Mathematical
Inequalities
& Applications
Volume 15, Number 1 (2012), 165–198

CONCENTRATION–COMPACTNESS PRINCIPLE FOR EMBEDDING INTO MULTIPLE EXPONENTIAL SPACES

ROBERT ČERNÝ

(*Communicated by V. Volenec*)

Abstract. Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be a bounded domain and let $\alpha < n-1$. We prove the Concentration-Compactness Principle for the embedding of the Orlicz-Sobolev space $W_0^1 L^n \log^{n-1} L \log^{\alpha} \log L(\Omega)$ into the Orlicz space corresponding to a Young function that behaves like $\exp(\exp(t^{\frac{n}{n-1-\alpha}}))$ for large t. We also give the result for the case of the embedding into triple and other multiple exponential spaces.

1. Introduction

Throughout the paper Ω denotes an open bounded set in $\mathbb{R}^n$, $n \geqslant 2$. Let ω_{n-1} be the measure of the surface of the unit sphere in $\mathbb{R}^n$. By ∇f we denote the generalized derivative of f. The space $W_0^{1,n}(\Omega)$ or $W_0 L^{\Phi}(\Omega)$ stands for the closure of $C_0^{\infty}(\Omega)$ in $W^{1,n}(\Omega)$ or $WL^{\Phi}(\Omega)$, respectively.

The classical Sobolev embedding theorem states that $W_0^{1,p}(\Omega)$ is continuously embedded into $L^{p^*}(\Omega)$ if $1 \leqslant p < n$ and $p^* = \frac{pn}{n-p}$. If $p > n$ then every function from $W_0^{1,p}(\Omega)$ is bounded (i.e. belongs to $L^{\infty}(\Omega)$) and in the limiting case $p = n$, it is known that every function from $W_0^{1,n}(\Omega)$ belongs to $L^q(\Omega)$ for every $1 \leqslant q < \infty$, but not necessarily to $L^{\infty}(\Omega)$. A famous result by Trudinger (see [15], [22], [25] and [26]) implies that the space $W_0^{1,n}(\Omega)$ is continuously embedded into the Orlicz space $L^{\Phi}(\Omega)$ with the Young function Φ of an exponential type $\Phi(t) = \exp(t^{\frac{n}{n-1}}) - 1, t > 0$.

In [18] Moser proved that for $K \leqslant n\omega_{n-1}^{\frac{1}{n-1}}$ we have

$$\sup\Big\{\int_{\Omega} \exp\big(K|f(x)|^{\frac{n}{n-1}}\big)\,dx : f \in W_0^{1,n}(\Omega), ||\nabla f||_{L^n(\Omega)} \leqslant 1\Big\} < \infty \tag{1}$$

but that for $K > n\omega_{n-1}^{\frac{1}{n-1}}$ the supremum in (1) is not finite.

For $\alpha < n-1$ set

$$\gamma = \frac{n}{n-1-\alpha} > 0 \qquad \text{and} \qquad B = 1 - \frac{\alpha}{n-1} = \frac{n}{(n-1)\gamma} > 0\,. \tag{2}$$

Mathematics subject classification (2010): 46E35, 46E30, 49J99.

Keywords and phrases: Orlicz spaces, Orlicz-Sobolev spaces, embedding theorems, sharp constants, concentration-compactness principle.

Paper MIA-15-15

The space $W_0L^n\log^\alpha L(\Omega)$ of the (first order) Sobolev type, modeled on the Zygmund space $L^n\log^\alpha L(\Omega)$, is continuously embedded into the Orlicz space with the Young function that behaves like $\exp(t^\gamma)$ for large t. These results are due to Fusco, Lions, Sbordone [14] for $\alpha<0$ and Edmunds, Gurka, Opic [6] in general. Moreover it is shown in [6] (see also [5] and [7]) that in the limiting case $\alpha=n-1$ we have the embedding into a double exponential space, i.e. the space $W_0L^n\log^{n-1}L\log^\alpha\log L(\Omega)$, $\alpha<n-1$, is continuously embedded into the Orlicz space with the Young function that behaves like $\exp(\exp(t^\gamma))$ for large t. Further in the limiting case $\alpha=n-1$ we have the embedding into triple exponential space and so on. The borderline case is always $\alpha=n-1$ and for $\alpha>n-1$ we have embedding into $L^\infty(\Omega)$. It is well-known that the Zygmund space $L^n\log^\alpha L(\Omega)$ coincides with the Orlicz space $L^\Phi(\Omega)$, where

$$\lim_{t\to\infty}\frac{\Phi(t)}{t^n\log^\alpha(t)}=1\ ,$$

the space $L^n\log^{n-1}L\log^\alpha\log L(\Omega)$ coincides with $L^\Phi(\Omega)$ where

$$\lim_{t\to\infty}\frac{\Phi(t)}{t^n\log^{n-1}(t)\log^\alpha(\log(t))}=1\ ,$$

and so on. For other results concerning these spaces we refer the reader to [7], [8], [9], [10], [13] and [20].

For $k\in\mathbb{N}$, let us write

$$\log_{[k]}(t)=\log(\log_{[k-1]}(t)),\qquad \text{where}\qquad \log_{[1]}(t)=\log(t)$$

and

$$\exp_{[k]}(t)=\exp(\exp_{[k-1]}(t)),\qquad \text{where}\qquad \exp_{[1]}(t)=\exp(t)\ .$$

Let $k\in\mathbb{N}$ and $\alpha<n-1$. Then we have above mentioned embedding results for any Young function Φ satisfying

$$\lim_{t\to\infty}\frac{\Phi(t)}{t^n\left(\prod_{j=1}^{k-1}\log_{[j]}^{n-1}(t)\right)\log_{[k]}^\alpha(t)}=1 \tag{3}$$

(for $k=1$ we read (3) as $\lim_{t\to\infty}\frac{\Phi(t)}{t^n\log_{[1]}^\alpha(t)}=1$). As Ω is bounded, all Young functions satisfying (3) give the same Orlicz-Sobolev space. In particular, we have from [6, Remarks 3.11(iv)]

PROPOSITION 1.1. *Let $n\geqslant 2$, $k\in\mathbb{N}$, $\alpha<n-1$ and let Φ be a Young function satisfying* (3). *Let $K\in\mathbb{R}$ and let $f\in W_0L^\Phi(\Omega)$. Then*

$$\int_\Omega \exp_{[k]}\left(K|f(x)|^\gamma\right)dx<\infty\ .$$

Moser-type results

Further for $K \in \mathbb{R}$ and Φ satisfying (3) let us set

$$S(K,\Phi) = \sup\Big\{ \int_\Omega \exp_{[k]}\big(K|f(x)|^\gamma\big)\,dx \ : \ f \in W_0L^\Phi(\Omega)\ ,\ ||\Phi(|\nabla f|)||_{L^1(\Omega)} \leqslant 1 \Big\}\ . \quad (4)$$

We have the following result.

THEOREM 1.2. *Let $n \geqslant 2$, $k \in \mathbb{N}$ and $\alpha < n-1$. Set*

$$K_{k,n,\alpha} = \begin{cases} B^{\frac{1}{B}} n \omega_{n-1}^{\frac{\gamma}{n}} & \text{for } k=1 \\ B^{\frac{1}{B}} \omega_{n-1}^{\frac{\gamma}{n}} & \text{for } k \geqslant 2\ . \end{cases} \quad (5)$$

Let Φ be a Young function satisfying (3). *Then the following statements hold.*
(i) *If $K < K_{k,n,\alpha}$, then $S(K,\Phi) < \infty$.*
(ii) *If $K > K_{k,n,\alpha}$, then $S(K,\Phi) = \infty$.*
(iii) *If $K = K_{k,n,\alpha}$ and there are $t_0 > \exp_{[k]}(1)$ and $a \in (0, \min\{1, \frac{1}{\gamma}\})$ such that Φ satisfies*

$$\Phi(t) \geqslant t^n \Big(\prod_{j=1}^{k-1} \log_{[j]}^{n-1}(t)\Big) \log_{[k]}^{\alpha}(t) \Big(1 + \log_{[k]}^{-a}(t)\Big) \qquad \text{for} \quad t > t_0\ , \quad (6)$$

then $S(K,\Phi) < \infty$.
(iv) *If $K = K_{k,n,\alpha}$ and there are $t_0 > \exp_{[k]}(1)$, $a \in (0, \min(1,B))$ and $C > 0$ such that*

$$\Phi(t) \leqslant \begin{cases} Ct^n & \text{for } t \in [0,t_0] \\ t^n \Big(\prod_{j=1}^{k-1} \log_{[j]}^{n-1}(t)\Big) \log_{[k]}^{\alpha}(t) \Big(1 - \log_{[k]}^{-a}(t)\Big) & \text{for } t \in [t_0,\infty)\ . \end{cases} \quad (7)$$

then $S(K,\Phi) = \infty$.

In the case $k \geqslant 2$, all four assertions of Theorem 1.2 follow from [3, Theorem 1.1, Theorem 1.2, Theorem 4.2 and Theorem 4.1]. In case $k = 1$, assertions (i), (ii), (iii) follow from [16, Theorem 1.1, Theorem 1.2 and Theorem 4.2]. A weaker version of (iv) is in [16, Theorem 4.1]. Our version of (iv) is a new result which we prove in Section 5, Example 5.1.

Let us briefly indicate why the borderline parameter $K_{k,n,\alpha}$ is the same for all $k \geqslant 2$ while for $k = 1$ it is n times larger (see (5)). For the value of $S(K,\Phi)$, the most important are radial functions of the growth $f(x) \sim (B^{\frac{1}{B}} \omega_{n-1}^{\frac{\gamma}{n}})^{-\frac{1}{\gamma}} \log_{[k]}^{\frac{1}{\gamma}}(\frac{1}{|x|})$. For such functions and $\tau > 0$ we have

$$\int_{B(R)} \exp_{[k]}\big(\tau B^{\frac{1}{B}} \omega_{n-1}^{\frac{\gamma}{n}} |f(x)|^\gamma\big)\,dx = C \int_0^R \exp_{[k]}\Big(\tau \log_{[k]}\Big(\frac{1}{r}\Big)\Big) r^{n-1}\,dr\ .$$

We observe that the last integral in the case $k \geqslant 2$ converges if and only if $\tau \leqslant 1$ while it is $\tau < n$ for $k = 1$.

Notice that even though all Young functions satisfying (3) with fixed $k \in \mathbb{N}$ and $\alpha < n-1$ give the same Orlicz-Sobolev space, they give different Moser-type results in the critical case $K = K_{k,n,\alpha}$.

Let us observe the borderline case $K = K_{k,n,\alpha}$ in more detail. Since there is a huge gap between (6) and (7), we see that Theorem 1.2 is not sharp in the critical case (for the Moser's case $k = 1$, $\alpha = 0$ it is shown in [4] that the borderline Young function is $\Phi(t) = t^n(1 - \log^{-1}(t))$).

Concentration-Compactness Principle

Next we are interested in the generalization of the Concentration-Compactness Alternative by Lions [17] for our spaces $W_0L^{\Phi}(\Omega)$.

Embeddings are usually not compact in the borderline cases. For example embedding of the Sobolev space $W_0^{1,p}(\Omega)$ into $L^{\frac{pn}{n-p}}(\Omega)$ for $1 \leqslant p < n$ or into $L^{\Phi}(\Omega)$ for $p = n$ are not compact. However there is the Concentration-Compactness Principle (see [24] and references given there for the history and applications) telling us that some substitute for compactness is still available for many embeddings. By this principle, each bounded sequence can be decomposed into subsequences that either converge in the target space or have very special behavior. For example such a subsequence can concentrate around one point and in some sense converge to the Dirac mass at this point. This observation is often used in many problems from the Calculus of Variations (see e.g. [11], [12], [17], [24]). In particular, this principle gives the Compactness in the situations which do not allow the Concentration (see for example the proof a level-set version of the Palais-Smale condition in [11] and [19], or the proof of our Theorem 1.5 bellow).

In recent paper [2], there are the techniques for obtaining the Concentration-Compactness Principle for $W_0^{1,n}(\Omega)$ (see [17]) modified for the embedding of the space $W_0L^n\log^{\alpha}L(\Omega)$ into the corresponding single exponential space. The main problem when adapting the proof was the fact that working with the Luxemburg norm corresponding to a general Young function satisfying the Δ_2-condition is much more difficult than working with the nice $L^n(\Omega)$-norm (in particular, homogeneity of the function $t \mapsto t^n$ is usually crucial for the techniques used in [17]). Moreover, the proof in [17] is based on (1) while the proof in [2] requires the Moser-like result for embedding of the space $W_0L^n\log^{\alpha}L(\Omega)$ into the Orlicz space corresponding to the Young function $\exp(t^{\frac{n}{n-1-\alpha}}) - 1$ given in [15] (i.e. Theorem 1.2 for $k = 1$).

The aim of this paper is to extend the Concentration-Compactness Principle for embeddings into multiple exponential spaces. For these spaces our main tool is a Moser-like result from [3] (i.e. Theorem 1.2 for $k \geqslant 2$).

THEOREM 1.3. *Let $n \geqslant 2$, $k \in \mathbb{N}$, $\alpha < n-1$ and let Φ be a Young function satisfying* (3)*, $S(K_{k,n,\alpha}, \Phi) < \infty$ and suppose there are $T \geqslant \exp_{[k]}(1)$ and $\ell > k$ such that*

$$\Phi(t) \geqslant t^n\Big(\prod_{j=1}^{k-1}\log_{[j]}^{n-1}(t)\Big)\log_{[k]}^{\alpha}(t)\Big(1 - \log_{[\ell]}^{-1}(t)\Big) \qquad \textit{for} \quad t \geqslant T\,. \tag{8}$$

Let $\{u_j\}_{j=1}^{\infty} \subset W_0L^{\Phi}(\Omega)$ *satisfy* $\|\Phi(|\nabla u_j|)\|_{L^1(\Omega)} \leqslant 1$ *and*

$$u_j \rightharpoonup u \quad \text{in } W_0L^{\Phi}(\Omega)\,,\ u_j \to u \text{ a.e. in } \Omega \text{ and } \Phi(|\nabla u_j|) \overset{*}{\rightharpoonup} \mu \text{ in } \mathscr{M}(\bar{\Omega})\,. \tag{9}$$

(i) *If* $u = 0$ *and* $\mu = \delta_{x_0}$ *for some* $x_0 \in \bar{\Omega}$*, then the sequence* $\{\exp_{[k]}(K_{k,n,\alpha}|u_j|^{\gamma})\}_{j=1}^{\infty}$ *is relatively compact with respect to the weak* convergence in* $\mathscr{M}(\bar{\Omega})$ *and the limits of convergent subsequences belong to* $\{\exp_{[k]}(0)\mathscr{L}_n|_{\Omega} + c\delta_{x_0} : c \geqslant 0\}$.
(ii) *If* $u \neq 0$ *or* μ *is not a Dirac mass at one point, then there is* $\delta > 0$ *such that* $\exp_{[k]}(K_{k,n,\alpha}(1+\delta)|u_j|^{\gamma})$ *is bounded in* $L^1(\Omega)$ *and*

$$\exp_{[k]}(K_{k,n,\alpha}|u_j|^{\gamma}) \overset{j\to\infty}{\to} \exp_{[k]}(K_{k,n,\alpha}|u|^{\gamma}) \qquad \text{in } L^1(\Omega)\,.$$

Theorem 1.3 tells us that each sequence of normalized functions from $W_0L^{\Phi}(\Omega)$ can be decomposed into subsequences (notice that, since $W_0L^{\Phi}(\Omega)$ is reflexive, assumptions (9) are obtained just passing to a subsequence) that either concentrate around a point in $\bar{\Omega}$, or the integrability of such a subsequence is better than Theorem 1.2(i) says.

The version of Theorem 1.3 given in paper [2] (case $k = 1$) with our assumptions $S(K_{k,n,\alpha}, \Phi) < \infty$ and (8) replaced by assumption (6) is much weaker. Let us explain why.

