

HYPONORMAL TOEPLITZ OPERATORS ON THE WEIGHTED BERGMAN SPACES

IN SUNG HWANG AND JONGRAK LEE

(Communicated by J. Pečarić)

Abstract. In this note we consider the hyponormality of Toeplitz operators T_φ on the Weighted Bergman space $A_\alpha^2(\mathbb{D})$ with symbol in the class of functions $f + \bar{g}$ with polynomials f and g of degree 2.

1. Introduction

Let \mathbb{D} be the open unit disk in the complex plane. The normalized area measure on \mathbb{D} will be denoted by dA . For $-1 < \alpha < \infty$, the weighted Bergman space $A_\alpha^2(\mathbb{D})$ of the disk is the space of analytic functions in $L^2(\mathbb{D}, dA_\alpha)$, where

$$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z).$$

The space $L^2(\mathbb{D}, dA_\alpha)$ is a Hilbert space with the inner product

$$\langle f, g \rangle_\alpha = \int_{\mathbb{D}} f(z) \overline{g(z)} dA_\alpha(z) \quad (f, g \in L^2(\mathbb{D}, dA_\alpha)).$$

If $\alpha = 0$ then $A_0^2(\mathbb{D})$ is the Bergman space $A^2(\mathbb{D})$. For any nonnegative integer n , let

$$e_n(z) = \sqrt{\frac{\Gamma(n + \alpha + 2)}{\Gamma(n + 1)\Gamma(\alpha + 2)}} z^n \quad (z \in \mathbb{D}),$$

where $\Gamma(s)$ stand for the usual Gamma function. It is easy to check that $\{e_n\}$ is an orthonormal basis for $A_\alpha^2(\mathbb{D})$ ([10]). The reproducing kernel in $A_\alpha^2(\mathbb{D})$ is given by

$$K_z^{(\alpha)}(\omega) = \frac{1}{(1 - \bar{z}\omega)^{2+\alpha}},$$

for $z, \omega \in \mathbb{D}$. For $\varphi \in L^\infty(\mathbb{D})$, the multiplication operator M_φ on $A_\alpha^2(\mathbb{D})$ is defined by $M_\varphi(f) = \varphi \cdot f$. The orthogonal projection P_α of $L^2(\mathbb{D}, dA_\alpha)$ onto $A_\alpha^2(\mathbb{D})$ is given by

$$(P_\alpha f)(z) = \langle f, k_z^{(\alpha)} \rangle_\alpha = \int_{\mathbb{D}} \frac{f(\omega)}{(1 - \bar{z}\omega)^{2+\alpha}} dA_\alpha(\omega),$$

Mathematics subject classification (2010): 47B20, 47B35.

Keywords and phrases: Toeplitz operators; hyponormal; weighted Bergman space.

This work was supported by National Research Foundation of Korea Grant funded by the Korean Government(2009-0075890).

for $f \in L^2(\mathbb{D}, dA_\alpha)$. For $\varphi \in L^\infty(\mathbb{D})$, the Toeplitz operator T_φ with symbol φ is defined on $A^2_\alpha(\mathbb{D})$ by $T_\varphi f = P_\alpha(\varphi \cdot f)$. We thus have

$$(T_\varphi f)(z) = \int_{\mathbb{D}} \frac{\varphi(\omega)f(\omega)}{(1 - z\bar{\omega})^{2+\alpha}} dA_\alpha(\omega),$$

for $f \in A^2_\alpha(\mathbb{D})$ and $\omega \in \mathbb{D}$. It is clear that those operators are bounded if φ is in $L^\infty(\mathbb{D})$. The Hankel operator on $A^2_\alpha(\mathbb{D})$ is defined by $H_\varphi f = (I - P_\alpha)(\varphi \cdot f)$.

A bounded linear operator A on a Hilbert space is said to be hyponormal if its selfcommutator $[A^*, A] := A^*A - AA^*$ is positive semidefinite. The hyponormality of Toeplitz operators on the Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T} = \partial\mathbb{D}$ has been studied by C. Cowen [1], R.E. Curto and W.Y. Lee [2], [3], [4] and others. Very recently, in [6] and [7], the hyponormality of T_φ on the weighted Bergman space $A^2_\alpha(\mathbb{D})$ was studied. In [1], Cowen characterized the hyponormality of Toeplitz operator T_φ on $H^2(\mathbb{T})$ by properties of the symbol $\varphi \in L^\infty(\mathbb{T})$. Here we shall employ an equivalent variant of Cowen's theorem that was first proposed by T. Nakazi and K. Takahashi [8].

