Mathematical
Inequalities
& Applications
Volume 15, Number 2 (2012), 331–341
doi:10.7153/mia-15-27

SOME NEW INTEGRAL INEQUALITIES ON TIME SCALES

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(*Communicated by M. Bohner*)

Abstract. By introducing two adjusting parameters, we investigate some new nonlinear integral inequalities on time scales, which provide explicit bounds on unknown functions, and can be used as tools in the qualitative theory of certain classes of dynamic equations on time scales.

1. Introduction

Following Hilger's landmark paper [1], there have been plenty of references focused on theory of time scales in order to unify continuous and discrete analysis, where a time scale is an arbitrary closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, e.g., $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ for $q > 1$ (which has important applications in quantum theory), $\mathbb{T} = h\mathbb{N}$ with $h > 0$, $\mathbb{T} = \mathbb{N}^2$ and $\mathbb{T} = \mathbb{T}_n$ the space of the harmonic numbers.

Recently, many authors have extended some continuous and discrete integral inequalities to arbitrary time scales. For example, see [2-14] and the references cited therein. However, all the integral inequalities in the literature only involved the unknown function $x(t)$ but not $x^{\sigma}(t)$, where σ is the forward jump operator defined in the next section. Thus, the results in the literature can not be applied to the qualitative theory of the dynamic equation of the form $x^{\Delta}(t) = f(t, x(t), x^{\sigma}(t))$.

The purpose of this paper is to investigate some nonlinear integral inequalities on time scales that involves both the unknown functions $x(t)$ and $x^{\sigma}(t)$. By introducing two adjusting parameters α and β, we first generalize a basic inequality that plays a fundamental role in the proofs of some existing results in [6, 9, 17, 18]. Then we provide explicit bounds on unknown functions for a class of nonlinear integral inequalities involving $x^{\sigma}(t)$, which can be used as tools in the qualitative theory of certain classes of dynamic equations on time scales such as $x^{\Delta}(t) = f(t, x(t), x^{\sigma}(t))$.

Mathematics subject classification (2010): 26D15, 26E70.
Keywords and phrases: Integral inequality, time scale, adjusting parameters.
This paper was supported by the natural science foundation of Shandong Province under the grant ZR2010AL002.

Paper MIA-15-27

2. Time scale essentials

The definitions below merely serve as a preliminary introduction to the time-scale calculus; they can be found in the context of a much more robust treatment than is allowed here in the text [15] and [16] and the references therein.

DEFINITION 1. Define the forward (backward) jump operator $\sigma(t)$ at t for $t < \sup\mathbb{T}$ (respectively $\rho(t)$ at t for $t > \inf\mathbb{T}$) by

$$\sigma(t) = \inf\{s > t : s \in \mathbb{T}\}, \quad (\rho(t) = \sup\{s < t : t \in \mathbb{T}\}), \quad t \in \mathbb{T}.$$

Also define $\sigma(\sup\mathbb{T}) = \sup\mathbb{T}$, if $\sup\mathbb{T} < \infty$, and $\rho(\inf\mathbb{T}) = \inf\mathbb{T}$, if $\inf\mathbb{T} > -\infty$. The graininess functions are given by $\mu(t) = \sigma(t) - t$ and $\nu(t) = t - \rho(t)$. The set $\mathbb{T}^\kappa$ is derived from $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

Throughout this paper, the assumption is made that $\mathbb{T}$ inherits from the standard topology on the real numbers $\mathbb{R}$. The jump operators σ and ρ allow the classification of points in a time scale in the following way: If $\sigma(t) > t$ the point t is right-scattered, while if $\rho(t) < t$ then t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. If $t < \sup\mathbb{T}$ and $\sigma(t) = (t)$ the point t is right-dense; if $t > \inf\mathbb{T}$ and $\rho(t) = t$ then t is left-dense. Points that are right-dense and left-dense at the same time are called dense. The composition $f \circ \sigma$ is often denoted f^σ.