First, the assumption $S(K_{k,n,\alpha}, \Phi) < \infty$ is necessary because otherwise we can find a sequence satisfying

$$\exp_{[k]}(K_{k,n,\alpha}|u_j|^{\gamma}) \overset{j\to\infty}{\to} \infty > \exp_{[k]}(K_{k,n,\alpha}|u|^{\gamma})$$

(see the concentrating sequences from Example 5.1 and [3, Theorem 4.1]) and thus neither (i) nor (ii) can hold. Second, let us compare the assumptions (8) and (6). Concerning (6), in view of remarks following Theorem 1.2 we see that assumption (6) is very restrictive. Concerning condition (8), the larger $\ell > k$ is, the more permissive this condition is. Moreover, from Theorem 1.2(iv) we see that already for $\ell = k+1$ there are many Young functions satisfying (8) even though $S(K_{k,n,\alpha}, \Phi) = \infty$. Therefore this condition is practically harmless and always satisfied by reasonable Young functions satisfying (3) and $S(K_{k,n,\alpha}, \Phi) < \infty$.

The author would like to know, if it is possible to remove condition (8).

Notice that the assertion of Theorem 1.3 cannot be valid for $K > K_{k,n,\alpha}$ because by Theorem 1.2 the integral from $\exp_{[k]}(K|u|^{\gamma})$ can be arbitrarily large if $K > K_{k,n,\alpha}$. Conversely, if $K < K_{k,n,\alpha}$ then we have just the Compactness as an easy corollary of the Moser-type result.

COROLLARY 1.4. *Let* $n \geqslant 2$, $\alpha < n-1$, $k \in \mathbb{N}$, $K < K_{k,n,\alpha}$ *and let* Φ *be a Young function satisfying* (3)*. Let* $\{u_j\}_{j=1}^{\infty} \subset W_0L^{\Phi}(\Omega)$ *satisfy* $\|\Phi(|\nabla u_j|)\|_{L^1(\Omega)} \leqslant 1$*. Further suppose that*

$$u_j \to u \quad \text{a.e. in } \Omega\,.$$

Then

$$\exp_{[k]}(K|u_j|^{\gamma}) \overset{j\to\infty}{\to} \exp_{[k]}(K|u|^{\gamma}) \qquad \text{in } L^1(\Omega)\,.$$

Having a version of the Concentration-Compactness Alternative for embedding into multiple exponential spaces we also have the corresponding result concerning the maximum of a functional with the sub-critical growth.

THEOREM 1.5. *Let $n \geqslant 2$, $\alpha < n-1$, $k \in \mathbb{N}$ and let Φ be a Young function satisfying* (3). *Suppose that the function $F : \mathbb{R} \mapsto \mathbb{R}$ is even and continuous. Further suppose that either*

$$\lim_{t\to\infty} \frac{F(t)}{\exp_{[k]}(K|t|^{\gamma})} = 0 \quad \text{for some } K < K_{k,n,\alpha} \tag{10}$$

or Φ satisfies additional condition (8) *and*

$$\lim_{t\to\infty} \frac{F(t)}{\exp_{[k]}(K_{k,n,\alpha}|t|^{\gamma})} = 0\,. \tag{11}$$

Then the functional

$$\Lambda_F(u) = \int_\Omega F(u(x))\,dx$$

attains its maximum on the set $\{u \in W_0L^{\Phi}(\Omega) : \|\Phi(|\nabla u|)\|_{L^1(\Omega)} \leqslant 1\}$.

Notice that we do not need to assume $S(K_{k,n,\alpha},\Phi) < \infty$ in Theorem 1.5.

The paper is organized in the following way. The third section is devoted to Lemma 3.2, which provides us with an estimate enabling us to use Hölders inequality much more effectively than in paper [2]. In fact this is the crucial point for us when replacing assumption (6) often used in [2] by our assumptions $S(K_{k,n,\alpha},\Phi) < \infty$ and (8).

In the fourth section we apply Lemma 3.2 in the proof of Theorem 1.3. We use the techniques from [2] modified for embedding into multiple exponential spaces. We can use some partial results concerning general Young functions satisfying the Δ_2-condition from [2]. Therefore we sketch some of our proofs. We also prove Corollary 1.4 and Theorem 1.5 in the fourth section.

In the fifth section we construct an example proving Theorem 1.2(iv) in the case $k = 1$.

In the sixth section we give a summary concerning the Concentration phenomenon.

2. Preliminaries

The n-dimensional Lebesgue measure is denoted by $\mathscr{L}_n$. Further, $\mathscr{L}_n|_\Omega$ is its restriction to Ω, i.e. $\mathscr{L}_n|_\Omega(A) = \mathscr{L}_n(A \cap \Omega)$ for every measurable set $A \subset \mathbb{R}^n$. If u is a measurable function on Ω, then by $u = 0$ (or $u \neq 0$) we mean that u is equal (or not equal) to the zero function a.e. on Ω.

By $\mathscr{M}(A)$ we denote the set of all Radon measures on a compact set A. We write that $\mu_j \overset{*}{\rightharpoonup} \mu$ in $\mathscr{M}(A)$ if $\int_A \psi\, d\mu_j \to \int_A \psi\, d\mu$ for every $\psi \in C(A)$. It is well known that each sequence bounded in $L^1(A)$ contains a subsequence converging weakly* in $\mathscr{M}(A)$.

By $B(x_0,R)$ we denote an open Euclidean ball in $\mathbb{R}^n$ centered at x_0 with the radius $R>0$. If $x_0=0$ we simply write $B(R)$.

By C we denote a generic positive constant which may depend on k, n, α, $\mathscr{L}_n(\Omega)$ and Φ. This constant may vary from expression to expression as usual. Sometimes we say that for every $\varepsilon>0$ something is true. Then the constants C in such a case may depend also on fixed $\varepsilon>0$.

For given functions g,h we say that $g(t)>>h(t)$ for t big enough if we have $\lim_{t\to\infty}\frac{g(t)}{h(t)}=\infty$. Analogously $g(t)>>h(t)$ for t small enough if $\lim_{t\to 0_+}\frac{g(t)}{h(t)}=\infty$.

The following lemma from [3, Lemma 2.2] shows that the function $\log_{[k]}$ has similar asymptotic behavior as the function $\log$.

LEMMA 2.1. *Let $t_1,p,q,\delta,E,L>0$ and $k\in\mathbb{N}$ and let functions $f,h:\mathbb{R}\mapsto(0,\infty)$ and $g:\mathbb{R}\mapsto\mathbb{R}$ satisfy*

$$g(t)+Ef(t)>\exp_{[k]}(0) \qquad \textit{and} \qquad Eh^q(t)f^p(t)>\exp_{[k]}(0) \quad \textit{on } (t_1,\infty),$$

$$\lim_{t\to\infty}f(t)=\infty, \quad \frac{g(t)}{f(t)}\in[-E+\delta,L] \quad \textit{and} \quad \frac{\log(h(t))}{\log(f(t))}\in\left[-\frac{p}{q}+\delta,L\right] \quad \textit{on } (t_1,\infty)\,.$$

Then there is $t_0>t_1$ such that if $t>t_0$ then

$$1-\frac{C}{\log_{[k]}\big(f(t)\big)}<\frac{\log_{[j]}\big(g(t)+Ef(t)\big)}{\log_{[j]}\big(f(t)\big)}<1+\frac{C}{\log_{[k]}\big(f(t)\big)} \quad \textit{for} \quad j\in\{1,\dots,k\} \tag{12}$$

and

$$1-\frac{C}{\log_{[k]}\big(f(t)\big)}<\frac{\log_{[j]}\big(Eh^q(t)f^p(t)\big)}{\log_{[j]}\big(f(t)\big)}<1+\frac{C}{\log_{[k]}\big(f(t)\big)} \quad \textit{for} \quad j\in\{2,\dots,k\}\,. \tag{13}$$

Young functions and Orlicz spaces

A function $\Phi:\mathbb{R}^+\to\mathbb{R}^+$ is a Young function if Φ is increasing, convex, $\Phi(0)=0$ and $\lim_{t\to\infty}\frac{\Phi(t)}{t}=\infty$.

Denote by $L^\Phi(A,d\mu)$ the Orlicz space corresponding to a Young function Φ on a set A with a measure μ. If $\mu=\mathscr{L}_n$ we simply write $L^\Phi(A)$. From some technical reasons we prefer the space $L^\Phi(A,d\mu)$ to be equipped with the norm

$$||f||_{L^\Phi(A,d\mu)}=\inf\Big\{\lambda>0:\int_A\Phi\Big(\frac{|f(x)|}{\lambda}\Big)\,d\mu(x)\leqslant\Phi(1)\Big\}\,. \tag{14}$$

This is different from the definition of the Luxemburg norm where we have the inequality $\int_A\Phi\big(\frac{|f(x)|}{\lambda}\big)\,d\mu(x)\leqslant 1$. We use (14) to have the Hölder's inequality (15) with a sharp constant.

Given a differentiable Young function Φ we can define the generalized inverse function to $\phi(u) = \Phi'(u)$ by

$$\psi(s) = \inf\{u : \phi(u) > s\} \qquad \text{for} \quad s > 0$$

and further we define the associated Young function Ψ by

$$\Psi(t) = \int_0^t \psi(s)ds \qquad \text{for} \quad t \geqslant 0 .$$

The dual space to $L^{\Phi}(A, d\mu)$ can be identified as the Orlicz space $L^{\Psi}(A, d\mu)$.

If we have $\Phi(1) + \Psi(1) = 1$ then the following generalization of Hölder's inequality is valid (see [21] page 58 for the proof)

$$\int_A |f(y)g(y)| \, d\mu(y) \leqslant ||f||_{L^{\Phi}(A,d\mu)} ||g||_{L^{\Psi}(A,d\mu)} . \tag{15}$$

We use this inequality for a measurable subset $A \subset \mathbb{R}$ and the measure $d\mu(y) = \omega_{n-1} y^{n-1} dy$. For an introduction to Orlicz spaces see e.g. [21].

Δ_2-condition

We say that the Young function Φ satisfies the Δ_2-condition, if there are $t_\Delta \geqslant 0$ and $C_\Delta > 1$ such that

$$\Phi(2t) \leqslant C_\Delta \Phi(t) \qquad \text{whenever } t \geqslant t_\Delta.$$

It is easy to see that if Φ satisfies the Δ_2-condition for one fixed $t_\Delta > 0$ then it satisfies this condition with arbitrary $\tilde{t}_\Delta > 0$ with a different constant $\tilde{C}_\Delta > 1$. From the Δ_2-condition one easily proves that for any $\eta > 0$ we can find $\varepsilon > 0$ so that

$$\Phi((1+\varepsilon)t) \leqslant (1+\eta)\Phi(t) , \qquad \text{for every } t \geqslant t_\Delta . \tag{16}$$

It is not difficult to check the Δ_2-condition for our Young functions satisfying (3). Therefore one easily proves

$$||f||_{L^{\Phi}(A,d\mu)} = 1 \quad \Longleftrightarrow \quad \int_A \Phi(|f|) \, d\mu(x) = \Phi(1) , \tag{17}$$

$$||f_j||_{L^{\Phi}(A,d\mu)} \stackrel{j\to\infty}{\to} 0 \quad \Longleftrightarrow \quad \int_A \Phi(|f_j|) \, d\mu(x) \stackrel{j\to\infty}{\to} 0 , \tag{18}$$

for every $\xi \in (0,1)$ there is $\eta \in (0,1)$ such that

$$||f||_{L^{\Phi}(A,d\mu)} \leqslant 1 - \eta \quad \Longrightarrow \quad \int_A \Phi(|f|) \, d\mu(x) \leqslant (1-\xi)\Phi(1) , \tag{19}$$

and for every $\eta \in (0,1)$ there is $\xi \in (0,1)$ such that

$$\int_A \Phi(|f|) \, d\mu(x) \leqslant (1-\xi)\Phi(1) \quad \Longrightarrow \quad ||f||_{L^{\Phi}(A,d\mu)} \leqslant 1 - \eta . \tag{20}$$

Orlicz-Sobolev spaces

Let A be a nonempty open set in $\mathbb{R}^n$ and let Φ be a Young function satisfying condition (3). In this subsection we consider Orlicz spaces only with the Lebesgue measure. We define the Orlicz-Sobolev space $WL^{\Phi}(A)$ as the set

$$WL^{\Phi}(A) := \{u\colon u, |\nabla u| \in L^{\Phi}(A)\}$$

equipped with the norm

$$\|u\|_{WL^{\Phi}(A)} := \|u\|_{L^{\Phi}(A)} + \|\nabla u\|_{L^{\Phi}(A)} ,$$

where ∇u is the gradient of u and we use its Euclidean norm in $\mathbb{R}^n$.

We put $W_0L^{\Phi}(A)$ for the closure of $C_0^{\infty}(A)$ in $WL^{\Phi}(A)$. For this space we prefer to use throughout the paper the equivalent norm

$$\|u\|_{W_0L^{\Phi}(A)} := \|\nabla u\|_{L^{\Phi}(A)} .$$

The space $W_0L^{\Phi}(A)$ is a reflexive Banach space and it is compactly embedded into $L^{\Phi}(A)$.

We write that $f_j \rightharpoonup f$ in $W_0L^{\Phi}(A)$, if

$$\int_A \frac{\partial f_j}{\partial x_i} g\, dx \to \int_A \frac{\partial f}{\partial x_i} g\, dx \quad \text{for every } g \in L^{\Psi}(A) \text{ and } i \in \{1,\dots,n\} .$$

Non-increasing rearrangement

The non-increasing rearrangement f^* of a measurable function f on Ω is defined by

$$f^*(t) = \inf\big\{s > 0 : \mathscr{L}_n(\{x \in \Omega : |f(x)| > s\}) \leqslant t\big\},\ t > 0 .$$

We also define the non-increasing radially symmetric rearrangement $f^{\#}$ by

$$f^{\#}(x) = f^*\Big(\frac{\omega_{n-1}}{n}|x|^n\Big) \quad \text{for } x \in B(R),\ \mathscr{L}_n(B(R)) = \mathscr{L}_n(\Omega) .$$

For an introduction to these rearrangements see e.g. [23].

We also use the Polya-Szegö principle (see e.g. Talenti [23] for the proof).