COWEN'S THEOREM ([1],[8]). For $\varphi \in L^\infty(\mathbb{T})$, write

$$\mathcal{E}(\varphi) := \{k \in H^\infty : \|k\|_\infty \leq 1 \text{ and } \varphi - k\bar{\varphi} \in H^\infty(\mathbb{T})\}.$$

Then T_φ is hyponormal if and only if $\mathcal{E}(\varphi)$ is nonempty.

The solution is based on a dilation theorem of Sarason [9]. For the weighted Bergman space, no dilation theorem(similar to Sarason's theorem) is available. The first named author characterized the hyponormality of T_φ on $A^2(\mathbb{D})$ in terms of the coefficients of the trigonometric polynomial φ under certain assumptions about the coefficients of φ ([5]).

THEOREM A ([5]). Let $\varphi(z) = \overline{g(z)} + f(z)$, where $f(z) = a_m z^m + a_N z^N$, $g(z) = a_{-m} z^m + a_{-N} z^N$ ($0 < m < N$). If $a_m \bar{a}_N = a_{-m} \bar{a}_{-N}$ then

T_φ on $A^2(\mathbb{D})$ is hyponormal

$$\iff \begin{cases} \frac{1}{N+1}(|a_N|^2 - |a_{-N}|^2) \geq \frac{1}{m+1}(|a_m|^2 - |a_{-m}|^2) & \text{if } |a_{-N}| \leq |a_N| \\ N^2(|a_{-N}|^2 - |a_N|^2) \leq m^2(|a_m|^2 - |a_{-m}|^2) & \text{if } |a_N| \leq |a_{-N}|. \end{cases}$$

In this note we consider the hyponormality of Toeplitz operators T_φ on $A^2_\alpha(\mathbb{D})$ with symbol in the class of functions $f + \bar{g}$ with polynomials f and g of degree 2. The hyponormality of T_φ is independent of the constant term $\varphi(0)$. Thus whenever we consider the hyponormality of T_φ we may assume that $\varphi(0) = 0$. The following relations can be easily proved:

$$T_{\varphi+\psi} = T_\varphi + T_\psi \quad (\varphi, \psi \in L^\infty); \tag{1.1}$$

$$T_\varphi^* = T_{\bar{\varphi}} \quad (\varphi \in L^\infty); \tag{1.2}$$

$$T_{\bar{\varphi}} T_\psi = T_{\bar{\varphi}\psi} \text{ if } \varphi \text{ or } \psi \text{ is analytic.} \tag{1.3}$$

The purpose of this paper is to prove the *weighted Bergman space* version of Theorem A for the Toeplitz operators on $A^2_\alpha(\mathbb{D})$ when $\alpha \geq 0$. Our main theorem is as follows.

THEOREM 1. Let $\varphi(z) = \overline{g(z)} + f(z)$, where $f(z) = a_1z + a_2z^2$ and $g(z) = a_{-1}z + a_{-2}z^2$. If $a_1\overline{a_2} = a_{-1}\overline{a_{-2}}$ and $\alpha \geq 0$, then T_φ on $A_\alpha^2(\mathbb{D})$ is hyponormal

$$\iff \begin{cases} \frac{1}{\alpha+3}(|a_2|^2 - |a_{-2}|^2) \geq \frac{1}{2}(|a_{-1}|^2 - |a_1|^2) & \text{if } |a_{-2}| \leq |a_2| \\ 4(|a_{-2}|^2 - |a_2|^2) \leq |a_1|^2 - |a_{-1}|^2 & \text{if } |a_2| \leq |a_{-2}|. \end{cases}$$

2. The proof of the main theorem

We need several auxiliary lemmas to prove Theorem 1. We begin with:

LEMMA 1. For any s, t nonnegative integers,

$$P_\alpha(\overline{z^s} z^s) = \begin{cases} \frac{\Gamma(s+1)\Gamma(s-t+\alpha+2)}{\Gamma(s+\alpha+2)\Gamma(s-t+1)} z^{s-t} & \text{if } s \geq t \\ 0 & \text{if } s < t. \end{cases}$$