DEFINITION 2. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous (denoted $f \in C_{rd}(\mathbb{T}, \mathbb{R})$) if it is continuous at each right-dense point and if there exists a finite left limit in all left-dense points.

Every right-dense continuous function has a delta antiderivative [15, Theorem 1.74]. This implies that the delta definite integral of any right-dense continuous function exists. Likewise every left-dense continuous function f on the time scale, denoted $f \in C_{ld}(\mathbb{T}, \mathbb{R})$, has a nabla antiderivative [15, Theorem 8.45]

DEFINITION 3. Fix $t \in \mathbb{T}$ and let $y : \mathbb{T}^\kappa \to \mathbb{R}$. Define $y^\Delta(t)$ to be the number (if it exists) with the property that given $\varepsilon > 0$ there is a neighborhood U of t such that, for all $s \in U$,

$$|[y(\sigma(t)) - y(s)] - y^\Delta(t)[\sigma(t) - s]| \leqslant \varepsilon|\sigma(t) - s|.$$

Call $y^\Delta(t)$ the (delta) derivative of y at t. It is easy to see that f^Δ is the usual derivative f' for $\mathbb{T} = \mathbb{R}$ and the usual forward difference Δf for $\mathbb{T} = \mathbb{Z}$.

DEFINITION 4. If $F^\Delta(t) = f(t)$ then define the (Cauchy) delta integral by

$$\int_a^b f(s)\Delta s = F(b) - F(a).$$

DEFINITION 5. Say $p:\mathbb{T}\to\mathbb{R}$ is regressive provided that $1+\mu(t)p(t)\neq 0$ for all $t\in\mathbb{T}$. For $h>0$, define the cylinder transformation $\xi_h:\mathbb{C}_h\to\mathbb{Z}_h$ by $\xi_h(z)=\frac{1}{h}\mathrm{Log}(1+zh)$, where Log is the principal logarithm function, $\mathbb{C}_h=\{z\in\mathbb{C}:z\neq -1/h\}$, and $\mathbb{Z}_h=\{z\in\mathbb{C}:-\pi/h<\mathrm{Im}(z)\leqslant\pi/h\}$. For $h=0$, define $\xi_0(z)=z$. Define the exponential function by

$$e_p(t,s)=\exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right),\quad t,s\in\mathbb{T}.$$

3. Problem statements

In the sequel, we always assume that $0<\lambda<1$ is a constant, $\mathbb{T}$ is a time scale with $t_0\in\mathbb{T}$. The following nonlinear integral inequalities on time scales will be considered.

$$x(t)\leqslant a(t)+b(t)\int_{t_0}^t [g(s)x(s)+h(s)(x^\sigma(s))^\lambda]\Delta s,\quad t\in\mathbb{T}^\kappa, \tag{I}$$

$$x(t)\leqslant a(t)+b(t)\int_{t_0}^t w(t,s)[g(s)x(s)+h(s)(x^\sigma(s))^\lambda]\Delta s,\quad t\in\mathbb{T}^\kappa, \tag{II}$$

$$x(t)\leqslant a(t)+b(t)\int_{t_0}^t f(s,(x^\sigma(s))^\lambda)\Delta s,\quad t\in\mathbb{T}^\kappa, \tag{III}$$

where $a,b,g,h,x:\mathbb{T}^\kappa\to\mathbb{R}_+=[0,\infty)$ are rd-continuous functions, $w:\mathbb{T}\times\mathbb{T}^\kappa\to\mathbb{R}_+$ is continuous, and $f:\mathbb{T}^\kappa\to\mathbb{R}_+$ is continuous.

If we let $x(t)=u^p(t)$, $\sigma(t)=t$ and $\lambda p=q$, then inequalities (I)-(III) reduce to those inequalities studied in [6]. We say inequalities (I)-(III) are sublinear since $0<\lambda<1$.