THEOREM 2.2. *Let Ω be an open bounded set and let $R > 0$ satisfy $\mathscr{L}_n(B(R)) = \mathscr{L}_n(\Omega)$. Let Φ be a Young function. Suppose that the function $f : \Omega \to \mathbb{R}$ is Lipschitz continuous and supported in Ω. Then f^* is locally absolutely continuous and*

$$\int_\Omega \Phi(|\nabla f(x)|)\, dx \geqslant \int_{B(R)} \Phi(|\nabla f^{\#}(x)|)\, dx . \tag{21}$$

Finally, let us recall the Hardy-Littlewood inequality, i.e. for f, g measurable on Ω we have

$$\int_\Omega |f(x)g(x)|\, dx \leqslant \int_0^\infty f^*(t)g^*(t)\, dt . \tag{22}$$

Tools from Measure Theory

LEMMA 2.3. *Let $k \in \mathbb{N}$, let $\{u_j\}_{j=1}^{\infty}$ be a sequence of measurable functions and let $u_j \to u$ a.e. in Ω. Suppose that there are $\tilde{K}, \delta, \gamma, C_1 > 0$ such that*

$$\|\exp_{[k]}(\tilde{K}(1+\delta)|u_j|^{\gamma})\|_{L^1(\Omega)} < C_1 \qquad \text{for all } j \in \mathbb{N}\,. \tag{23}$$

Let F be an even continuous function such that

$$\sup_{t\in(t_0,\infty)} \frac{|F(t)|}{\exp_{[k]}(\tilde{K}|t|^{\gamma})} < \infty \qquad \text{for some } t_0 > 0\,.$$

Then

$$F(u_j) \stackrel{j\to\infty}{\to} F(u) \qquad \text{in the } L^1(\Omega)\text{-norm}\,.$$

In particular

$$\exp_{[k]}(\tilde{K}|u_j|^{\gamma}) \stackrel{j\to\infty}{\to} \exp_{[k]}(\tilde{K}|u|^{\gamma}) \qquad \text{in the } L^1(\Omega)\text{-norm}\,.$$

Proof. Fist, we observe that if $s_1, s_2 \geqslant 2$, then $s_1 + s_2 \leqslant s_1 s_2$. Hence we easily obtain by induction

$$s_1, s_2 \geqslant 2, k \in \mathbb{N} \quad \Longrightarrow \quad \exp_{[k]}(s_1+s_2) \geqslant \exp_{[k]}(s_1)\exp_{[k]}(s_2)\,. \tag{24}$$

Further, as $\exp_{[k]}(\tilde{K}|t|^{\gamma}) \geqslant 1$ on $\mathbb{R}$, from the assumptions on F we obtain $L > 0$ such that

$$|F(t)| \leqslant L\exp_{[k]}(\tilde{K}|t|^{\gamma}) \qquad \text{for every } t \in \mathbb{R}\,. \tag{25}$$

By Fatou's lemma we have $\exp_{[k]}(\tilde{K}|u|^{\gamma}) \in L^1(\Omega)$, and thus for every $\varepsilon > 0$ there is $\xi > 0$ so that

$$\int_A \exp_{[k]}(\tilde{K}|u|^{\gamma}) < \frac{\varepsilon}{L} \qquad \text{provided} \quad \mathcal{L}_n(A) < \xi\,. \tag{26}$$

Next, since obviously $u \in L^1(\Omega)$, we find $M_1 > 0$ such that

$$\mathcal{L}_n(\{x \in \Omega : |u(x)| > M_1\}) < \xi\,. \tag{27}$$

Fix $M \geqslant M_1$ large enough so that $\frac{C_1}{\exp_{[k]}(\tilde{K}\delta M^{\gamma})} < \frac{\varepsilon}{L}$, $\tilde{K}M^{\gamma} \geqslant 2$ and $\tilde{K}\delta M^{\gamma} \geqslant 2$. We have by (25), (26) and (27)

$$\int_{\{|u|\geqslant M\}} |F(u)| \leqslant L\int_{\{|u|\geqslant M\}} \exp_{[k]}(\tilde{K}|u|^{\gamma}) < L\frac{\varepsilon}{L} = \varepsilon$$

and similarly we use (23), (24) and (25) to obtain

$$\begin{aligned}\int_{\{|u_j|\geqslant M\}} |F(u_j)| &\leqslant L\int_{\{|u_j|\geqslant M\}} \exp_{[k]}(\tilde{K}|u_j|^{\gamma})\\ &\leqslant L\int_{\{|u_j|\geqslant M\}} \frac{\exp_{[k]}(\tilde{K}(1+\delta)|u_j|^{\gamma})}{\exp_{[k]}(\tilde{K}\delta|u_j|^{\gamma})}\\ &\leqslant L\frac{C_1}{\exp_{[k]}(\tilde{K}\delta M^{\gamma})} < L\frac{\varepsilon}{L} = \varepsilon\,.\end{aligned}$$

Finally, the assumption $u_j \to u$ a.e. in Ω and the continuity of F imply

$$g_j := F(u_j)\chi_{\{|u_j|<M\}} - F(u)\chi_{\{|u|<M\}} \overset{j\to\infty}{\to} 0 \qquad \text{a.e. in } \Omega \ .$$

Moreover

$$\begin{aligned} |g_j(x)| &\leqslant L\exp_{[k]}(\tilde{K}|u_j|^\gamma)\chi_{\{|u_j|<M\}} + L\exp_{[k]}(\tilde{K}|u|^\gamma)\chi_{\{|u|<M\}} \\ &\leqslant 2L\exp_{[k]}(\tilde{K}M^\gamma) \quad \in L^1(\Omega) \ . \end{aligned}$$

Hence the Lebesgue dominated convergence theorem gives for $j \in \mathbb{N}$ large enough

$$\int_\Omega |F(u_j) - F(u)| \leqslant \int_{\{|u_j|\geqslant M\}} |F(u_j)| + \int_{\{|u|\geqslant M\}} |F(u)| + \int_\Omega |g_j| < 3\varepsilon$$

and the result follows. $\square$

3. Estimates concerning the associated function

In this section we follow the ideas of [16, Proof of Theorem 4.2].

Suppose that the function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies (8). In a standard way we can prove that there is a Young function $\Phi_1 : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\begin{gathered} \Phi_1' \text{ is continuous and increasing on } (0,\infty) \ , \\ \Phi_1(t) = \frac{1}{n}t^n \text{ for } t \in [0,1] \ , \\ \text{there is } G > T \text{ such that for every } t \geqslant G \text{ we have} \\ \Phi_1(t) = \frac{1}{n}t^n\Big(\prod_{j=1}^{k-1}\log_{[j]}^{n-1}(t)\Big)\log_{[k]}^{\alpha}(t)\Big(1-\log_{[\ell]}^{-1}(t)\Big) \leqslant \frac{1}{n}\Phi(t) \ . \end{gathered} \tag{28}$$

Denote by Ψ the Young function associated to the function Φ_1. Clearly $\Psi(t) = \frac{n-1}{n}t^{\frac{n}{n-1}}$ for $t \in [0,1]$. Hence $\Phi_1(1) + \Psi(1) = 1$. Therefore (Φ_1, Ψ) is a normalized complementary Young pair and we can use inequality (15).

Let us first estimate the growth of Ψ.

LEMMA 3.1. *There is $E > 0$ such that for every $t \in \mathbb{R}$ we have*

$$\Psi(t) < \hat{\Psi}(t) := Et^{\frac{n}{n-1}}\left(1 + |\log t|^E\right) \ . \tag{29}$$

Moreover there is $\tilde{T} > G$ such that for every $t \in [\tilde{T},\infty)$ we have in the case $k \geqslant 2$

$$\Psi(t) \leqslant \tilde{\Psi}(t) := \frac{(n-1)^2}{n}t^{\frac{n}{n-1}}\Big(\prod_{j=1}^{k-1}\log_{[j]}^{-1}(t)\Big)\log_{[k]}^{-\frac{\alpha}{n-1}}(t)\Big(1+\log_{[\ell]}^{-\frac{1}{2}}(t)\Big) \tag{30}$$

and in the case $k = 1$

$$\Psi(t) \leqslant \tilde{\Psi}(t) := \frac{(n-1)^{1+\frac{\alpha}{n-1}}}{n}t^{\frac{n}{n-1}}\log_{[1]}^{-\frac{\alpha}{n-1}}(t)\Big(1+\log_{[\ell]}^{-\frac{1}{2}}(t)\Big) \ . \tag{31}$$

Proof. Let us start with the case $k \geqslant 2$. Put

$$\tilde{\Psi}_1(t) = \frac{(n-1)^2}{n} t^{\frac{n}{n-1}} \Big(\prod_{j=1}^{k-1} \log_{[j]}^{-1}(t)\Big) \log_{[k]}^{-\frac{\alpha}{n-1}}(t) \Big(1 + \log_{[\ell]}^{-\frac{3}{4}}(t)\Big) .$$

Denote $\tilde{\psi}_1 = \tilde{\Psi}_1'$, $\phi = \Phi_1'$ and $\psi = \phi^{-1}$, hence $\Psi(t) = \int_0^t \psi$.

By (28) there is $B_1 > G$ such that for every $t > B_1$ we have

$$\begin{aligned} \phi(t) = t^{n-1} \Big(\prod_{j=1}^{k-1} \log_{[j]}^{n-1}(t)\Big) \log_{[k]}^{\alpha}(t) \Bigg[1 - \log_{[\ell]}^{-1}(t) + \sum_{j=1}^{k-1} \frac{n-1}{n} \frac{1 - \log_{[\ell]}^{-1}(t)}{\prod_{i=1}^{j} \log_{[i]}(t)} \\ + \frac{\alpha}{n} \frac{1 - \log_{[\ell]}^{-1}(t)}{\prod_{j=1}^{k} \log_{[j]}(t)} + \frac{1}{n} \frac{\log_{[\ell]}^{-1}(t)}{\prod_{j=1}^{\ell} \log_{[j]}(t)} \Bigg] \quad (32) \\ \geqslant t^{n-1} \Big(\prod_{j=1}^{k-1} \log_{[j]}^{n-1}(t)\Big) \log_{[k]}^{\alpha}(t) \Big(1 - 2\log_{[\ell]}^{-1}(t)\Big) = \tilde{\phi}(t) . \end{aligned}$$

Analogously there is $B_2 > B_1$ such that for every $t > B_2$

$$\begin{aligned} \tilde{\psi}_1(t) = (n-1) t^{\frac{1}{n-1}} \Big(\prod_{j=1}^{k-1} \log_{[j]}^{-1}(t)\Big) \log_{[k]}^{-\frac{\alpha}{n-1}}(t) \Bigg[1 + \log_{[\ell]}^{-\frac{3}{4}}(t) \\ - \sum_{j=1}^{k-1} \frac{n-1}{n} \frac{1 + \log_{[\ell]}^{-\frac{3}{4}}(t)}{\prod_{i=1}^{j} \log_{[i]}(t)} - \frac{\alpha}{n} \frac{1 + \log_{[\ell]}^{-\frac{3}{4}}(t)}{\prod_{j=1}^{k} \log_{[j]}(t)} - \frac{3(n-1)}{4n} \frac{\log_{[\ell]}^{-\frac{3}{4}}(t)}{\prod_{j=1}^{\ell} \log_{[j]}(t)} \Bigg] \\ \geqslant (n-1) t^{\frac{1}{n-1}} \Big(\prod_{j=1}^{k-1} \log_{[j]}^{-1}(t)\Big) \log_{[k]}^{-\frac{\alpha}{n-1}}(t) \Big(1 + \frac{1}{2} \log_{[\ell]}^{-\frac{3}{4}}(t)\Big) = \tilde{\psi}(t) . \end{aligned} \quad (33)$$

Using (33) and $\frac{\log_{[2]}(t)}{\log(t)} << \frac{1}{\log_{[\ell]}(t)}$ we find $B_3 > B_2$ such that for $t > B_3$ we have

$$\begin{aligned} \log_{[1]}\big(\tilde{\psi}(t)\big) \geqslant \log_{[1]}\Big(t^{\frac{1}{n-1}} \frac{1}{\log_{[1]}^2(t)}\Big) = \Big(\frac{1}{n-1} \log_{[1]}(t) - 2\log_{[2]}(t)\Big) \\ = \frac{1}{(n-1)} \log_{[1]}(t) \Big(1 - 2\frac{(n-1)\log_{[2]}(t)}{\log_{[1]}(t)}\Big) \quad (34) \\ \geqslant \frac{1}{(n-1)} \log_{[1]}(t) \Big(1 - \log_{[\ell]}^{-1}(t)\Big) . \end{aligned}$$

By Lemma 2.1 there is $B_4 > B_3$ such that for $t > B_4$ we obtain

$$\log_{[j]}^{n-1}\big(\tilde{\psi}(t)\big) \geqslant \log_{[j]}^{n-1}\big(t^{\frac{1}{n}}\big) \geqslant \log_{[j]}^{n-1}(t)\Big(1 - \log_{[\ell]}^{-1}(t)\Big) \qquad \text{for} \quad j \in \{2, \ldots, k-1\} , \quad (35)$$

analogously

$$\log_{[k]}^{\alpha}\big(\tilde{\psi}(t)\big) \geqslant \log_{[k]}^{\alpha}(t)\Big(1 - \log_{[\ell]}^{-1}(t)\Big) \quad (36)$$

and trivially

$$1-2\log_{[\ell]}^{-1}\big(\tilde{\psi}(t)\big)\geqslant 1-2\log_{[\ell]}^{-1}(t)\;. \tag{37}$$

From (32), (33), (34), (35), (36) and (37) we can see that there is $B_5>B_4$ such that for all $t>B_5$ we have

$$\begin{aligned}\tilde{\phi}(\tilde{\psi}(t)) &= \tilde{\psi}^{n-1}(t)\Big(\prod_{j=1}^{k-1}\log_{[j]}^{n-1}(\tilde{\psi}(t))\Big)\log_{[k]}^{\alpha}(\tilde{\psi}(t))\Big(1-2\log_{[\ell]}^{-1}(\tilde{\psi}(t))\Big)\\ &\geqslant\Big[(n-1)t^{\frac{1}{n-1}}\Big(\prod_{j=1}^{k-1}\log_{[j]}^{-1}(t)\Big)\log_{[k]}^{-\frac{\alpha}{n-1}}(t)\Big(1+\frac12\log_{[\ell]}^{-\frac34}(t)\Big)\Big]^{n-1}\frac{\log_{[1]}^{n-1}(t)}{(n-1)^{n-1}}\\ &\qquad\Big(\prod_{j=2}^{k}\log_{[j]}^{n-1}(t)\Big)\log_{[k]}^{\alpha}(t)\Big(1-\log_{[\ell]}^{-1}(t)\Big)^{n+k-2}\Big(1-2\log_{[\ell]}^{-1}(t)\Big)\\ &\geqslant t\Big(1+\frac12\log_{[\ell]}^{-\frac34}(t)\Big)\Big(1-\log_{[k]}^{-1}(t)\Big)^{n+k-2}\Big(1-2\log_{[\ell]}^{-1}(t)\Big)\\ &> t\;.\end{aligned}$$

It follows that $\phi(\tilde{\psi}_1(t))>t$ for $t>B_5$ and thus

$$t>B_5\quad\Rightarrow\quad\tilde{\psi}_1(t)>\phi^{-1}(t)=\psi(t)\;. \tag{38}$$

Hence for $t>B_5$ we have

$$\Psi(t)<\tilde{\Psi}_1(t)+C\;.$$

Together with $\log_{[\ell]}^{-\frac34}(t)<<\log_{[\ell]}^{-\frac12}(t)$ for t large this implies that there is $\tilde{T}>B_5$ such that for all $t\geqslant\tilde{T}$ we have

$$\Psi(t)<\tilde{\Psi}(t)\;.$$

Since Ψ is increasing and $\Psi(t)=\frac{t^{n'}}{n'}$ for $t\in[0,1]$, (30) obviously implies (29).

Now suppose that $k=1$. This time we consider

$$\tilde{\Psi}_1(t)=\frac{(n-1)^{1+\frac{\alpha}{n-1}}}{n}t^{\frac{n}{n-1}}\log_{[1]}^{-\frac{\alpha}{n-1}}(t)\Big(1+\log_{[\ell]}^{-\frac34}(t)\Big)\;.$$

Estimate (32) for $t>B_1$ and $k=1$ reads

$$\phi(t)\geqslant t^{n-1}\log_{[1]}^{\alpha}(t)\Big(1-2\log_{[\ell]}^{-1}(t)\Big)=\tilde{\phi}(t)\;. \tag{39}$$

There is $B_2>B_1$ such that for every $t>B_2$

$$\begin{aligned}\tilde{\psi}_1(t) &= (n-1)^{\frac{\alpha}{n-1}}t^{\frac{1}{n-1}}\log_{[1]}^{-\frac{\alpha}{n-1}}(t)\Bigg[1+\log_{[\ell]}^{-\frac34}(t)-\frac{\alpha}{n}\frac{1+\log_{[\ell]}^{-\frac34}(t)}{\log_{[1]}(t)}\\ &\qquad -\frac{3(n-1)}{4n}\frac{\log_{[\ell]}^{-\frac34}(t)}{\prod_{j=1}^{\ell}\log_{[j]}(t)}\Bigg] \qquad (40)\\ &\geqslant (n-1)^{\frac{\alpha}{n-1}}t^{\frac{1}{n-1}}\log_{[1]}^{-\frac{\alpha}{n-1}}(t)\Big(1+\frac12\log_{[\ell]}^{-\frac34}(t)\Big)=\tilde{\psi}(t)\;.\end{aligned}$$

Estimates (34) and (37) are still valid in this case provided t is large enough. Next we need to prove that there is $B_4 > B_3$ such that for $t > B_4$ we have

$$\log_{[1]}^{\alpha}\big(\tilde{\psi}(t)\big) \geqslant \frac{1}{(n-1)^{\alpha}} \log_{[1]}^{\alpha}\Big(1 + C\log_{[\ell]}^{-1}(t)\Big) . \tag{41}$$

If $\alpha \geqslant 0$ then (41) easily follows from (34). Otherwise for t large we use (40) and estimate $\tilde{\psi}(t) \leqslant Ct^{\frac{1}{n-1}} \log_{[1]}^{-\frac{\alpha}{n-1}}(t)$ to obtain

$$\log_{[1]}\big(\tilde{\psi}(t)\big) \leqslant C + \frac{1}{n-1}\log_{[1]}(t) + \frac{|\alpha|}{n-1}\log_{[2]}(t) \leqslant \frac{1}{n-1}\log_{[1]}(t)\Big(1 + \frac{C\log_{[2]}(t)}{\log_{[1]}(t)}\Big) .$$

This implies (recall $\alpha < 0$) for t large enough

$$\begin{aligned}\log_{[1]}^{\alpha}\big(\tilde{\psi}(t)\big) &\geqslant \frac{1}{(n-1)^{\alpha}}\log_{[1]}^{\alpha}(t)\Big(1 + \frac{C\log_{[2]}(t)}{\log_{[1]}(1)}\Big)^{\alpha} \\ &\geqslant \frac{1}{(n-1)^{\alpha}}\log_{[1]}^{\alpha}(t)\Big(1 - C\log_{[\ell]}^{-1}(t)\Big) .\end{aligned}$$

Thus (41) is proved.