Proof. Let m be a nonnegative integer. A direct calculation shows that

$$\langle P_\alpha(\overline{z^s} z^s), z^m \rangle_\alpha = 0 \quad \text{if } m \neq s - t.$$

If $s \geq t$, then

$$\langle P_\alpha(\overline{z^s} z^s), z^{s-t} \rangle_\alpha = \|z^s\|_\alpha^2 = \frac{\Gamma(s+1)\Gamma(\alpha+2)}{\Gamma(s+\alpha+2)}.$$

Thus we have that

$$\begin{aligned} P_\alpha(\overline{z^s} z^s) &= \frac{\Gamma(s-t+\alpha+2)}{\Gamma(s-t+1)\Gamma(\alpha+2)} \cdot \frac{\Gamma(s+1)\Gamma(\alpha+2)}{\Gamma(s+\alpha+2)} z^{s-t} \\ &= \frac{\Gamma(s+1)\Gamma(s-t+\alpha+2)}{\Gamma(s+\alpha+2)\Gamma(s-t+1)} z^{s-t}, \end{aligned}$$

which gives the result. \square

Write

$$k_i(z) := \sum_{n=0}^\infty c_{2n+i} z^{2n+i} \quad (i = 0, 1)$$

We then have:

LEMMA 2. For $0 \leq m \leq 2$ and $i = 0, 1$, we have

$$\begin{aligned} \text{(i)} \quad \|\overline{z^m} k_i(z)\|_\alpha^2 &= \sum_{n=0}^\infty \frac{(2n+m+i)! \Gamma(\alpha+2)}{\Gamma(2n+m+i+\alpha+2)} |c_{2n+i}|^2 \\ \text{(ii)} \quad \|P_\alpha(\overline{z^m} k_i(z))\|_\alpha^2 &= \begin{cases} \sum_{n=0}^\infty \frac{(2n+i)!^2 \Gamma(2n+i-m+\alpha+2) \Gamma(\alpha+2)}{\Gamma(2n+i+\alpha+2)^2 \Gamma(2n+i-m+1)} |c_{2n+i}|^2 & \text{if } m \leq i \\ \sum_{n=1}^\infty \frac{(2n+i)!^2 \Gamma(2n+i-m+\alpha+2) \Gamma(\alpha+2)}{\Gamma(2n+i+\alpha+2)^2 \Gamma(2n+i-m+1)} |c_{2n+i}|^2 & \text{if } m > i. \end{cases} \end{aligned}$$

Proof. Let $0 \leq m \leq 2$. Then we have

$$\begin{aligned} \|\bar{z}^m k_i(z)\|_\alpha^2 &= \left\| \sum_{n=0}^\infty c_{2n+i} z^{2n+i+m} \right\|_\alpha^2 \\ &= \sum_{n=0}^\infty |c_{2n+i}|^2 \|z^{2n+i+m}\|_\alpha^2 \\ &= \sum_{n=0}^\infty \frac{(2n+m+i)\Gamma(\alpha+2)}{\Gamma(2n+m+i+\alpha+2)} |c_{2n+i}|^2. \end{aligned}$$

This proves the equation (i). For the equation (ii) if $m \leq i$ then by Lemma 1 we have

$$\begin{aligned} \|P_\alpha(\bar{z}^m k_i(z))\|_\alpha^2 &= \left\| \sum_{n=0}^\infty \frac{(2n+i)\Gamma(2n+i-m+\alpha+2)}{\Gamma(2n+i+\alpha+2)\Gamma(2n+i-m+1)} c_{2n+i} z^{2n+i-m} \right\|_\alpha^2 \\ &= \sum_{n=0}^\infty \frac{(2n+i)!^2 \Gamma(2n+i-m+\alpha+2)\Gamma(\alpha+2)}{\Gamma(2n+i+\alpha+2)^2 \Gamma(2n+i-m+1)} |c_{2n+i}|^2. \end{aligned}$$

If instead $m > i$, a similar argument gives the result. \square

LEMMA 3. Let $f(z) = a_1z + a_2z^2$ and $g(z) = a_{-1}z + a_{-2}z^2$. If $a_1\bar{a}_2 = a_{-1}\bar{a}_{-2}$, then

$$\langle H_{\bar{f}}k_0(z), H_{\bar{f}}k_1(z) \rangle_\alpha = \langle H_{\bar{g}}k_0(z), H_{\bar{g}}k_1(z) \rangle_\alpha.$$