The reason for studying inequalities of type (I)-(III) is that sometimes we may need to estimate the solutions of the following dynamic equations

$$x^\Delta(t)=f(t,(x^\sigma(t))^\lambda) \tag{1}$$

and

$$x^{\Delta\Delta}(t)=g(t)x(t)+h(t)(x^\sigma(t))^\lambda \tag{2}$$

with the initial condition $x(t_0)=x_0$ and $x^\Delta(t_0)=x_0'$. Integrating (1) and (2) from t_0 to t, we get

$$x(t)=x_0+\int_{t_0}^t f(s,(x^\sigma(s))^\lambda)\Delta s \tag{3}$$

and

$$x(t)=x_0+x_0'(t-t_0)+\int_{t_0}^t (t-s)[g(s)x(s)+h(s)(x^\sigma(s))^\lambda]\Delta s. \tag{4}$$

It is obvious that inequalities (3) and (4) are special cases of (II) and (III). Thus, it is necessary and significant to provide explicit bounds on $x(t)$ satisfying inequalities (I)-(III).

To the best of our knowledge, inequalities (I)-(III) receive less attention at present. Most of existing results in the literature are focused on the case $\sigma(t)=t$. Generally speaking, those results can only be extended to inequalities involving $x^{\sigma}(t)$ by imposing additional conditions. For example, consider the following Gronwall's inequality

$$x(t)\leqslant a(t)+\int_{t_0}^{t} b(s)x^{\sigma}(s)\Delta s,\quad t\in\mathbb{T}^{\kappa}.$$

As usual, let $y(t)=\int_{t_0}^{t} b(s)x^{\sigma}(s)\Delta s$. Noticing that $x(t)\leqslant a(t)+y(t)$, we have

$$y^{\Delta}(t)=b(t)x^{\sigma}(t)\leqslant b(t)[a^{\sigma}(t)+y^{\sigma}(t)]=a^{\sigma}(t)b(t)+b(t)y(t)+\mu(t)b(t)y^{\Delta}(t),$$

i.e.,

$$(1-\mu(t)b(t))y^{\Delta}(t)\leqslant a^{\sigma}(t)b(t)+b(t)y(t).$$

In the general case, explicit bounds of $y(t)$ satisfying the above inequality can not be given by the following well-known Lemma 1 until an additional assumption $\mu(t)b(t)<1$ is imposed. However, for sublinear inequalities (I)-(III), we will show that explicit bounds of $x(t)$ can be obtained without imposing such additional assumptions.

Before establishing our main results, we need the following lemmas.

LEMMA 1. [15, Th. 6.1, p. 255] *Let y,p and q be rd-continuous on $\mathbb{T}$ with $p(t)\geqslant 0$ for $t\in\mathbb{T}$. Then*

$$y^{\Delta}(t)\leqslant p(t)y(t)+q(t),\quad t\in\mathbb{T}$$

implies

$$y(t)\leqslant y(t_0)e_p(t,t_0)+\int_{t_0}^{t} e_p(t,\sigma(s))q(s)\Delta s,\quad t\in\mathbb{T}.$$

LEMMA 2. *Let y,p_1,p_2 and q be rd-continuous on $\mathbb{T}$ with $p_i(t)\geqslant 0$ for $i=1,2$ and $t\in\mathbb{T}$. Then*

$$y^{\Delta}(t)\leqslant p_1(t)y(t)+\frac{p_2(t)}{1+\mu(t)p_2(t)}y^{\sigma}(t)+q(t),\quad t\in\mathbb{T}\tag{5}$$

implies

$$y(t)\leqslant y(t_0)e_{\tilde{p}}(t,t_0)+\int_{t_0}^{t} e_{\tilde{p}}(t,\sigma(s))\tilde{q}(s)\Delta s,\quad t\in\mathbb{T},\tag{6}$$

where

$$\tilde{p}(t)=p_1(t)\oplus p_2(t)=p_1(t)+p_2(t)+\mu(t)p_1(t)p_2(t)\quad \textit{and}\quad \tilde{q}(t)=[1+\mu(t)p_2(t)]q(t).$$

Proof. From (5), we have

$$y^{\Delta}(t)\leqslant p_1(t)y(t)+\frac{p_2(t)}{1+\mu(t)p_2(t)}[y(t)+\mu(t)y^{\Delta}(t)]+q(t).$$