From (37), (39), (40) and (41) we can see that there is $B_5 > B_4$ such that for all $t > B_5$ we have

$$\begin{aligned}\tilde{\phi}(\tilde{\psi}(t)) &= \tilde{\psi}^{n-1}(t)\log_{[1]}^{\alpha}(\tilde{\psi}(t))\Big(1 - 2\log_{[\ell]}^{-1}(\tilde{\psi}(t))\Big) \\ &\geqslant \Big[(n-1)^{\frac{\alpha}{n-1}} t^{\frac{1}{n-1}} \log_{[1]}^{-\frac{\alpha}{n-1}}(t)\Big(1 + \frac{1}{2}\log_{[\ell]}^{-\frac{3}{4}}(t)\Big)\Big]^{n-1} \frac{1}{(n-1)^{\alpha}}\log_{[1]}^{\alpha}(t) \\ &\qquad \Big(1 - C\log_{[\ell]}^{-1}(t)\Big)\Big(1 - 2\log_{[\ell]}^{-1}(t)\Big) \\ &\geqslant t\Big(1 + \frac{1}{2}\log_{[\ell]}^{-\frac{3}{4}}(t)\Big)\Big(1 - C\log_{[\ell]}^{-1}(t)\Big)\Big(1 - 2\log_{[\ell]}^{-1}(t)\Big) \\ &> t .\end{aligned}$$

Now, we conclude the proof the same way as in the case $k \geqslant 2$. □

LEMMA 3.2. *There is $t_0 \in (0,1)$ such that for $0 < t \leqslant t_0$ we have*

$$\Big\|\frac{1}{y^{n-1}}\Big\|_{L^{\Psi}((t,R),\omega_{n-1}y^{n-1}dy)} \leqslant D\log_{[k]}^{\frac{1}{\gamma}}\Big(\frac{1}{t}\Big)\Big(1 + \log_{[\ell]}^{-\frac{1}{4}}\Big(\frac{1}{t}\Big)\Big) , \tag{42}$$

$$\text{where } D = \Big(\frac{\omega_{n-1}}{B}\Big)^{\frac{n-1}{n}} . \tag{43}$$

Proof. We want to prove that for

$$\lambda = D\log_{[k]}^{\frac{1}{\gamma}}\Big(\frac{1}{t}\Big)\Big(1 + \log_{[\ell]}^{-\frac{1}{4}}\Big(\frac{1}{t}\Big)\Big)$$

we have

$$\int_t^R \Psi\Big(\frac{1}{\lambda y^{n-1}}\Big)\omega_{n-1}y^{n-1}\,dy \leqslant \Psi(1) = \frac{n-1}{n}$$

for $t>0$ small enough.

For $t\in\big(0,\exp_{[k]}^{-1}(1)\big)$ set $M=M(t)=\exp\Big(-\log_{[k]}^{\frac{\min\{1,\frac{1}{\gamma}\}}{(E+2)(n-1)}}(\frac{1}{t})\Big)$. We can clearly find $t_1\in\big(0,\exp_{[\ell]}^{-1}(2)\big)$ such that for $0<t<t_1$ we have

$$t<M<R \qquad \text{and} \qquad \frac{1}{\lambda M^{n-1}}>\tilde{T}\ . \tag{44}$$

Hence Lemma 3.1 gives us that

$$\begin{aligned}&\int_t^R \Psi\Big(\frac{1}{\lambda y^{n-1}}\Big)y^{n-1}\,dy\\ &\qquad\leqslant \int_t^M \tilde{\Psi}\Big(\frac{1}{\lambda y^{n-1}}\Big)y^{n-1}\,dy+\int_M^R \hat{\Psi}\Big(\frac{1}{\lambda y^{n-1}}\Big)y^{n-1}\,dy = I_1+I_2\ .\end{aligned} \tag{45}$$

By (29) we have

$$\begin{aligned}I_2 &= E\int_M^R \frac{1}{\lambda^{\frac{n}{n-1}}}\Big(1+\Big|\log\Big(\frac{1}{\lambda y^{n-1}}\Big)\Big|^E\Big)\frac{dy}{y}\\ &\leqslant \frac{C}{\lambda^{\frac{n}{n-1}}}\int_M^R\Big(1+|\log(\lambda)|^E+|\log(y)|^E\Big)\frac{dy}{y} = J_1+J_2\ ,\end{aligned}$$

where (we observe that $(1-\log_{[\ell]}^{-\frac{1}{4}}(\frac{1}{t}))^{-1}<C$ on $(0,t_1)\subset(0,\exp_{[\ell]}^{-1}(2))$)

$$\begin{aligned}J_1 &= \frac{C}{\lambda^{\frac{n}{n-1}}}\int_M^R\Big(1+|\log(\lambda)|^E\Big)\frac{dy}{y}\\ &\leqslant \frac{C}{\log_{[k]}^{\frac{n}{(n-1)\gamma}}(\frac{1}{t})}\Big(1+\log^E\Big(\log_{[k]}\Big(\frac{1}{t}\Big)\Big)\Big)\Big(1+\log\Big(\frac{1}{M}\Big)\Big)\end{aligned}$$

and

$$J_2 = \frac{C}{\lambda^{\frac{n}{n-1}}}\int_M^R|\log(y)|^E\frac{dy}{y}\leqslant \frac{C}{\log_{[k]}^{\frac{n}{(n-1)\gamma}}(\frac{1}{t})}\Big(1+\log^{E+1}\Big(\frac{1}{M}\Big)\Big)$$

Hence we obtain

$$I_2\leqslant \frac{C}{\log_{[k]}^{\frac{n}{(n-1)\gamma}}(\frac{1}{t})}\Big(1+\log^E\Big(\log_{[k]}\Big(\frac{1}{t}\Big)\Big)\Big)\Big(1+\log\Big(\frac{1}{M}\Big)+\log^{E+1}\Big(\frac{1}{M}\Big)\Big)\ .$$

Thus there is $t_2\in(0,t_1)$ such that if $0<t<t_2$ we have

$$I_2\leqslant \frac{C}{\log_{[k]}^{\frac{n}{(n-1)\gamma}}(\frac{1}{t})}\log^{E+2}\Big(\frac{1}{M}\Big)\leqslant \frac{C}{\log_{[k]}^{\frac{n}{(n-1)\gamma}-\frac{\min\{1,\frac{1}{\gamma}\}}{(n-1)}}(\frac{1}{t})}\leqslant \frac{1}{\log_{[\ell]}(\frac{1}{t})}\ . \tag{46}$$

Next we need to estimate I_1. Our goal is to prove for $t>0$ small enough

$$I_1 \leqslant \frac{n-1}{\omega_{n-1} n}\Big(1-\frac{1}{2}\log_{[\ell]}^{-\frac{1}{4}}\Big(\frac{1}{t}\Big)\Big)\,. \tag{47}$$

First, let us prove (47) for $k \geqslant 2$. Since $\log_{[1]}^{\frac{1}{2}}(\frac{1}{M}) >> \log_{[1]}(\lambda) > 1$ for small $t>0$, we can choose $t_3 \in (0,t_2)$ such that if $0<t<t_3$ and $y \in [t,M]$ we have

$$\begin{aligned}
\log_{[1]}^{-1}\Big(\frac{1}{\lambda y^{n-1}}\Big) &= \log_{[1]}^{-1}\Big(\frac{1}{y^{n-1}}\Big)\frac{1}{1+\frac{\log_{[1]}(\frac{1}{\lambda})}{\log_{[1]}(\frac{1}{y^{n-1}})}}\\
&\leqslant \frac{1}{n-1}\log_{[1]}^{-1}\Big(\frac{1}{y}\Big)\Big(1+\frac{C\log_{[1]}(\lambda)}{\log_{[1]}(\frac{1}{y})}\Big)\\
&\leqslant \frac{1}{n-1}\log_{[1]}^{-1}\Big(\frac{1}{y}\Big)\Big(1+\frac{C\log_{[1]}(\lambda)}{\log_{[1]}^{\frac{1}{2}}(\frac{1}{M})}\frac{1}{\log_{[1]}^{\frac{1}{2}}(\frac{1}{y})}\Big)\\
&\leqslant \frac{1}{n-1}\log_{[1]}^{-1}\Big(\frac{1}{y}\Big)\Big(1+\log_{[\ell]}^{-1}\Big(\frac{1}{y}\Big)\Big)\,.
\end{aligned} \tag{48}$$

Further, estimate (13) from Lemma 2.1 gives us $t_4 \in (0,t_3)$ such that if $0<t<t_4$ and $y \in [t,M]$, then we have for $j \in \{2,\dots,k-1\}$

$$\log_{[j]}^{-1}\Big(\frac{1}{\lambda y^{n-1}}\Big) \leqslant \log_{[j]}^{-1}\Big(\frac{1}{y}\Big)\Big(1+\frac{C}{\log_{[k]}(\frac{1}{y})}\Big) \leqslant \log_{[j]}^{-1}\Big(\frac{1}{y}\Big)\Big(1+\log_{[\ell]}^{-1}\Big(\frac{1}{y}\Big)\Big)\,, \tag{49}$$

$$\log_{[k]}^{-\frac{\alpha}{n-1}}\Big(\frac{1}{\lambda y^{n-1}}\Big) \leqslant \log_{[k]}^{-\frac{\alpha}{n-1}}\Big(\frac{1}{y}\Big)\Big(1+\frac{C}{\log_{[k]}(\frac{1}{y})}\Big) \leqslant \log_{[k]}^{-\frac{\alpha}{n-1}}\Big(\frac{1}{y}\Big)\Big(1+\log_{[\ell]}^{-1}\Big(\frac{1}{y}\Big)\Big)\,, \tag{50}$$

$$1+\log_{[\ell]}^{-\frac{1}{2}}\Big(\frac{1}{\lambda y^{n-1}}\Big) \leqslant 1+2\log_{[\ell]}^{-\frac{1}{2}}\Big(\frac{1}{y}\Big) \tag{51}$$

and

$$\Big(1+\log_{[\ell]}^{-\frac{1}{4}}\Big(\frac{1}{t}\Big)\Big)^{-\frac{n}{n-1}} \leqslant 1-\log_{[\ell]}^{-\frac{1}{4}}\Big(\frac{1}{t}\Big)\,. \tag{52}$$

Hence (30), (45), (48), (49), (50), (51) and (52) give us that

$$\begin{aligned}
I_1 &= \frac{(n-1)^2}{n}\int_t^M \Big(\frac{1}{\lambda y^{n-1}}\Big)^{\frac{n}{n-1}}\Big(\prod_{j=1}^{k-1}\log_{[j]}^{-1}\Big(\frac{1}{\lambda y^{n-1}}\Big)\Big)\log_{[k]}^{-\frac{\alpha}{n-1}}\Big(\frac{1}{\lambda y^{n-1}}\Big)\\
&\qquad \Big(1+\log_{[\ell]}^{-\frac{1}{2}}\Big(\frac{1}{\lambda y^{n-1}}\Big)\Big)y^{n-1}\,dy\\
&\leqslant \frac{\frac{n-1}{n}(1-\log_{[\ell]}^{-\frac{1}{4}}(\frac{1}{t}))}{D^{\frac{n}{n-1}}\log_{[k]}^{\frac{n}{(n-1)\gamma}}(\frac{1}{t})}\int_t^M \Big(\prod_{j=1}^{k-1}\log_{[j]}^{-1}\Big(\frac{1}{y}\Big)\Big)\log_{[k]}^{-\frac{\alpha}{n-1}}\Big(\frac{1}{y}\Big)\\
&\qquad \Big(1+\log_{[\ell]}^{-1}\Big(\frac{1}{y}\Big)\Big)^k\Big(1+2\log_{[\ell]}^{-\frac{1}{2}}\Big(\frac{1}{y}\Big)\Big)\frac{dy}{y}\,.
\end{aligned}$$

Further there is $t_5 \in (0,t_4)$ such that for $0<t<t_5$ and $y \in [t,M]$ we have

$$\Big(1-\log_{[\ell]}^{-\frac14}\Big(\frac1t\Big)\Big)\Big(1+\log_{[\ell]}^{-1}\Big(\frac1y\Big)\Big)^k\Big(1+2\log_{[\ell]}^{-\frac12}\Big(\frac1y\Big)\Big) \leqslant 1-\frac12\log_{[\ell]}^{-\frac14}\Big(\frac1t\Big).$$

Therefore (43) and $-\frac{\alpha}{n-1}=B-1\neq -1$ imply

$$\begin{aligned} I_1 &\leqslant \frac{n-1}{n}\frac{B}{\omega_{n-1}}\frac{1-\frac12\log_{[\ell]}^{-\frac14}(\frac1t)}{\log_{[k]}^{\frac{n}{(n-1)\gamma}}(\frac1t)}\int_t^M\Big(\prod_{j=1}^{k-1}\log_{[j]}^{-1}\Big(\frac1y\Big)\Big)\log_{[k]}^{-\frac{\alpha}{n-1}}\Big(\frac1y\Big)\frac{dy}{y} \\ &= \frac{n-1}{n}\frac{B}{\omega_{n-1}}\frac{1-\frac12\log_{[\ell]}^{-\frac14}(\frac1t)}{\log_{[k]}^{\frac{n}{(n-1)\gamma}}(\frac1t)}\left[-\frac{\log_{[k]}^{1-\frac{\alpha}{n-1}}(\frac1y)}{1-\frac{\alpha}{n-1}}\right]_t^M. \end{aligned}$$

Since $1-\frac{\alpha}{n-1}=B=\frac{n}{(n-1)\gamma}$ and $\log_{[k]}(\frac1M)>0$, we have (47) in case $k\geqslant 2$.

Now, let us prove (47) for $k=1$. We need the estimate

$$\log_{[1]}^{-\frac{\alpha}{n-1}}\Big(\frac{1}{\lambda y^{n-1}}\Big) \leqslant \frac{1}{(n-1)^{\frac{\alpha}{n-1}}}\log_{[1]}^{-\frac{\alpha}{n-1}}\Big(\frac1y\Big)\Big(1+\log_{[\ell]}^{-1}\Big(\frac1y\Big)\Big). \tag{53}$$

For $\alpha\geqslant 0$ estimate (53) easily follows from (48). Let $\alpha<0$. This time $\log_{[1]}(\lambda)>0$ implies

$$\log_{[1]}\Big(\frac{1}{\lambda y^{n-1}}\Big)=\log_{[1]}\Big(\frac{1}{y^{n-1}}\Big)\Big(1+\frac{\log_{[1]}(\frac1\lambda)}{\log_{[1]}(\frac{1}{y^{n-1}})}\Big)\leqslant (n-1)\log_{[1]}\Big(\frac1y\Big)$$

and (53) follows trivially.