Proof. Observe that

$$M_{\bar{f}}k_0(z) = \bar{a}_1 \sum_{n=0}^\infty c_{2n} z^{2n\bar{z}} + \bar{a}_2 \sum_{n=0}^\infty c_{2n} z^{2n\bar{z}^2},$$

and

$$M_{\bar{f}}k_1(z) = \bar{a}_1 \sum_{n=0}^\infty c_{2n+1} z^{2n+1\bar{z}} + \bar{a}_2 \sum_{n=0}^\infty c_{2n+1} z^{2n+1\bar{z}^2},$$

and

$$\left\langle \sum_{n=0}^\infty c_{2n} z^{2n\bar{z}}, \sum_{n=0}^\infty c_{2n+1} z^{2n+1\bar{z}} \right\rangle_\alpha = \left\langle \sum_{n=0}^\infty c_{2n} z^{2n\bar{z}^2}, \sum_{n=0}^\infty c_{2n+1} z^{2n+1\bar{z}^2} \right\rangle_\alpha = 0,$$

which implies that

$$\begin{aligned} \langle M_{\bar{f}}k_0(z), M_{\bar{f}}k_1(z) \rangle_\alpha &= \bar{a}_1 \bar{a}_2 \left\langle \sum_{n=0}^\infty c_{2n} z^{2n\bar{z}}, \sum_{n=0}^\infty c_{2n+1} z^{2n+1\bar{z}^2} \right\rangle_\alpha \\ &\quad + \bar{a}_1 \bar{a}_2 \left\langle \sum_{n=0}^\infty c_{2n} z^{2n\bar{z}^2}, \sum_{n=0}^\infty c_{2n+1} z^{2n+1\bar{z}} \right\rangle_\alpha. \end{aligned} \tag{2.1}$$

Similarly,

$$\begin{aligned} \left\langle M_{\bar{g}}k_0(z), M_{\bar{g}}k_1(z) \right\rangle_{\alpha} &= \overline{a_{-1}}a_{-2} \left\langle \sum_{n=0}^{\infty} c_{2n}z^{2n}\bar{z}, \sum_{n=0}^{\infty} c_{2n+1}z^{2n+1}\bar{z}^2 \right\rangle_{\alpha} \\ &\quad + a_{-1}\overline{a_{-2}} \left\langle \sum_{n=0}^{\infty} c_{2n}z^{2n}\bar{z}^2, \sum_{n=0}^{\infty} c_{2n+1}z^{2n+1}\bar{z} \right\rangle_{\alpha}. \end{aligned} \tag{2.2}$$

Combining (2.1), (2.2) and the assumption $a_1\overline{a_2} = a_{-1}\overline{a_{-2}}$, we get

$$\left\langle M_{\bar{f}}k_0(z), M_{\bar{f}}k_1(z) \right\rangle_{\alpha} = \left\langle M_{\bar{g}}k_0(z), M_{\bar{g}}k_1(z) \right\rangle_{\alpha}. \tag{2.3}$$

On the other hand, it follows from Lemma 2 that

$$\left\langle P_{\alpha}(\bar{z}^p k_0(z)), P_{\alpha}(\bar{z}^p k_1(z)) \right\rangle_{\alpha} = 0 \text{ for all } p = 0, 1, 2, \dots,$$

so that

$$\begin{aligned} \left\langle T_{\bar{f}}k_0(z), T_{\bar{f}}k_1(z) \right\rangle_{\alpha} &= \left\langle \overline{a_1}P_{\alpha}(\bar{z}k_0(z)) + \overline{a_2}P_{\alpha}(\bar{z}^2k_0(z)), \overline{a_1}P_{\alpha}(\bar{z}k_1(z)) + \overline{a_2}P_{\alpha}(\bar{z}^2k_1(z)) \right\rangle_{\alpha} \\ &= \overline{a_1}a_2 \left\langle P_{\alpha}(\bar{z}^1k_0(z)), P_{\alpha}(\bar{z}^2k_1(z)) \right\rangle_{\alpha} \\ &\quad + a_1\overline{a_2} \left\langle P_{\alpha}(\bar{z}^2k_0(z)), P_{\alpha}(\bar{z}k_1(z)) \right\rangle_{\alpha}. \end{aligned}$$