That is

$$y^{\Delta}(t)\leqslant \tilde{p}(t)y(t)+\tilde{q}(t).$$

By Lemma 1, we get (6). □

LEMMA 3. *Let* $c \geqslant 0$, $x \geqslant 0$ *and* $0 < \lambda < 1$. *Then for any* $k > 0$

$$cx^{\lambda} \leqslant \lambda k^{\lambda-1} c^{\alpha} x + (1-\lambda) k^{\lambda} c^{\beta} \tag{7}$$

holds, where α *and* β *are nonnegative constants satisfying* $\lambda\alpha + (1-\lambda)\beta = 1$.

Proof. For nonnegative constants a and b, positive constants p and q with $1/p + 1/q = 1$, the basic inequality

$$\frac{a}{p} + \frac{b}{q} \geqslant a^{1/p} b^{1/q}$$

holds. Let $1/p = \lambda$, $1/q = 1-\lambda$, $a = k^{\lambda-1}c^{\alpha}$ and $b = k^{\lambda}c^{\beta}$. Then inequality (7) is met. □

REMARK 1. When $c = 1$, Lemma 2 reduces to Lemma 3.1 with $\lambda = q/p$ in [6, 9, 17, 18].

LEMMA 4. [15, Th. 1.117, p. 46] *Suppose that for each* $\varepsilon > 0$ *there exists a neighborhood* U *of* t, *independent of* $\tau \in [t_0, \sigma(t)]$, *such that*

$$|w(\sigma(t),\tau) - w(s,\tau) - w_1^{\Delta}(t,\tau)(\sigma(t)-s)| \leqslant \varepsilon|\sigma(t)-s|, \quad s \in U, \tag{8}$$

where $w : \mathbb{T} \times \mathbb{T}^{\kappa} \to \mathbb{R}_+$ *is continuous at* (t,t), $t \in \mathbb{T}^{\kappa}$ *with* $t > t_0$, *and* $w_1^{\Delta}(t,\cdot)$ *(the derivative of* w *with respect to the first variable) is rd-continuous on* $[t_0, \sigma(t)]$. *Then*

$$v(t) := \int_{t_0}^{t} w(t,\tau)\Delta\tau$$

implies

$$v^{\Delta}(t) = \int_{t_0}^{t} w_1^{\Delta}(t,\tau)\Delta\tau + w(\sigma(t),t).$$

4. Main results

Now, let's give the main results of this paper.

THEOREM 1. *Assume that* $a,b,g,h,x : \mathbb{T}^{\kappa} \to \mathbb{R}_+$ *are rd-continuous functions. Then for any nonnegative constants* α *and* β *satisfying* $\lambda\alpha + (1-\lambda)\beta = 1$, *inequality (I) implies*

$$x(t) \leqslant a(t) + b(t)\int_{t_0}^{t} e_P(t,\sigma(s))Q(s)\Delta s, \quad t \in \mathbb{T}^{\kappa}, \tag{9}$$

where

$$P(t) = P_1(t) + P_2(t) + \mu(t)P_1(t)P_2(t),$$

$$P_1(t) = g(t)b(t), \quad P_2(t) = \lambda h^{\alpha}(t)b^{\sigma}(t), \quad Q(t) = [1+\mu(t)P_2(t)]Q_1(t)$$

$$Q_1(t) = g(t)a(t) + \frac{\lambda h^{\alpha}(t)a^{\sigma}(t)}{1+\mu(t)P_2(t)} + (1-\lambda)[1+\mu(t)P_2(t)]^{\lambda/(1-\lambda)}h^{\beta}(t).$$

Proof. Set

$$y(t)=\int_{t_0}^{t}[g(s)x(s)+h(s)(x^{\sigma}(s))^{\lambda}]\Delta s,\quad t\in\mathbb{T}^{\kappa}.$$