Hence (31), (45), (51), (52) and (53) give us that

$$\begin{aligned} I_1 &= \frac{(n-1)^{1+\frac{\alpha}{n-1}}}{n}\int_t^M\Big(\frac{1}{\lambda y^{n-1}}\Big)^{\frac{n}{n-1}}\log_{[1]}^{-\frac{\alpha}{n-1}}\Big(\frac{1}{\lambda y^{n-1}}\Big) \\ &\qquad\qquad \Big(1+\log_{[\ell]}^{-\frac12}\Big(\frac{1}{\lambda y^{n-1}}\Big)\Big)y^{n-1}\,dy \\ &\leqslant \frac{\frac{(n-1)}{n}(1-\log_{[\ell]}^{-\frac14}(\frac1t))}{D^{\frac{n}{n-1}}\log_{[k]}^{\frac{n}{(n-1)\gamma}}(\frac1t)}\int_t^M\log_{[1]}^{-\frac{\alpha}{n-1}}\Big(\frac1y\Big)\Big(1+\log_{[\ell]}^{-1}\Big(\frac1y\Big)\Big) \\ &\qquad\qquad \Big(1+2\log_{[\ell]}^{-\frac12}\Big(\frac1y\Big)\Big)\frac{dy}{y}. \end{aligned}$$

Further there is $t_5 \in (0,t_4)$ such that for $0<t<t_5$ and $y \in [t,M]$ we have

$$\Big(1-\log_{[\ell]}^{-\frac14}\Big(\frac1t\Big)\Big)\Big(1+\log_{[\ell]}^{-1}\Big(\frac1y\Big)\Big)\Big(1+2\log_{[\ell]}^{-\frac12}\Big(\frac1y\Big)\Big) \leqslant 1-\frac12\log_{[\ell]}^{-\frac14}\Big(\frac1t\Big).$$

Therefore (43) and $-\frac{\alpha}{n-1} = B - 1 \neq -1$ imply

$$I_1 \leqslant \frac{n-1}{n}\frac{B}{\omega_{n-1}}\frac{1-\frac{1}{2}\log_{[\ell]}^{-\frac{1}{4}}(\frac{1}{t})}{\log_{[k]}^{\frac{n}{(n-1)\gamma}}(\frac{1}{t})}\int_t^M \log_{[1]}^{-\frac{\alpha}{n-1}}\Big(\frac{1}{y}\Big)\frac{dy}{y}$$
$$= \frac{n-1}{n}\frac{B}{\omega_{n-1}}\frac{1-\frac{1}{2}\log_{[\ell]}^{-\frac{1}{4}}(\frac{1}{t})}{\log_{[k]}^{\frac{n}{(n-1)\gamma}}(\frac{1}{t})}\left[-\frac{\log_{[1]}^{1-\frac{\alpha}{n-1}}(\frac{1}{y})}{1-\frac{\alpha}{n-1}}\right]_t^M .$$

Since $1-\frac{\alpha}{n-1} = B = \frac{n}{(n-1)\gamma}$ and $\log_{[k]}(\frac{1}{M}) > 0$, we have (47) also for $k=1$.

Finally, from (45), (46) and (47) we obtain that there is $t_0 \in (0,t_5)$ such that for $0 < t < t_0$ we have the desired inequality

$$\int_t^R \Psi\Big(\frac{1}{\lambda y^{n-1}}\Big)y^{n-1}\,dy \leqslant I_1 + I_2 \leqslant \log_{[\ell]}^{-1}\Big(\frac{1}{t}\Big) + \frac{n-1}{\omega_{n-1}n}\Big(1-\frac{1}{2}\log_{[\ell]}^{-\frac{1}{4}}\Big(\frac{1}{t}\Big)\Big)$$
$$\leqslant \frac{n-1}{\omega_{n-1}n} = \frac{1}{\omega_{n-1}}\Psi(1) . \qquad \square$$

4. Concentration-Compactness Alternative

Proof of Corollary 1.4. Since $K < K_{k,n,\alpha}$, we can find $\delta > 0$ such that $K(1+\delta) < K_{k,n,\alpha}$. Hence, by Theorem 1.2(i) we see that the assumptions of Lemma 2.3 are satisfied (with $\tilde{K} = K$ and $C_1 = S(K(1+\delta),\Phi) < \infty$) and thus Lemma 2.3 concludes the proof. $\square$

In the proof of Theorem 1.3 we distinguish three cases. These cases are studied separately in Propositions 4.2, 4.3 and 4.4 bellow.

Case 1

In this subsection we prove the Compactness in the case $u = 0$ and $\mu \neq \delta_{x_0}$.

LEMMA 4.1. *Let $n \geqslant 2$, $\alpha < n-1$, $k \in \mathbb{N}$ and let Φ be a Young function satisfying* (3)*. Let $\{u_j\}_{j=1}^{\infty} \subset W_0L^{\Phi}(\Omega)$ satisfy $\|\Phi(|\nabla u_j|)\|_{L^1(\Omega)} \leqslant 1$. Suppose that*

$$u_j \rightharpoonup 0 \quad \text{in } W_0L^{\Phi}(\Omega) \qquad \text{and} \qquad \Phi(|\nabla u_j|) \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathscr{M}(\bar{\Omega}) .$$

Let $F,N \subset \bar{\Omega}$ be compact sets such that $F \cap N = \emptyset$ and $\mu(N) > 0$. Then there is $\delta > 0$ such that

$$\|\exp_{[k]}(K_{k,n,\alpha}(1+\delta)|u_j|^{\gamma})\|_{L^1(F)} \qquad \text{is bounded} . \tag{54}$$

Similarly, if $\mu(\bar{\Omega}) < 1$, then there is $\delta > 0$ such that (54) *is satisfied with F replaced by Ω.*

Sketch of proof. In [2, proof of Lemma 3.1] it is shown that if $\mu(N) > 0$, Φ is a Young function satisfying the Δ_2-condition and $\{u_j\}_{j=1}^{\infty} \subset W_0L^{\Phi}(\Omega)$ satisfy $\|\Phi(|\nabla u_j|)\|_{L^1(\Omega)} \leqslant 1$, then there are $\tilde{\delta} > 0$, $j_1 \in \mathbb{N}$ and $\{v_j\}_{j=j_1+1}^{\infty} \subset W_0L^{\Phi}(\Omega)$ such that

$$\int_{\Omega} \Phi(|\nabla v_j|) \leqslant 1 \qquad \text{and} \qquad v_j = (1+2\tilde{\delta})u_j \text{ on } F \qquad \text{for all } j > j_1 . \tag{55}$$

Thus using Theorem 1.2(i) with $K = \frac{1+\tilde{\delta}}{(1+2\tilde{\delta})} K_{k,n,\alpha} < K_{k,n,\alpha}$ we obtain for $j > j_1$

$$\begin{aligned}\|\exp_{[k]}(K_{k,n,\alpha}(1+\tilde{\delta})^{\gamma}|u_j|^{\gamma})\|_{L^1(F)} &= \|\exp_{[k]}(K(1+2\tilde{\delta})^{\gamma}|u_j|^{\gamma})\|_{L^1(F)} \\ &\leqslant \|\exp_{[k]}(K|v_j|^{\gamma})\|_{L^1(\Omega)} \leqslant S(K,\Phi) < \infty .\end{aligned}$$

Moreover, by Proposition 1.1, for every fixed $j \in \{1,\dots,j_1\}$ there is C_j such that $\|\exp_{[k]}(K_{k,n,\alpha}(1+\tilde{\delta})|u_j|^{\gamma})\|_{L^1(F)} \leqslant C_j$. Hence we obtain (54) for $\delta = (1+\tilde{\delta})^{\gamma} - 1$ with the bound $\max(C_1,\dots,C_{j_1}, S(K,\Phi))$.

If $\mu(\bar{\Omega}) < 1$, then using $\Phi(|\nabla u_j|) \stackrel{*}{\rightharpoonup} \mu$ in $\mathscr{M}(\bar{\Omega})$ with the test-function $\psi \equiv 1$ and (20) we find $\tilde{\delta} > 0$ and $j_1 \in \mathbb{N}$ such that (55) is satisfied with $v_j = (1+2\tilde{\delta})u_j$ on Ω and we conclude the proof the same way as before. $\square$

PROPOSITION 4.2. *Let $n \geqslant 2$, $\alpha < n-1$, $k \in \mathbb{N}$ and let Φ be a Young function satisfying* (3). *Let $\{u_j\}_{k=1}^{\infty} \subset W_0L^{\Phi}(\Omega)$ satisfy $\|\Phi(|\nabla u_j|)\|_{L^1(\Omega)} \leqslant 1$. Further suppose that*

$$u_j \rightharpoonup 0 \quad \textit{in } W_0L^{\Phi}(\Omega)\,, \quad u_j \to 0 \quad \textit{a.e. in } \Omega \quad \textit{and} \quad \Phi(|\nabla u_j|) \stackrel{*}{\rightharpoonup} \mu \quad \textit{in } \mathscr{M}(\bar{\Omega})\,,$$

where μ is not a Dirac mass at one point. Then there is $\delta > 0$ such that

$$\exp_{[k]}(K_{k,n,\alpha}(1+\delta)|u_j|^{\gamma}) \qquad \textit{is bounded in } L^1(\Omega)$$

and

$$\exp_{[k]}(K_{k,n,\alpha}|u_j|^{\gamma}) \stackrel{j\to\infty}{\to} \exp_{[k]}(K_{k,n,\alpha}|u|^{\gamma}) \qquad \textit{in} \quad L^1(\Omega)\,.$$

Sketch of proof. If $\mu(\bar{\Omega}) < 1$, then the proof follows from Lemma 4.1.

If $\mu(\bar{\Omega}) = 1$, then the same way as in [2, proof of Proposition 3.2] we can find compact sets $F_1, F_2 \subset \bar{\Omega}$ such that $F_1 \cup F_2 = \bar{\Omega}$ and the complement of each of these sets contains a compact set with a positive μ-measure. Hence we conclude easily by Lemma 4.1.

The case $\mu(\bar{\Omega}) > 1$ is impossible as one can see using $\Phi(|\nabla u_j|) \stackrel{*}{\rightharpoonup} \mu$ in $\mathscr{M}(\bar{\Omega})$ with the test-function $\psi \equiv 1$. $\square$

Case 2

In this subsection we prove the Concentration in the case $u=0$ and $\mu=\delta_{x_0}$.

PROPOSITION 4.3. *Let $n\geqslant 2$, $\alpha<n-1$, $k\in\mathbb{N}$ and let Φ be a Young function satisfying* (3). *Let $\{u_j\}_{j=1}^{\infty}\subset W_0L^{\Phi}(\Omega)$ satisfy $\|\Phi(|\nabla u_j|)\|_{L^1(\Omega)}\leqslant 1$. Further suppose that*

$$u_j\rightharpoonup 0 \quad \text{in } W_0L^{\Phi}(\Omega)\,,\quad u_j\to 0 \quad \text{a.e. in } \Omega \quad \text{and} \quad \Phi(|\nabla u_j|)\overset{*}{\rightharpoonup}\delta_{x_0} \quad \text{in } \mathscr{M}(\bar{\Omega})\,,$$

where $x_0\in\bar{\Omega}$.
(i) *If $K\geqslant 0$ and u_j satisfy*

$$\int_{\Omega}\exp_{[k]}(K|u_j|^{\gamma})-\exp_{[k]}(0)\overset{j\to\infty}{\to} c\in[0,\infty)\,, \tag{56}$$

then

$$\exp_{[k]}(K|u_j|^{\gamma})-\exp_{[k]}(0)\overset{*}{\rightharpoonup} c\delta_{x_0} \quad \text{in } \mathscr{M}(\bar{\Omega})\,.$$

(ii) *In addition, if Φ satisfies the condition $S(K_{k,n,\alpha},\Phi)<\infty$, then the sequence $\{\exp_{[k]}(K_{k,n,\alpha}|u_j|^{\gamma})-\exp_{[k]}(0)\}_{j=1}^{\infty}$ is relatively compact with respect to the weak* convergence in $\mathscr{M}(\bar{\Omega})$ and the limits of convergent subsequences belong to*

$$\{c\delta_{x_0}: c\in[0,S(K_{k,n,\alpha},\Phi)-\exp_{[k]}(0)\mathscr{L}_n(\Omega)]\}\,.$$

The proof of Proposition 4.3 is very similar to [2, proof of Proposition 3.3]. However, sketching the proof might be a bit confusing because of the constant 1 in [2] which sometimes comes from the estimate of the modular in $W_0L^{\Phi}(\Omega)$ and sometimes it is $\exp(0)$ and it needs to be replaced by $\exp_{[k]}(0)$ in such a case. Moreover we need to replace condition (6) by the assumption $S(K_{k,n,\alpha},\Phi)<\infty$. Therefore we give a detailed proof for the convenience of the reader.

Proof. Let us prove the first assertion. For $K=0$ we can only have $c=0$ and the proof is trivial. Thus suppose $K>0$. First, we claim that

$$\eta>0 \quad\Longrightarrow\quad \int_{\Omega\setminus B(x_0,\eta)}\exp_{[k]}(K|u_j|^{\gamma})-\exp_{[k]}(0)\overset{j\to\infty}{\to} 0\,. \tag{57}$$

From Lemma 4.1 for $N=\overline{B(x_0,\frac{\eta}{2})}$ and $F=\bar{\Omega}\setminus B(x_0,\eta)$ we obtain that

$$\int_{\Omega\setminus B(x_0,\eta)}\exp_{[k]}(K(1+\delta)|u_j|^{\gamma})$$

is bounded for some $\delta>0$ and thus we may use Lemma 2.3 to obtain (57).

Further we observe that (57) and assumption (56) imply

$$\eta>0 \quad\Longrightarrow\quad \int_{B(x_0,\eta)}\exp_{[k]}(K|u_j|^{\gamma})-\exp_{[k]}(0)\overset{j\to\infty}{\to} c\,. \tag{58}$$

Fix arbitrary test function $\psi \in C(\bar{\Omega})$ and let $\varepsilon > 0$. Then there is $\eta > 0$ such that

$$|\psi(x) - \psi(x_0)| < \frac{\varepsilon}{2\max(c,1)} \qquad \text{whenever } |x - x_0| < \eta \; . \tag{59}$$

We have

$$\begin{aligned} I := & \left| \int_{\bar{\Omega}} \psi \, d(c\delta_{x_0}) - \int_\Omega \psi \big(\exp_{[k]}(K|u_j|^\gamma) - \exp_{[k]}(0)\big) \right| \\ = & \left| c\psi(x_0) - \int_\Omega \psi \big(\exp_{[k]}(K|u_j|^\gamma) - \exp_{[k]}(0)\big) \right| \\ \leqslant & \int_{\Omega \setminus B(x_0,\eta)} |\psi| \big(\exp_{[k]}(K|u_j|^\gamma) - \exp_{[k]}(0)\big) \\ & + \int_{B(x_0,\eta)} |\psi - \psi(x_0)| \big(\exp_{[k]}(K|u_j|^\gamma) - \exp_{[k]}(0)\big) \\ & + |\psi(x_0)| \left| c - \int_{B(x_0,\eta)} \big(\exp_{[k]}(K|u_j|^\gamma) - \exp_{[k]}(0)\big) \right| = I_1 + I_2 + I_3 \; . \end{aligned}$$

By (57) and $\sup_\Omega |\psi| < \infty$ we see that there is $j_1 \in \mathbb{N}$ such that $I_1 < \varepsilon$ for $j > j_1$. Further, using (58) and (59) we obtain

$$\begin{aligned} I_2 = & \int_{B(x_0,\eta)} |\psi - \psi(x_0)| (\exp_{[k]}(K|u_j|^\gamma) - \exp_{[k]}(0)) \\ \leqslant & \frac{\varepsilon}{2\max(c,1)} \int_{B(x_0,\eta)} \exp_{[k]}(K|u_j|^\gamma) - \exp_{[k]}(0) \overset{j\to\infty}{\to} \frac{\varepsilon}{2} \frac{c}{\max(c,1)} \leqslant \frac{\varepsilon}{2} \; . \end{aligned}$$

Therefore we can find $j_2 > j_1$ such that $I_2 < \varepsilon$ for $j > j_2$. Finally, from (58) and $|\psi(x_0)| < \infty$ we obtain $j_3 > j_2$ such that $I_3 < \varepsilon$ for $j > j_3$. Hence we have $I < 3\varepsilon$ for j large and the first assertion is proved.