Similarly, we also have that

$$\begin{aligned} \left\langle T_{\bar{g}}k_0(z), T_{\bar{g}}k_1(z) \right\rangle_{\alpha} &= \overline{a_{-1}}a_{-2} \left\langle P_{\alpha}(\bar{z}k_0(z)), P_{\alpha}(\bar{z}^2k_1(z)) \right\rangle_{\alpha} \\ &\quad + a_{-1}\overline{a_{-2}} \left\langle P_{\alpha}(\bar{z}^2k_0(z)), P_{\alpha}(\bar{z}k_1(z)) \right\rangle_{\alpha}. \end{aligned}$$

Hence, again by assumption $a_1\overline{a_2} = a_{-1}\overline{a_{-2}}$, we get

$$\left\langle T_{\bar{f}}k_0(z), T_{\bar{f}}k_1(z) \right\rangle_{\alpha} = \left\langle T_{\bar{g}}k_0(z), T_{\bar{g}}k_1(z) \right\rangle_{\alpha}. \tag{2.4}$$

Combining (2.3) and (2.4) it follows that

$$\begin{aligned} \left\langle H_{\bar{f}}k_0(z), H_{\bar{f}}k_1(z) \right\rangle_{\alpha} &= \left\langle M_{\bar{f}}k_0(z), M_{\bar{f}}k_1(z) \right\rangle_{\alpha} - \left\langle T_{\bar{f}}k_0(z), T_{\bar{f}}k_1(z) \right\rangle_{\alpha} \\ &= \left\langle H_{\bar{g}}k_0(z), H_{\bar{g}}k_1(z) \right\rangle_{\alpha}. \end{aligned}$$

This completes the proof. \square

We are ready for:

Proof of Theorem 1. Put $K_i := \{k_i(z) \in A_{\alpha}^2(\mathbb{D}) : k_i(z) = \sum_{n=0}^{\infty} c_{2n+i}z^{2n+i}\}$ for $i = 0, 1$. Then it is easy to see that T_{φ} is hyponormal if and only if

$$\left\langle (H_{\bar{f}}^*H_{\bar{f}} - H_{\bar{g}}^*H_{\bar{g}})(k_0(z) + k_1(z)), (k_0(z) + k_1(z)) \right\rangle_{\alpha} \geq 0 \text{ for all } k_i \in K_i \ (i = 0, 1). \tag{2.5}$$

Also we have that

$$\begin{aligned} & \left\langle H_{\mathcal{F}}^* H_{\mathcal{F}}(k_0(z) + k_1(z)), (k_0(z) + k_1(z)) \right\rangle_{\alpha} \\ &= \left\langle H_{\mathcal{F}} k_0(z), H_{\mathcal{F}} k_0(z) \right\rangle_{\alpha} + \left\langle H_{\mathcal{F}} k_0(z), H_{\mathcal{F}} k_1(z) \right\rangle_{\alpha} \\ & \quad + \left\langle H_{\mathcal{F}} k_1(z), H_{\mathcal{F}} k_0(z) \right\rangle_{\alpha} + \left\langle H_{\mathcal{F}} k_1(z), H_{\mathcal{F}} k_1(z) \right\rangle_{\alpha}, \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} & \left\langle H_{\mathcal{G}}^* H_{\mathcal{G}}(k_0(z) + k_1(z)), (k_0(z) + k_1(z)) \right\rangle_{\alpha} \\ &= \left\langle H_{\mathcal{G}} k_0(z), H_{\mathcal{G}} k_0(z) \right\rangle_{\alpha} + \left\langle H_{\mathcal{G}} k_0(z), H_{\mathcal{G}} k_1(z) \right\rangle_{\alpha} \\ & \quad + \left\langle H_{\mathcal{G}} k_1(z), H_{\mathcal{G}} k_0(z) \right\rangle_{\alpha} + \left\langle H_{\mathcal{G}} k_1(z), H_{\mathcal{G}} k_1(z) \right\rangle_{\alpha}. \end{aligned} \tag{2.7}$$

