left(\frac{1}{2} \int_{-1}^{1} r(t) s(t)\, dt - \sum_{k=0}^{n-1} \frac{2k+1}{4} \int_{-1}^{1} P_n(t) r(t)\, dt \int_{-1}^{1} P_n(t) s(t)\, dt \right), \qquad (18)$$

Proof. Expressing A_{n+1} as

$$A_{n+1}(r,s) = \begin{vmatrix} c_{2n} & c_n & c_{n+1} & \cdots & c_{2n-1} & \tau_n \\ c_n & c_0 & c_1 & \cdots & c_{n-1} & \tau_0 \\ \vdots & \vdots & \vdots & & \vdots & \\ c_{2n-1} & c_{n-1} & c_n & \cdots & c_{2n-2} & \tau_{n-1} \\ \phi_n & \phi_0 & \phi_1 & \ldots & \phi_{n-1} & \frac{1}{2}\int_{-1}^{1} r(t) s(t)\, dt \end{vmatrix},$$

and applying the Desnanot-Jacobi identity (see [4], pp. 11), we get

$$H_nA_{n+1}(r,s) = H_{n+1}A_n(r,s) - \begin{vmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_1 & c_2 & c_3 & \cdots & c_{n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{n-1} & c_n & c_{n+1} & \cdots & c_{2n-1} \\ \phi_0 & \phi_1 & \phi_2 & \ldots & \phi_n \end{vmatrix} \begin{vmatrix} c_0 & c_1 & \cdots & c_{n-1} & \tau_0 \\ c_1 & c_2 & \cdots & c_{n+1} & \tau_1 \\ \vdots & \vdots & & \vdots & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-1} & \tau_{n-1} \\ c_n & c_{n+1} & \cdots & c_{2n-1} & \tau_n \end{vmatrix}.$$

Using Lemma 3 and (16), we get the result. This completes the proof. □

Now, we are able to establish a sharp lower bound involving Legendre polynomials, for the Čebyšev functional (2), where f and g are n-convex functions.

THEOREM 1. *If f,g are n-convex (n-concave) functions on the interval $[a,b]$, then the following inequality holds*

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f(x)\,dx\int_a^b g(x)\,dx \geqslant \sum_{k=1}^{n-1}\frac{2k+1}{(b-a)^2}\int_a^b P_k\left(\frac{2x-a-b}{b-a}\right)f(x)\,dx\int_a^b P_k\left(\frac{2x-a-b}{b-a}\right)g(x)\,dx, \quad (19)$$

where $P_k(x)$ are Legendre polynomials. The equality holds when at least one of the functions f or g is a polynomial function of degree at most $n-1$ on $[a,b]$. The reverse inequality holds when f is n-convex (n-concave) and g is n-concave (n-convex).

Proof. We translate (18) to the interval $[a,b]$ and use (6), we get (19). □

THEOREM 2. *If the functions f and g are n-convex (n-concave) functions on $[a,b]$, respectively one of them is n-convex and the other is n-concave. Then, we obtain (19), respectively the reverse of (19). The equality in (19) holds when at least one of the functions f,g is a polynomial function of degree at most $n-1$ on $[a,b]$ and the reverse inequality holds if f is n-convex (n-concave) and g is n-concave (n-convex). Moreover, if f,g are differentiable to the $(n-1)$ order on $[a,b]$, then the following inequality holds*

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f(x)\,dx\int_a^b g(x)\,dx \geqslant \sum_{k=1}^{n-1}\frac{2^{2k}(2k+1)}{(k!)^2(b-a)^{4k+2}}\int_a^b (x-a)^k(x-b)^k f^{(k)}(x)\,dx\int_a^b (x-a)^k(x-b)^k g^{(k)}(x)\,dx. \quad (20)$$

Proof. We use (12) to obtain (20). □

COROLLARY 1. *If f,g are n-convex (n-concave) functions on $[a,b]$, for $n \geqslant 2$ we have*

$$\frac{1}{b-a}\int_a^b f^{(l)}(x)\,g^{(l)}(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f^{(l)}(x)\,dx\int_a^b g^{(l)}(x)\,dx$$
$$\geqslant \sum_{k=1}^{n-l-1}\frac{2k+1}{(b-a)^2}\int_a^b P_k\left(\frac{2x-a-b}{b-a}\right)f^{(l)}(x)\,dx\int_a^b P_k\left(\frac{2x-a-b}{b-a}\right)g^{(l)}(x)\,dx. \quad (21)$$

Proof. We use the fact that, if f,g are n-convex functions on $[a,b]$ for $n \geqslant 2$, then the functions $f^{(l)}, g^{(l)}$ exist and are $(n-l)$-convex, for $1 \leqslant l \leqslant n-2$. □

REMARK 1. A function f is convex of order (n,n) if for every $x_i, y_j, i,j = 1,\ldots,n$, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{f(x_i,y_j)}{u'(x_i)\,v'(y_j)} \geqslant 0,$$

where $u(x) = \prod_{i=1}^{n}(x-x_i)$, $v(y) = \prod_{j=1}^{n}(y-y_j)$. Note that the inequality (19) can be generalized for convex function of order (n,n), obtaining a result similar to that of Pečarić [8], from where the inequality

$$\frac{1}{b-a}\int_a^b f(x,x)\,dx - \frac{1}{(b-a)^2}\int_a^b\int_a^b f(x,y)\,dxdy$$
$$\geqslant \sum_{k=1}^{n-1}\frac{2k+1}{(b-a)^2}\int_a^b\int_a^b P_k\left(\frac{2x-a-b}{b-a}\right)P_k\left(\frac{2y-a-b}{b-a}\right)f(x,y)\,dxdy,$$

holds for all integer numbers $n \geqslant 2$.

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