Let us prove the second assertion. By assumption $S(K_{k,n,\alpha}, \Phi) < \infty$ we have

$$\| \exp_{[k]}(K_{k,n,\alpha}|u_j|^\gamma) - \exp_{[k]}(0) \|_{L^1(\Omega)} \leqslant S(K_{k,n,\alpha}, \Phi) - \exp_{[k]}(0) \mathscr{L}_n(\Omega) \; . \tag{60}$$

Now, we use the fact that every bounded set in the $L^1(\Omega)$-norm is relatively compact in $\mathscr{M}(\bar{\Omega})$ with respect to the weak*-convergence. Further, suppose that $\{v_j\}_{j=1}^\infty \subset \{u_j\}_{j=1}^\infty$ is such that

$$\exp_{[k]}(K_{k,n,\alpha}|v_j|^\gamma) - \exp_{[k]}(0) \overset{*}{\rightharpoonup} \nu \qquad \text{in } \mathscr{M}(\bar{\Omega}) \; .$$

Choosing the test function $\psi \equiv 1$ we obtain

$$\begin{aligned} & \int_\Omega \exp_{[k]}(K|v_j|^\gamma) - \exp_{[k]}(0) \\ & \qquad = \int_{\bar{\Omega}} \psi(\exp_{[k]}(K|v_j|^\gamma) - \exp_{[k]}(0)) \overset{j\to\infty}{\to} \int_{\bar{\Omega}} \psi \, d\nu = \nu(\bar{\Omega}) \; . \end{aligned}$$

Thus the sequence $\{v_j\}_{j=1}^\infty$ satisfies the assumptions of the first part of our proposition with $c = \nu(\bar{\Omega}) \in [0, S(K_{k,n,\alpha}, \Phi) - \exp_{[k]}(0)\mathscr{L}_n(\Omega)]$ (for the upper estimate of c we use (60)), thus the first assertion concludes the proof. □

Case 3

In this subsection we prove the Compactness for $u \neq 0$.

PROPOSITION 4.4. *Let $n \geqslant 2$, $\alpha < n-1$, $k \in \mathbb{N}$ and let Φ be a Young function satisfying* (3) *and* (8). *Assume that $\{u_j\}_{j=1}^{\infty} \subset W_0L^{\Phi}(\Omega)$ is such that*

$$\|\Phi(|\nabla u_j|)\|_{L^1(\Omega)} \leqslant 1\ , \quad u_j \rightharpoonup u \quad \text{in } W_0L^{\Phi}(\Omega) \quad \text{and} \quad u \neq 0\ .$$

Then there is $\delta > 0$ such that $\|\exp_{[k]}(K_{k,n,\alpha}(1+\delta)|u_j|^{\gamma})\|_{L^1(\Omega)}$ is bounded and

$$\exp_{[k]}(K_{k,n,\alpha}|u_j|^{\gamma}) \overset{j\to\infty}{\to} \exp_{[k]}(K_{k,n,\alpha}|u|^{\gamma}) \qquad \text{in} \quad L^1(\Omega)\ .$$

The key ingredient of the proof of Proposition 4.4 is the following lemma telling us that if the sequence $\{u_j\}_{j=1}^{\infty}$ satisfies condition (61) (which is what we do not want in Proposition 4.4), then we actually have $u = 0$. Notice that we do not assume $S(K_{k,n,\alpha}, \Phi) < \infty$ in the lemma. It is important for us in the last section.

LEMMA 4.5. *Let $n \geqslant 2$, $\alpha < n-1$, $k \in \mathbb{N}$ and let Φ be a Young function satisfying* (3) *and* (8). *Let $R > 0$ and let $\{g_j\}_{j=1}^{\infty} \subset C^1([0,R])$ be non-increasing functions satisfying $g_j(R) = 0$. Set*

$$u_j(x) = g_j(|x|)\ , \qquad x \in B(R)\ ,\ j \in \mathbb{N}$$

and assume that $||\Phi(|\nabla u_j|)||_{L^1(B(R))} \leqslant 1$. If

$$\lim_{j\to\infty} ||\exp(K_{k,n,\alpha}(1+\delta)|u_j|^{\gamma}||_{L^1(B(R))} = \infty \qquad \text{for every } \delta > 0\ , \tag{61}$$

then

$$\int_{B(R)\setminus B(r)} \Phi(|\nabla u_j|) \overset{j\to\infty}{\to} 0 \qquad \text{for every } r \in (0,R) \tag{62}$$

and

$$u_j \overset{j\to\infty}{\to} 0 \quad \text{uniformly on } B(R)\setminus B(r) \qquad \text{for every } r \in (0,R)\ .$$

Proof. First let us prove (62). If (62) is not true then passing to a subsequence we can find $\tau > 0$ and $r_0 \in (0,R)$ such that

$$\int_{B(R)\setminus B(r_0)} \Phi(|\nabla u_j|) \geqslant \tau \qquad \text{for all } j \in \mathbb{N}$$

and thus

$$\int_{B(r_0)} \Phi(|\nabla u_j|) \leqslant 1-\tau \qquad \text{for all } j \in \mathbb{N}\ . \tag{63}$$

Put $d\mu(y) = \omega_{n-1}y^{n-1}dy$ and let Φ_1 be the Young function from (28). Fix $t \in (0,r_0)$ and for every $j \in \mathbb{N}$ set

$$A_j = \{y \in (t,r_0) : |g_j'(y)| > G\}\ , \qquad \tilde{A}_j = \{y \in (r_0,R) : |g_j'(y)| > G\}$$

(Recall that the constant G comes from (28)). From (28) and (63) we obtain

$$\begin{aligned}\int_{A_j}\Phi_1(|g_j'(y)|)\omega_{n-1}y^{n-1}\,dy &\leqslant \frac{\omega_{n-1}}{n}\int_{A_j}\Phi(|g_j'(y)|)y^{n-1}dy\\ &\leqslant \frac{\omega_{n-1}}{n}\int_0^{r_0}\Phi(|g_j'(y)|)y^{n-1}dy=\frac{1}{n}\int_{B(r_0)}\Phi(|\nabla u_j(x)|)\,dx \quad (64)\\ &\leqslant \frac{1}{n}(1-\tau)=(1-\tau)\Phi_1(1)\,.\end{aligned}$$

Thus (20) gives $\tilde{\tau}>0$ independent of $j\in\mathbb{N}$ such that

$$||g_j'(y)||_{L^{\Phi_1}(A_j,d\mu)}\leqslant 1-2\tilde{\tau}\,,\quad j\in\mathbb{N}\,. \tag{65}$$

The same way we obtain from $||\Phi(|\nabla u_j|)||_{L^1(B(R))}\leqslant 1$

$$||g_j'(y)||_{L^{\Phi_1}(\tilde{A}_j,d\mu)}\leqslant 1\,,\quad j\in\mathbb{N}\,. \tag{66}$$

The generalized Hölder's inequality (15) gives

$$\begin{aligned}g_j(t)&\leqslant\int_t^R|g_j'(y)|dy=\int_{(r_0,R)\setminus\tilde{A}_j}+\int_{\tilde{A}_j}+\int_{(t,r_0)\setminus A_j}+\int_{A_j}\\ &\leqslant GR+\int_{\tilde{A}_j}|g_j'(y)|\frac{1}{\omega_{n-1}y^{n-1}}d\mu(y)+Gr_0+\int_{A_j}|g_j'(y)|\frac{1}{\omega_{n-1}y^{n-1}}d\mu(y)\\ &\leqslant C+\frac{1}{\omega_{n-1}}||g_j'(y)||_{L^{\Phi_1}(\tilde{A}_j,d\mu)}\left|\left|\frac{1}{y^{n-1}}\right|\right|_{L^{\Psi}((r_0,R),d\mu)}\\ &\qquad+\frac{1}{\omega_{n-1}}||g_j'(y)||_{L^{\Phi_1}(A_j,d\mu)}\left|\left|\frac{1}{y^{n-1}}\right|\right|_{L^{\Psi}((t,R),d\mu)}\,.\end{aligned}$$

Therefore Lemma 3.2, (65), (66) and $||\frac{1}{y^{n-1}}||_{L^{\Psi}((r_0,R),d\mu)}\leqslant C$ ($\frac{1}{y^{n-1}}$ is bounded on (r_0,R)) imply in case $t\in(0,\min(t_0,r_0))$

$$g_j(t)\leqslant C+\frac{1}{\omega_{n-1}}(1-2\tilde{\tau})D\log_{[k]}^{\frac{1}{\gamma}}\left(\frac{1}{t}\right)\left(1+\log_{[\ell]}^{-\frac{1}{4}}\left(\frac{1}{t}\right)\right)\,.$$

Therefore there is $t_1\in(0,\min(t_0,r_0))$ such that

$$g_j(t)\leqslant(1-\tilde{\tau})\frac{1}{\omega_{n-1}}D\log_{[k]}^{\frac{1}{\gamma}}\left(\frac{1}{t}\right)\qquad\text{for } t\in(0,t_1)\,. \tag{67}$$

Finally pick $\delta_0>0$ small enough so that

$$(1+\delta_0)(1-\tilde{\tau})^{\gamma}=\eta<1 \tag{68}$$

and let us show that we have a contradiction with (61). If $k \geqslant 2$, then (67), (68) and $K_{k,n,\alpha}(\frac{D}{\omega_{n-1}})^\gamma = 1$ (see (5) and (43)) imply

$$\begin{aligned}
&\int_{B(R)} \exp_{[k]}(K_{k,n,\alpha}(1+\delta_0)|u_j(x)|^\gamma)\,dx \\
&\quad = \omega_{n-1}\int_0^R \exp_{[k]}(K_{k,n,\alpha}(1+\delta_0)|g_j(y)|^\gamma)y^{n-1}\,dy \\
&\quad \leqslant C\int_{t_1}^R \exp_{[k]}(C|g_j(t_1)|^\gamma)y^{n-1}\,dy + C\int_0^{t_1} \exp_{[k]}\Big(\eta\log_{[k]}\Big(\frac{1}{y}\Big)\Big)y^{n-1}\,dy \\
&\quad \leqslant C + C\int_0^{t_1} y^{n-2}\,dy \leqslant C\,.
\end{aligned}$$

If $k=1$, then (67), (68), $\eta < 1$ and $K_{1,n,\alpha}(\frac{D}{\omega_{n-1}})^\gamma = n$ imply

$$\begin{aligned}
&\int_{B(R)} \exp_{[1]}(K_{1,n,\alpha}(1+\delta_0)|u_j(x)|^\gamma)\,dx \\
&\quad = \omega_{n-1}\int_0^R \exp_{[1]}(K_{1,n,\alpha}(1+\delta_0)|g_j(y)|^\gamma)y^{n-1}\,dy \\
&\quad \leqslant C\int_{t_1}^R \exp_{[1]}(C|g_j(t_1)|^\gamma)y^{n-1}\,dy + C\int_0^{t_1} \exp_{[1]}\Big(\eta n\log_{[1]}\Big(\frac{1}{y}\Big)\Big)y^{n-1}\,dy \\
&\quad \leqslant C + C\int_0^{t_1} y^{n-1-\eta n}\,dy \leqslant C\,.
\end{aligned}$$

Hence we have a contradiction with (61) in both cases and (62) is proved. Now, fix $r \in (0,R)$ and let us check the uniform convergence. From (18) and (62) we obtain $\|\nabla u_j\|_{L^\Phi(B(R)\setminus B(r))} \to 0$. Hence $\|\nabla u_j\|_{L^1(B(R)\setminus B(r))} \to 0$ and thus the radial symmetry of u_j, $u_j\big|_{|x|=R} = 0$ and the monotonicity with respect to $|x|$ imply the uniform convergence. $\square$

Proof of Proposition 4.4. We prove Proposition 4.4 by contradiction. Suppose that

$$\sup_{j\in\mathbb{N}} \|\exp_{[k]}(K_{k,n,\alpha}(1+\delta)|u_j|^\gamma)\|_{L^1(\Omega)} = \infty \qquad \text{for every } \delta > 0\,.$$

Recall that for fixed $j \in \mathbb{N}$ and $\delta > 0$ we have

$$\|\exp_{[k]}(K_{k,n,\alpha}(1+\delta)|u_j|^\gamma)\|_{L^1(\Omega)} < \infty$$

by Proposition 1.1. Thus passing to a subsequence, we can suppose that

$$\|\exp_{[k]}(K_{k,n,\alpha}(1+\delta)|u_j|^\gamma)\|_{L^1(\Omega)} \overset{j\to\infty}{\longrightarrow} \infty \qquad \text{for every } \delta > 0\,. \tag{69}$$

By a standard symmetrization argument based on Theorem 2.2 and the density of C_0^∞-functions in $W_0L^\Phi(\Omega)$ (cf. [2, Proof of Proposition 3.4]) we may assume that $\Omega = B(R)$, $R > 0$, u_j, u are continuous, spherically symmetric, non-negative, non-increasing with respect to $|x|$ and differentiable almost everywhere. Hence the assumptions of Lemma 4.5 are satisfied and we obtain that u_j converge uniformly to the zero function on $B(R)\setminus B(r)$ for every $r\in(0,R)$. This implies $u=0$ a.e. and we have a contradiction with $u\neq 0$.

The last assertion of Proposition 4.4 follows from Lemma 2.3. □

Proof of Theorem 1.3. Theorem 1.3 follows from Propositions 4.2, 4.3 and 4.4. □

Proof of Theorem 1.5. Put

$$S:=\sup\{\Lambda_F(u): u\in W_0L^\Phi(\Omega), \|\Phi(|\nabla u|)\|_{L^1(\Omega)}\leqslant 1\}\,.$$

There is a sequence

$$\{u_j\}_{j=1}^\infty\subset\{u\in W_0L^\Phi(\Omega): \|\Phi(|\nabla u|)\|_{L^1(\Omega)}\leqslant 1\}$$

such that $\Lambda_F(u_j)\overset{j\to\infty}{\to} S$. We can further suppose that

$$u_j\rightharpoonup u \quad \text{in } W_0L^\Phi(\Omega), \quad u_j\to u \quad \text{a.e. in }\Omega \quad \text{and} \quad \Phi(|\nabla u_j|)\overset{*}{\rightharpoonup}\mu \quad \text{in } \mathscr{M}(\bar{\Omega})\,,$$

otherwise we pass to a subsequence. Obviously we have $\|\Phi(|\nabla u|)\|_{L^1(\Omega)}\leqslant 1$, thus all we need to show is $\Lambda_F(u)=S$.

If (10) is satisfied, then let us find $\delta>0$ such that $K(1+\delta)<K_{k,n,\alpha}$. Now, it is enough to use Lemma 2.3 (with $\tilde{K}=K$ and $C_1=S(K(1+\delta),\Phi)<\infty$) and we are done.

If we have (11), we have to be more careful than in [2]. Let us distinguish the three cases that were important in the proof of Theorem 1.3:
(a) $u=0$ and μ is not a Dirac mass at one point,
(b) $u=0$ and $\mu=\delta_{x_0}$,
(c) $u\neq 0$.

If (a) or (c) is satisfied, then the proof follows from Proposition 4.2 and Proposition 4.4, respectively, combined with Lemma 2.3.

Finally, suppose that (b) is satisfied. This time we prove

$$S=\lim_{j\to\infty}\Lambda_F(u_j)=\mathscr{L}_n(\Omega)F(0)=\Lambda_F(0)$$

the same way as in [2, Proof of Theorem 1.2] and we are done. □

5. Example

In this section we carefully modify the construction and estimates from [16, proof of Theorem 4.1] to obtain an example proving Theorem 1.2(iv) in the case $k=1$.