Substituting (2.6) and (2.7) into (2.5), it follows from Lemma 3 that

$$\begin{aligned} T_{\varphi} : \text{hyponormal} & \iff \left\langle (H_{\mathcal{F}}^* H_{\mathcal{F}} - H_{\mathcal{G}}^* H_{\mathcal{G}})(k_0(z) + k_1(z)), (k_0(z) + k_1(z)) \right\rangle_{\alpha} \geq 0 \\ & \iff \sum_{i=0}^1 \left(\|\overline{f}k_i\|_{\alpha}^2 - \|\overline{g}k_i\|_{\alpha}^2 + \|P_{\alpha}(\overline{g}k_i)\|_{\alpha}^2 - \|P_{\alpha}(\overline{f}k_i)\|_{\alpha}^2 \right) \geq 0. \end{aligned}$$

Therefore it follows from Lemma 2 that T_{φ} is hyponormal if and only if

$$\begin{aligned} & (|a_1|^2 - |a_{-1}|^2) \left\{ \frac{1}{\Gamma(\alpha+3)} |c_0|^2 + \sum_{n=1}^{\infty} \left(\frac{\Gamma(2n+2)}{\Gamma(2n+\alpha+3)} - \frac{(2n)^2 \Gamma(2n)}{(2n+\alpha+1)^2 \Gamma(2n+\alpha+1)} \right) |c_{2n}|^2 \right. \\ & \quad \left. + \sum_{n=0}^{\infty} \left(\frac{\Gamma(2n+3)}{\Gamma(2n+\alpha+4)} - \frac{(2n+1)^2 \Gamma(2n+1)}{(2n+\alpha+2)^2 \Gamma(2n+\alpha+2)} \right) |c_{2n+1}|^2 \right\} \\ & + (|a_2|^2 - |a_{-2}|^2) \left\{ \frac{2}{\Gamma(\alpha+4)} |c_0|^2 + \sum_{n=1}^{\infty} \left(\frac{\Gamma(2n+3)}{\Gamma(2n+\alpha+4)} - \frac{(2n-1)^2 (2n)^2 \Gamma(2n-1)}{(2n+\alpha+1)^2 (2n+\alpha)^2 \Gamma(2n+\alpha)} \right) \right. \\ & \quad \times |c_{2n}|^2 + \frac{6}{\Gamma(\alpha+5)} |c_1|^2 + \sum_{n=1}^{\infty} \left(\frac{\Gamma(2n+4)}{\Gamma(2n+\alpha+5)} - \frac{(2n)^2 (2n+1)^2 \Gamma(2n)}{(2n+\alpha+1)^2 (2n+\alpha+2)^2 \Gamma(2n+\alpha+1)} \right) \\ & \quad \left. \times |c_{2n+1}|^2 \right\} \geq 0, \end{aligned}$$

or equivalently

$$\begin{aligned}
 & (|a_1|^2 - |a_{-1}|^2) \left\{ \frac{1}{\Gamma(\alpha+3)} |c_0|^2 + \sum_{n=1}^{\infty} \left(\frac{\Gamma(n+2)}{\Gamma(n+\alpha+3)} - \frac{n^2\Gamma(n)}{(n+\alpha+1)^2\Gamma(n+\alpha+1)} \right) |c_n|^2 \right\} \\
 & + (|a_2|^2 - |a_{-2}|^2) \left\{ \sum_{n=0}^1 \frac{\Gamma(n+3)}{\Gamma(n+\alpha+4)} |c_n|^2 \right. \\
 & \left. + \sum_{n=2}^{\infty} \left(\frac{\Gamma(n+3)}{\Gamma(n+\alpha+4)} - \frac{(n-1)^2n^2\Gamma(n-1)}{(n+\alpha)^2(n+\alpha+1)^2\Gamma(n+\alpha)} \right) |c_n|^2 \right\} \geq 0.
 \end{aligned} \tag{2.8}$$

For $n \in \mathbb{N}$, define ζ_α by

$$\zeta_\alpha(n) := \frac{\frac{\Gamma(n+2)}{\Gamma(n+\alpha+3)} - \frac{n^2\Gamma(n)}{(n+\alpha+1)^2\Gamma(n+\alpha+1)}}{\frac{\Gamma(n+3)}{\Gamma(n+\alpha+4)} - \frac{(n-1)^2n^2\Gamma(n-1)}{(n+\alpha)^2(n+\alpha+1)^2\Gamma(n+\alpha)}}.$$

A direct calculation gives that

$$\zeta_\alpha(n) = \frac{(n+\alpha)(n+\alpha+3)}{4n^2 + 4(\alpha+2)n + 2\alpha}.$$