EXAMPLE 5.1. Let $n\geqslant 2$, $\alpha<n-1$ and $K=K_{1,n,\alpha}$. If there are $t_0>1$, $a\in(0,\min(1,B))$ and $C>0$ such that the Young function Φ satisfies

$$\Phi(t)\leqslant\begin{cases} Ct^n & \text{for } t\in[0,t_0]\\ t^n\log^\alpha(t)\Big(1-\log^{-a}(t)\Big) & \text{for } t\in[t_0,\infty)\,, \end{cases} \tag{70}$$

then for every $m \in \mathbb{N}$ there is $f \in W_0L^\Phi(B(R))$ such that $\int_{B(R)} \Phi(|\nabla f|)\,dx \leqslant 1$ but

$$\int_{B(R)} \exp(K|f(x)|)^\gamma)\,dx > m\,.$$

Proof. For $s > 1$ we define $f_s(x) = g_s(|x|)$ where

$$g_s(y) = \begin{cases} \left(-\frac{2}{R}y+2\right)K^{-\frac{1}{\gamma}}n^B\log^B(2)s^{\frac{1}{\gamma}-B}\left(1+\frac{\log(s)}{s}\right)^{\frac{1}{\gamma}} & \text{for } y \in [\frac{R}{2},R] \\ K^{-\frac{1}{\gamma}}n^B\log^B\left(\frac{R}{y}\right)s^{\frac{1}{\gamma}-B}\left(1+\frac{\log(s)}{s}\right)^{\frac{1}{\gamma}} & \text{for } y \in [Re^{-\frac{s}{n}},\frac{R}{2}] \\ K^{-\frac{1}{\gamma}}s^{\frac{1}{\gamma}}\left(1+\frac{\log(s)}{s}\right)^{\frac{1}{\gamma}} & \text{for } y \in [0,Re^{-\frac{s}{n}}]\,. \end{cases}$$

An easy computation gives us

$$\begin{aligned}\int_{B(R)} \exp\bigl(K|f_s(x)|^\gamma\bigr)\,dx &\geqslant \int_{B(Re^{-\frac{s}{n}})} \exp\bigl(K|f_s(x)|^\gamma\bigr)\,dx \\ &= Ce^{-s}e^{(1+\frac{\log(s)}{s})s} \overset{s\to\infty}{\longrightarrow} \infty\,.\end{aligned}$$

It remains to prove that $\int_{B(R)} \Phi(|\nabla f_s|) \leqslant 1$ for s large enough. On $[Re^{-\frac{s}{n}},\frac{R}{2}]$ we have

$$|g_s'(y)| = K^{-\frac{1}{\gamma}}n^B B\log^{B-1}\left(\frac{R}{y}\right)\frac{1}{R}\frac{R}{y}s^{\frac{1}{\gamma}-B}\left(1+\frac{\log(s)}{s}\right)^{\frac{1}{\gamma}}\,. \tag{71}$$

Set $M = \frac{R}{s^{\log(s)}}$. Plainly there is $s_1 > 1$ such that for $s > s_1$ we have $Re^{-\frac{s}{n}} < M < \frac{R}{2}$ and therefore

$$\int_0^R \Phi(|g_s'(y)|)y^{n-1}dy = \int_{Re^{-\frac{s}{n}}}^M + \int_M^{\frac{R}{2}} + \int_{\frac{R}{2}}^R = I_1+I_2+I_3\,. \tag{72}$$

Obviously $(\frac{1}{\gamma}-B) < 0$ and $|g_s'(y)| \leqslant Cs^{\frac{1}{\gamma}-B}$ for $y \in (\frac{R}{2},R)$. Hence there is $s_2 > s_1$ such that for every $s > s_2$ and $y \in (\frac{R}{2},R)$ we have $|g_s'(y)| < t_0$. It follows from (70) that

$$I_3 \leqslant C\int_{\frac{R}{2}}^R |g_s'(y)|^n y^{n-1}\,dy \leqslant Cs^{(\frac{1}{\gamma}-B)n} = Cs^{-B}\,. \tag{73}$$

Using (71) on $[M,\frac{R}{2}]$ we obtain

$$|\log(|g_s'(y)|)|^n \leqslant C\Bigl(\log^n(s)+\log^n\left(\frac{R}{y}\right)\Bigr) \leqslant C\Bigl(\log^n(s)+\log^n\left(\frac{R}{M}\right)\Bigr) \leqslant C\log^{2n}(s)\,. \tag{74}$$

Now, if $\beta \neq -1$, then

$$\int_M^{\frac{R}{2}} \log^\beta\left(\frac{R}{y}\right)\frac{dy}{y} = \frac{1}{\beta+1}\Bigl[\log^{\beta+1}\left(\frac{R}{y}\right)\Bigr]_{\frac{R}{2}}^M \leqslant C + C\log^{\beta+1}\left(\frac{R}{M}\right).$$

Further

$$\int_M^{\frac{R}{2}} \log^{-1}\Big(\frac{R}{y}\Big)\frac{dy}{y} = \Big[\log\Big(\log\Big(\frac{R}{y}\Big)\Big)\Big]_{\frac{R}{2}}^M \leqslant C + C\log\Big(\log\Big(\frac{R}{M}\Big)\Big).$$

Thus in both cases we have

$$\int_M^{\frac{R}{2}} \log^{\beta}\Big(\frac{R}{y}\Big)\frac{dy}{y} \leqslant C + C\log^{\beta+2}\Big(\frac{R}{M}\Big) \leqslant C\log^{2(\beta+2)}(s)\ . \tag{75}$$

From (70) we obtain $\Phi(t)\leqslant Ct^n(1+|\log(t)|^n)$ on $[0,\infty)$, hence (71), (74), (75) and $\log(|g_s'(M)|)\leqslant C\log(s^{\log(s)}) = C\log^2(s)$ imply that

$$\begin{aligned} I_2 &\leqslant C\int_M^{\frac{R}{2}} |g_s'(y)|^n(1+|\log(|g_s'(y)|)|^n)y^{n-1}\,dy \\ &\leqslant Cs^{(\frac{1}{\gamma}-B)n}\log^{2n}(s)\int_M^{\frac{1}{2}}\log^{(B-1)n}\Big(\frac{1}{y}\Big)\frac{dy}{y} \\ &\leqslant Cs^{(\frac{1}{\gamma}-B)n}\log^{2n}(s)\log^{2Bn-2n+4}(s) \\ &= Cs^{-B}\log^{2Bn+4}(s)\ . \end{aligned} \tag{76}$$

Since $M << s^{\frac{1}{\gamma}-B} << 1$, by (71) we can find $s_3 > s_2$ such that if $s > s_3$ then

$$y\in[Re^{-\frac{s}{n}},M] \quad\Rightarrow\quad |g_s'(y)| > t_0\ .$$

Plainly there is $s_4 > s_3$ such that for $s > s_4$ we have by (71)

$$\sup_{y\in(Re^{-\frac{s}{n}},M]} |g_s'(y)| = \lim_{y\to(Re^{-\frac{s}{n}})_+} |g_s'(y)| \leqslant e^s\ .$$

Thus (70) gives us for $y\in[Re^{-\frac{s}{n}},M]$ and $s>s_4$ that

$$\Phi(|g_s'(y)|)\leqslant\Big(1-\frac{1}{s^a}\Big)|g_s'(y)|^n\log^{\alpha}(|g_s'(y)|)\ . \tag{77}$$

We are ready to estimate I_1 for $\alpha\geqslant 0$. Since $s^{\frac{1}{\gamma}-B} \overset{s\to\infty}{\to} 0$, there is $s_5 > s_4$ such that $|g_s'(y)|\leqslant\frac{R}{y}$ for $\alpha\geqslant 0$, $s>s_5$ and $y\in[Re^{-\frac{s}{n}},M]$. Therefore from (2), (5), (77) and $-(\frac{1}{\gamma}-B)n = B = (B-1)n+\alpha+1$ we obtain for $s>s_5$ and $\alpha\geqslant 0$ that

$$
\begin{aligned}
I_1 &\leqslant \Big(1-\frac{1}{s^a}\Big)\int_{Re^{-\frac{s}{n}}}^{M} |g_s'(y)|^n \log^{\alpha}(|g_s'(y)|) y^{n-1} dy \\
&= \Big(1-\frac{1}{s^a}\Big)\Big(1+\frac{\log(s)}{s}\Big)^{\frac{n}{\gamma}} K^{-\frac{n}{\gamma}} n^{Bn} B^n s^{-B} \int_{Re^{-\frac{s}{n}}}^{M} \log^{(B-1)n}\Big(\frac{R}{y}\Big)\log^{\alpha}(|g_s'(y)|)\frac{dy}{y} \\
&\leqslant \Big(1-\frac{1}{s^a}\Big)\Big(1+\frac{\log(s)}{s}\Big)^{\frac{n}{\gamma}} K^{-\frac{n}{\gamma}} n^{Bn} B^n s^{-B} \int_{Re^{-\frac{s}{n}}}^{M} \log^{B-1}\Big(\frac{R}{y}\Big)\frac{dy}{y} \\
&\leqslant \Big(1-\frac{1}{s^a}\Big)\Big(1+\frac{\log(s)}{s}\Big)^{\frac{n}{\gamma}} K^{-\frac{n}{\gamma}} n^{Bn} B^n s^{-B} \frac{(\frac{s}{n})^B}{B} \\
&\leqslant \frac{1}{\omega_{n-1}}\Big(1-\frac{1}{s^a}\Big)\Big(1+\frac{\log s}{s}\Big)^{\frac{n}{\gamma}} .
\end{aligned} \tag{78}
$$

Using (73), (76), (78) we obtain that for s large enough and $\alpha \geqslant 0$ we have

$$
\begin{aligned}
\int_{B(R)} \Phi(|\nabla f_s(x)|)dx &= \omega_{n-1}\int_0^R \Phi(|g_s'(y)|)y^{n-1}\, dy = \omega_{n-1}(I_1+I_2+I_3) \\
&\leqslant \Big(1-\frac{1}{s^a}\Big)\Big(1+\frac{\log(s)}{s}\Big)^{\frac{n}{\gamma}} + Cs^{-B}\log^{2Bn+4}(s) \leqslant 1 .
\end{aligned}
$$

Now, let $\alpha < 0$. Clearly we can find $s_5 > s_4$ such that for $s > s_5$ and $y \in [e^{-\frac{s}{n}}, M]$ we have

$$
\begin{aligned}
\frac{\log^{\alpha}(|g_s'(y)|)}{\log^{\alpha}(\frac{R}{y})} &= \Big(1+\frac{\log(\frac{y}{R}|g_s'(y)|)}{\log(\frac{R}{y})}\Big)^{\alpha} \leqslant 1 + C\frac{|\log(Cs^{\frac{1}{\gamma}-B}\log^{B-1}(\frac{R}{y}))|}{\log(\frac{R}{y})} \\
&\leqslant 1 + C\frac{\log(s)}{\log(\frac{R}{y})} + C\frac{\log(|\log^{B-1}(\frac{R}{y})|)}{\log(\frac{R}{y})} .
\end{aligned} \tag{79}
$$

From $\alpha < 0$ we have $B \neq 1$. Further estimate (79) implies that it is enough to replace the integral $\int_{Re^{-\frac{s}{n}}}^{M} \log^{B-1}(\frac{R}{y})\frac{dy}{y}$ in (78) by

$$
\begin{aligned}
&\int_{Re^{-\frac{s}{n}}}^{M} \log^{B-1}\Big(\frac{R}{y}\Big)\Big(1+C\frac{\log(s)}{\log(\frac{R}{y})} + C\frac{(B-1)\log(\log(\frac{R}{y}))}{\log(\frac{R}{y})}\Big)\frac{dy}{y} \\
&= \Big[\frac{1}{B}\log^{B}\Big(\frac{R}{y}\Big) + \frac{C}{B-1}\log^{B-1}\Big(\frac{R}{y}\Big) \\
&\qquad + C\Big(\frac{\log^{B-1}(\frac{R}{y})}{B-1} - \log^{B-1}\Big(\frac{R}{y}\Big)\ \log\Big(\log\Big(\frac{R}{y}\Big)\Big)\Big)\Big]_M^{Re^{-\frac{s}{n}}} \\
&\leqslant \frac{(\frac{s}{n})^B}{B}\Big(1+\frac{C}{s}+\frac{C}{s}+\frac{C}{s}\log(s)\Big) \leqslant \frac{(\frac{s}{n})^B}{B}\Big(1+\frac{C\log(s)}{s}\Big) .
\end{aligned}
$$

Hence finding s large enough we conclude also for $\alpha < 0$

$$\int_{B(R)} \Phi(|\nabla f_s(x)|)dx = \omega_{n-1}\int_0^R \Phi(|g_s'(y)|)y^{n-1}\,dy = \omega_{n-1}(I_1+I_2+I_3)$$
$$\leqslant \Big(1-\frac{1}{s^a}\Big)\Big(1+\frac{\log(s)}{s}\Big)^{\frac{n}{\gamma}}\Big(1+\frac{C\log(s)}{s}\Big)+Cs^{-B}\log^{2Bn+4}(s)\leqslant 1\,. \qquad \square$$

6. Some notes on concentrating sequences

In this section (similarly as in [12]), we say that a sequence $\{u_j\}_{j=1}^{\infty}\subset W_0L^{\Phi}(\Omega)$ is a *normalized concentrating sequence* if

$$\begin{aligned}&(i)\ ||\Phi(|\nabla u_j|)||_{L^1(\Omega)}=1 \qquad \text{for every } j\in\mathbb{N}\,,\\ &(ii)\ u_j \rightharpoonup 0 \quad \text{in } W_0L^{\Phi}(\Omega)\,,\\ &(iii)\ \text{there is } x_0\in\Omega \text{ such that } \int_{\Omega\setminus B(x_0,\rho)}\Phi(|\nabla u_j|)\to 0 \quad \text{for every } \rho>0\,.\end{aligned} \tag{80}$$

A sequence $\{u_j\}_{j=1}^{\infty}\subset W_0L^{\Phi}(\Omega)$ is a *normalized sequence* if it satisfies (i) of (80).

First, let us show that the sequences of functions constructed in [16, Proof of Theorem 1.2] and [3, Proof of Theorem 1.2] can be modified so that we have a normalized concentrating sequences proving Theorem 1.2(ii).

EXAMPLE 6.1. Let $n\geqslant 2$, $\alpha<n-1$, $k\in\mathbb{N}$, $K>K_{k,n,\alpha}$ and let Φ be a Young function satisfying (3). Suppose that $x_0\in\Omega$ and $R>0$ are such that $B(x_0,R)\subset\Omega$ and let $j\in\mathbb{N}$. Then there is a function $f_{K,j}\in W_0L^{\Phi}(\Omega)$ such that

$$\begin{aligned}\operatorname{spt} f_{K,j} = \overline{B(x_0,R)}\,, \qquad & ||\Phi(|\nabla f_{K,j}|)||_{L^1(\Omega)}=1\,,\\ ||\exp_{[k]}(K|f_{K,j}|^{\gamma})||_{L^1(\Omega)}\geqslant j\,, \qquad & ||\Phi(|\nabla f_{K,j}|)||_{L^1(\Omega\setminus B(x_0,\frac{1}{j}))}\leqslant \frac{1}{j}\,.\end{aligned} \tag{81}$$

Proof. We can assume that $x_0=0\in\Omega$ and $j>S(\frac{1}{2}K_{k,n,\alpha},\Phi)$ (we know that $S(\frac{1}{2}K_{k,n,\alpha},\Phi)<\infty$ by Theorem 1.2(i)).