Write

$$\zeta_\alpha(x) := \frac{(x+\alpha)(x+\alpha+3)}{4x^2 + 4(\alpha+2)x + 2\alpha} \quad (x \in \mathbb{R}^+).$$

Then we have that

$$\zeta'_\alpha(x) = -\frac{(\alpha+1)x^2 + (2\alpha^2 + 5\alpha)x + (\alpha^3 + 4\alpha^2 + \frac{9}{2}\alpha)}{(2x^2 + 2(\alpha+2)x + \alpha)^2} < 0,$$

which implies that $\zeta_\alpha(n)$ is strictly decreasing function. Let $|a_{-2}| \leq |a_2|$. Observe that

$$\lim_{n \rightarrow \infty} \zeta_\alpha(n) = \frac{1}{4} \tag{2.9}$$

and

$$\frac{\alpha+3}{2} \geq \frac{(\alpha+1)(\alpha+4)}{6(\alpha+2)} \geq \zeta_\alpha(2). \tag{2.10}$$

Therefore (2.8) and (2.10) give that T_φ is hyponormal if and only if

$$2(|a_2|^2 - |a_{-2}|^2) \geq (\alpha+3)(|a_{-1}|^2 - |a_1|^2).$$

If instead $|a_2| \leq |a_{-2}|$, since $\zeta_\alpha(n) \geq \frac{1}{4}$, it follows from (2.8), (2.9) and (2.10) that T_φ is hyponormal if and only if

$$4(|a_{-2}|^2 - |a_2|^2) \leq (|a_1|^2 - |a_{-1}|^2).$$

This completes the proof. \square

EXAMPLE 1. Consider the polynomial

$$\varphi(z) = 4\bar{z}^2 + 2\bar{z} + 5z^2.$$

A simple computation gives that

$$2(|a_2|^2 - |a_{-2}|^2) \geq (\alpha + 3)(|a_{-1}|^2 - |a_1|^2).$$

Observe that

$$\langle [T_\varphi^*, T_\varphi](1+z), (1+z) \rangle = -\frac{29}{12} < 0$$

which says that T_φ is not hyponormal. Note that $a_1\bar{a}_2 \neq a_{-1}\bar{a}_{-2}$. Thus this example shows that if the hypothesis $a_1\bar{a}_2 = a_{-1}\bar{a}_{-2}$ is not satisfied then Theorem 1 is not true.

REFERENCES

- [1] C. COWEN, *Hyponormality of Toeplitz operators*, Proc. Amer. Math. Soc., **103** (1988), 809–812.
- [2] R.E. CURTO AND W.Y. LEE, *Joint hyponormality of Toeplitz pairs*, Memoirs Amer. Math. Soc. **150** (2001), 712.
- [3] I.S. HWANG, I.H. KIM AND W.Y. LEE, *Hyponormality of Toeplitz operators with polynomial symbol*, Math. Ann., **313** (1999), 247–261.
- [4] I.S. HWANG AND W.Y. LEE, *Hyponormality of trigonometric Toeplitz operators*, Trans. Amer. Math., **354** (2002), 2461–2474.
- [5] I.S. HWANG, *Hyponormal Toeplitz operators on the Bergman space*, J. Korean Math. Soc., **42** (2005), 387–403.
- [6] YUFENG LU AND CHAOMEI LIU, *Commutativity and hyponormality of Toeplitz operators on the weighted Bergman space*, J. Korean Math. Soc., **46** (2009), 621–642.
- [7] YUFENG LU AND YANYUE SHI, *Hyponormal Toeplitz operators on the weighted Bergman space*, Integral Equations Operator Theory, **65** (2009), no 1, 115–129.
- [8] T. NAKAZI AND K. TAKAHASHI, *Hyponormal Toeplitz operators and extremal problems of Hardy spaces*, Trans. Amer. Math. Soc., **338** (1993), 753–769.
- [9] D. SARASON, *Generalized interpolation in H^∞* , Trans. Amer. Math. Soc., **127** (1967), 179–203.
- [10] K. ZHU, *Theory of Bergman spaces*, Springer-verlag, New York, 2000.

(Received August 24, 2010)

In Sung Hwang
Department of Mathematics
Sungkyunkwan University
Suwon 440-746, Korea
e-mail: ihwang@skku.edu

Jongrak Lee
Department of Mathematics
Sungkyunkwan University
Suwon 440-746, Korea,
e-mail: jjonglak@skku.ac.kr