Let us consider $k=1$ first. By the construction in [16, Proof of Theorem 1.2] there are $A>0$, $\varepsilon>0$ and $s_0>0$ such that for every $s>s_0$ the function $f_s(x)=g_s(|x|)$, where

$$g_s(y)=\begin{cases}0 & \text{for } y\in[R,\infty)\\ \big(2-\frac{2}{R}y\big)An^B\log^B(2)s^{\frac{1}{\gamma}-B} & \text{for } y\in[\frac{R}{2},R]\\ An^B\log^B\big(\frac{R}{y}\big)s^{\frac{1}{\gamma}-B} & \text{for } y\in[R\exp^{-\frac{1}{n}}(s),\frac{R}{2}]\\ As^{\frac{1}{\gamma}} & \text{for } y\in[0,R\exp^{-\frac{1}{n}}(s)]\,,\end{cases}$$

satisfies

$$||\Phi(|\nabla f_s|)||_{L^1(\Omega)}\leqslant 1\,, \qquad ||\exp(K|f_s|^{\gamma})||_{L^1(\Omega)}\geqslant \frac{\omega_{n-1}}{n}R^n e^{-s}e^{(1+\varepsilon)s}=Ce^{\varepsilon s}\,. \tag{82}$$

Let us show that if $s > s_0$ is large enough and if we set

$$f_{K,j} = \frac{f_s}{||\nabla f_s||_{L_L^\Phi(\Omega)}} ,$$

where $|| \, ||_{L_L^\Phi(\Omega)}$ denotes the Luxemburg norm in $L^\Phi(\Omega)$, then $f_{K,j}$ satisfies (81). The first property in (81) follows from the definition of g_s. The second is given by the basic properties of the Luxemburg norm (Φ satisfies the Δ_2-condition). The third property is obviously satisfied by (82) for s large enough. Only the proof of the fourth property corresponding to the Concentration phenomenon requires some work. First, let us show that we have for all s large enough

$$\frac{1}{2}K_{1,n,\alpha} < K||\nabla f_s||^{\gamma}_{L_L^\Phi(\Omega)} . \tag{83}$$

If (83) is not true, then (82), Theorem 1.2(i) and our assumption $S(\frac{1}{2}K_{k,n,\alpha},\Phi) < j$ give

$$j < ||\exp(K|f_s|^\gamma)||_{L^1(\Omega)} = \Big|\Big|\exp\Big(K||\nabla f_s||^{\gamma}_{L_L^\Phi(\Omega)}\Big|\frac{f_s}{||\nabla f_s||^{\gamma}_{L_L^\Phi(\Omega)}}\Big|^\gamma\Big)\Big|\Big|_{L^1(\Omega)}$$

$$\leqslant \Big|\Big|\exp\Big(\frac{1}{2}K_{1,n,\alpha}\Big|\frac{f_s}{||\nabla f_s||^{\gamma}_{L_L^\Phi(\Omega)}}\Big|^\gamma\Big)\Big|\Big|_{L^1(\Omega)} \leqslant S\Big(\frac{1}{2}K_{1,n,\alpha},\Phi\Big) < j$$

and we have a contradiction. Therefore for all s large enough we have by the definition of g_s, by the definition of $f_{K,j}$ and by (83)

$$|\nabla f_{K,s}(x)| \leqslant Cs^{\frac{1}{\gamma}-B} \quad \text{provided } |x| > \frac{1}{j} .$$

Since $\frac{1}{\gamma} - B < 0$, the fourth property in (81) follows.

For $k \geqslant 2$, we use the construction from [3, Proof of Theorem 1.2]: there are $A > 0$, $\varepsilon > 0$, $T > \exp_{[k]}(1)$ and $s_0 > 0$ such that for $s > s_0$ the function $f_s(x) = g_s(|x|)$, where

$$g_s(y) = \begin{cases} 0 & \text{for } y \in [R,\infty) \\ \big(2-\frac{2}{R}y\big)A\log^B_{[k]}(T+2)s^{\frac{1}{\gamma}-B} & \text{for } y \in [\frac{R}{2},R] \\ A\log^B_{[k]}\big(T+\frac{R}{y}\big)s^{\frac{1}{\gamma}-B} & \text{for } y \in [R\exp^{-\frac{1}{n}}_{[k]}(s),\frac{R}{2}] \\ A\log^B_{[k]}\Big(T+\exp^{\frac{1}{n}}_{[k]}(s)\Big)s^{\frac{1}{\gamma}-B} & \text{for } y \in [0,R\exp^{-\frac{1}{n}}_{[k]}(s)] , \end{cases}$$

satisfies

$$||\Phi(|\nabla f_s|)||_{L^1(\Omega)} \leqslant 1 , \qquad ||\exp_{[k]}(K|f_s|^\gamma)||_{L^1(\Omega)} \geqslant \frac{\omega_{n-1}}{n}R^n\frac{\exp_{[k]}((1+\varepsilon)s)}{\exp_{[k]}(s)} .$$

We conclude the proof the same way as in the case $k = 1$. □

Let us give a summary of the properties of the normalized concentrating sequences. In all following claims we suppose $n \geqslant 2$, $\alpha < n-1$, $k \in \mathbb{N}$ and that Φ is a Young function satisfying (3) and (8).

CLAIM 6.2. *Let $K < K_{k,n,\alpha}$. Then the following statements hold.*
(i) *For any normalized concentrating sequence*

$$||\exp_{[k]}(K|u_j|^\gamma)||_{L^1(\Omega)} \to \exp_{[k]}(0)\mathscr{L}_n(\Omega)\ .$$

(ii) *The supremum $S(K,\Phi)$ is attained.*

Proof. The first assertion follows from Corollary 1.4. Indeed, since $u_j \rightharpoonup 0$ in $W_0L^\Phi(\Omega)$, we have $u_j \to 0$ in $L^1(\Omega)$ and thus we conclude the proof using the fact that each subsequence of $\{u_j\}_{j=1}^\infty$ contains a subsequence converging almost everywhere in Ω.

For the proof of the second assertion we use the same argument for a maximizing sequence. □

CLAIM 6.3. *If $K > K_{k,n,\alpha}$, then there are both normalized concentrating and normalized non-concentrating sequences such that*

$$||\exp_{[k]}(K|u_j|^\gamma)||_{L^1(\Omega)} \to \infty\ .$$

Proof. As a normalized concentrating sequence we can take $\{u_j\}_{j=1}^\infty := \{f_{K,j}\}_{j=1}^\infty$, where $f_{K,j}$ are given by Example 6.1 (constructed on $B(x_0,R) \subset \Omega$). We pass to a weakly convergent subsequence, if necessary.

Let us show, how a non-concentrating sequence can be constructed. Fix $\tilde{K} \in (K_{k,n,\alpha},K)$ and a ball $B(x_0,2R) \subset \Omega$. Then set

$$\tilde{u}_j = f_{\tilde{K},j}\ , \qquad j \in \mathbb{N}$$

where $f_{\tilde{K},j}$ come from Example 6.1. Hence

$$||\exp_{[k]}(\tilde{K}|\tilde{u}_j|^\gamma)||_{L^1(B(x_0,R))} \to \infty\ .$$

For $v_j = (\frac{\tilde{K}}{K})^{\frac{1}{\gamma}}\tilde{u}_j$ we therefore have

$$||\exp_{[k]}(K|v_j|^\gamma)||_{L^1(B(x_0,R))} \to \infty$$

and, by the Δ_2-condition (we use a version of (19) for the Luxemburg norm)

$$||\Phi(|\nabla v_j|)||_{L^1(B(x_0,R))} \leqslant 1-\eta \qquad \text{for every } j \in \mathbb{N} \text{ and some } \eta > 0\ .$$

Now it is enough to take some $v \in W_0L^\Phi(B(x_0,2R))$ such that v is non-negative, $||\Phi(|\nabla v|)||_{L^1(B(x_0,2R))} \leqslant \eta$ and v is equal to a positive constant on $B(x_0,R)$. Obviously, the sequence $\{u_j\}_{j=1}^\infty := \{v_j + v\}_{j=1}^\infty$ has the desired properties. □

CLAIM 6.4. *If*

$$||\exp_{[k]}(K_{k,n,\alpha}(1+\delta)|u_j|^\gamma)||_{L^1(\Omega)} \to \infty \qquad \textit{for every } \delta > 0 \tag{84}$$

is satisfied for a normalized sequence, then this sequence contains a normalized concentrating subsequence. Moreover, such a sequence exists.

Proof. Proving the first assertion let us pass to a subsequence satisfying

$$u_j \rightharpoonup u \quad \text{in } W_0L^{\Phi}(\Omega)\ ,\ u_j \to u \text{ a.e. in } \Omega \text{ and } \Phi(|\nabla u_j|) \overset{*}{\rightharpoonup} \mu \text{ in } \mathscr{M}(\bar{\Omega})\ .$$

We have $u = 0$ and $\mu = \delta_{x_0}$ for some $x_0 \in \bar{\Omega}$, since otherwise Proposition 4.2 and Proposition 4.4, respectively, give a contradiction with (84).

Fix $\rho > 0$ and let $\psi \in C(\bar{\Omega})$ be a test function such that $\psi \geqslant 0$, $\psi = 0$ on $B(x_0, \frac{\rho}{2})$ and $\psi = 1$ on $\bar{\Omega} \setminus B(x_0,\rho)$. Hence we have

$$0 \leqslant \int_{\Omega\setminus B(x_0,\rho)} \Phi(|\nabla u_j|) = \int_{\bar{\Omega}\setminus B(x_0,\rho)} \psi\Phi(|\nabla u_j|)$$
$$\leqslant \int_{\bar{\Omega}} \psi\Phi(|\nabla u_j|) \overset{j\to\infty}{\to} \int_{\bar{\Omega}} \psi\, d\delta_{x_0} = \psi(x_0) = 0\ .$$

This is (80)(iii) and thus we have a normalized concentrating subsequence.

Let us prove the second assertion. Fix x_0 and $R > 0$ such that $B(x_0,R) \subset \Omega$. Now it is enough to set $\{u_j\}_{j=1}^{\infty} := \{f_{K_{k,n,\alpha}+\frac{1}{j},j}\}_{j=1}^{\infty}$, where $f_{K,j}$ are given by Example 6.1. □

CLAIM 6.5. *If $S(K_{k,n,\alpha},\Phi) = \infty$ and if $\{u_j\}_{j=1}^{\infty} \subset W_0L^{\Phi}(\Omega)$ is normalized and*

$$||\exp_{[k]}(K_{k,n,\alpha}|u_j|^{\gamma})||_{L^1(\Omega)} \to \infty$$

then this sequence contains a normalized concentrating subsequence.

Proof. The proof follows from Claim 6.4. □

CLAIM 6.6. *If $K \geqslant 0$ and there is a normalized concentrating sequence $\{u_j\}_{j=1}^{j=\infty} \subset W_0L^{\Phi}(\Omega)$ such that*

$$||\exp_{[k]}(K|u_j|^{\gamma})||_{L^1(\Omega)} \to c \in [0,\infty]\ ,$$

then for every $d \in [0,c]$ there is a normalized concentrating sequence $\{v_j\}_{j=1}^{\infty}$ such that

$$||\exp_{[k]}(K|v_j|^{\gamma})||_{L^1(\Omega)} \to d\ .$$

Proof. We can use the construction given in [2, Proof of Proposition 5.1]. Notice, that the assumption $K \leqslant K_{k,n,\alpha}$ of the Proposition is not used in the proof, and also the assumption $k = 1$ is also not important (the function $\exp_{[k]}$ is as well continuous as $\exp_{[1]}$). □

REMARK 6.7. We do not know much about normalized concentrating sequences in the case with $K = K_{k,n\alpha}$ and $S(K_{k,n,\alpha},\Phi) < \infty$. Of course, the modulars in the multiple exponential space cannot exceed $S(K_{k,n,\alpha},\Phi)$ and obviously $S(K_{k,n,\alpha},\Phi) > \exp_{[k]}(0)\mathscr{L}_n(\Omega)$. In the $W_0^{1,n}$-case (it is $k = 1$ and $\alpha = 0$) there is an interesting result by Carleson and Chang [1] telling us that

$$\limsup_{j\to\infty} ||\exp_{[1]}(K_{1,n,0}|u_j|^{\frac{n}{n-1}})||_{L^1(\Omega)} < S\Big(K_{1,n,0},t^n\Big)$$

for every normalized concentrating sequence and thus $S(K_{1,n,0},t^n)$ is attained.

Acknowledgment

The work is a part of the research project MSM 0021620839 financed by MŠMT.

The author would like to thank Stanislav Hencl for suggestions improving the readability of the paper.

The author would like to thank the referee for careful reading and for suggestions improving the readability of the paper.

REFERENCES

[1] L. CARLESON, S.-Y. A. CHANG, *On the existence of an extremal function for an inequality of J. Moser*, Bull. Sci. Math. (2), **110**, 2 (1986), 113–127.

[2] R. ČERNÝ, P. GURKA, S. HENCL, *Concentration-compactness principle for generalized Trudinger inequalities*, submitted (preprint is available at www.karlin.mff.cuni.cz/kma-preprints/).

[3] R. ČERNÝ, S. MAŠKOVÁ, *A sharp form of an embedding into multiple exponential spaces*, Czechoslovak Math. J., **60**, 3 (2010), 751–782.

[4] R. ČERNÝ, S. MAŠKOVÁ, *On generalization of Moser's theorem in the critical case*, Math. Inequal. Appl., **13**, 4 (2010), 785–802.

[5] A. CIANCHI, *A sharp embedding theorem for Orlicz-Sobolev spaces*, Indiana Univ. Math. J., **45** (1996), 39–65.

[6] D.E. EDMUNDS, P. GURKA, B. OPIC, *Double exponential integrability of convolution operators in generalized Lorentz-Zygmund spaces*, Indiana Univ. Math. J., **44** (1995), 19–43.

[7] D.E. EDMUNDS, P. GURKA, B. OPIC, *Double exponential integrability, Bessel potentials and embedding theorems*, Studia Math., **115** (1995), 151–181.

[8] D.E. EDMUNDS, P. GURKA, B. OPIC, *Sharpness of embeddings in logarithmic Bessel-potential spaces*, Proc. Roy. Soc. Edinburgh, **126A** (1996), 995–1009.

[9] D.E. EDMUNDS, P. GURKA, B. OPIC, *On embeddings of logarithmic Bessel potential spaces*, J. Functional Analysis, **146** (1997), 116–150.

[10] D.E. EDMUNDS, P. GURKA, B. OPIC, *Norms of embeddings in logarithmic Bessel-potential spaces*, Proc. Amer. Math. Soc., **126** (1998), 2417–2425.

[11] D.G. FIGUEIREDO, O. H. MIYAGAKI, B. RUF, *Elliptic equations in $\mathbb{R}^2$ with nonlinearities in the critical growth range*, Calc. Var., **3** (1995), 139–153.

[12] D.G. DE FIGUEIREDO, J. MARCOS DO Ó, B. RUF, *On an Inequality by N.Trudinger and J.Moser and Related Elliptic Equations*, Comm. Pure Appl. Math., **55** (2002), 135–152.

[13] D.E. EDMUNDS, M. KRBEC, *Two limiting cases of Sobolev imbeddings*, Houston J. Math., **21** (1995), 119–128.

[14] N. FUSCO, P. L. LIONS, C. SBORDONE, *Sobolev imbedding theorems in borderline cases*, Proc. Amer. Math. Soc., **124** (1996), 561–565.

[15] L. I. HEDBERG, *On certain convolution inequalities*, Proc. Amer. Math. Soc., **36** (1972), 505–512.

[16] S. HENCL, *A sharp form of an embedding into exponential and double exponential spaces*, J. Funct. Anal., **204** (2003), 196–227.

[17] P. L. LIONS, *The concentration-compactness principle in the calculus of variations. The limit case. I.*, Rev. Mat. Iberoamericana, **1**, 1 (1985), 145–201.

[18] J. MOSER, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J., **20** (1971), 1077–1092.

[19] J. MARCOS DO Ó, *Semilinear Dirichlet problems for the N-laplacian in $\mathbb{R}^n$ with nonlinearities in the critical growth range*, Differential Integral Equations, **9**, 5 (1996), 967–979.

[20] B. OPIC, L. PICK, *On generalized Lorentz-Zygmund spaces*, Math. Inequal. Appl., **2** (July 1999), 391–467.

[21] M. M. RAO, Z. D. REN, *Theory of Orlicz spaces*, Pure and applied mathematics, 1991.

[22] R. S. STRICHARTZ, *A note on Trudinger's extension of Sobolev's inequality*, Indiana Univ. Math. J., **21** (1972), 841–842.

[23] G. TALENTI, *Inequalities in rearrangement invariant function spaces*, Nonlinear Analysis, Function Spaces and Applications **5**, Prometheus Publ. House Prague, 1994, 177–230.

[24] K. TINTAREV, K. H. FIESELER, *Concentration compactness*, Imperial College Press, London, 2007.
[25] N. S. TRUDINGER, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech., **17** (1967), 473–484.
[26] V. I. YUDOVICH, *Some estimates connected with integral operators and with solutions of elliptic equations*, Soviet Math. Doklady, **2** (1961), 746–749.

(Received January 10, 2010)

Robert Černý
Charles University
Faculty of Mathematics and Physics
Department of Mathematical Analysis
Sokolovská 83
186 75 Praha 8
Czech Republic
e-mail: rcerny@karlin.mff.cuni.cz