Then $y(t_0)=0$ and (I) can be restated as

$$x(t)\leqslant a(t)+b(t)y(t),\quad t\in\mathbb{T}^{\kappa}.\tag{10}$$

Based on the straightforward computation and Lemma 3, for any rd-continuous function $k(t)>0$ we have

$$\begin{aligned}y^{\Delta}(t)&=g(t)x(t)+h(t)(x^{\sigma}(t))^{\lambda}\\&\leqslant g(t)x(t)+\lambda k^{\lambda-1}(t)h^{\alpha}(t)x^{\sigma}(t)+(1-\lambda)k^{\lambda}(t)h^{\beta}(t),\quad t\in\mathbb{T}^{\kappa}.\end{aligned}\tag{11}$$

Combining (10) and (11) yields

$$\begin{aligned}y^{\Delta}(t)\leqslant{}& g(t)b(t)y(t)+\lambda k^{\lambda-1}(t)h^{\alpha}(t)b^{\sigma}(t)y^{\sigma}(t)\\&+g(t)a(t)+\lambda k^{\lambda-1}(t)h^{\alpha}(t)a^{\sigma}(t)+(1-\lambda)k^{\lambda}(t)h^{\beta}(t),\quad t\in\mathbb{T}^{\kappa}.\end{aligned}\tag{12}$$

Let $k(t)=[1+\lambda\mu(t)h^{\alpha}(t)b^{\sigma}(t)]^{1/(1-\lambda)}$. Then (12) reduces to

$$y^{\Delta}(t)\leqslant P_1(t)y(t)+\frac{P_2(t)}{1+\mu(t)P_2(t)}y^{\sigma}(t)+Q_1(t).$$

This together with Lemma 2 yield

$$y(t)\leqslant\int_{t_0}^{t}e_P(t,\sigma(s))Q(s)\Delta s,\quad t\in\mathbb{T}^{\kappa}.$$

Thus, from (10) we get (9). □

REMARK 2. By choosing different α and β, some improved bounds on $x(t)$ can be obtained. For example, when $h(t)$ is sufficiently large, we may remove $h(t)$ out of $P(t)$ (i.e., set $\alpha=0$) since the value of $e_P(t,s)$ usually changes in a big way. When $h(t)$ is sufficiently small, we may remove $h(t)$ out of $Q(t)$, i.e., set $\beta=0$.

THEOREM 2. *Assume that $a,b,g,h,x:\mathbb{T}^{\kappa}\to\mathbb{R}_+$ are rd-continuous functions. Let $w:\mathbb{T}\times\mathbb{T}^{\kappa}\to\mathbb{R}_+$ be continuous such that $w_1^{\Delta}(t,s)\geqslant 0$ for $t\geqslant s$ and (8) holds. Suppose also that there exist rd-continuous functions $k_1(t)>0$ and $k_2(t)>0$ such that $\mu(t)W(t)<1$ for $t\in\mathbb{T}^{\kappa}$, where*

$$W(t)=\lambda\left[w(\sigma(t),t)k_1^{\lambda-1}(t)h^{\alpha}(t)b^{\sigma}(t)+\int_{t_0}^{t}w_1^{\Delta}(t,s)k_2^{\lambda-1}(s)h^{\alpha}(s)b^{\sigma}(s)\Delta s\right].$$

Then for any nonnegative constants α and β satisfying $\lambda\alpha+(1-\lambda)\beta=1$, inequality (II) implies

$$x(t)\leqslant a(t)+b(t)\int_{t_0}^{t}e_A(t,\sigma(s))B(s)\Delta s,\quad t\in\mathbb{T}^{\kappa},\tag{13}$$

where

$$A(t) = A_1(t) + A_2(t) + \mu(t)A_1(t)A_2(t),$$

$$A_1(t) = w(\sigma(t),t)g(t)b(t) + \int_{t_0}^{t} w_1^{\Delta}(t,s)g(s)b(s)\Delta s,$$

$$A_2(t) = \frac{W(t)}{1-\mu(t)W(t)}, \quad B(t) = [1+\mu(t)A_2(t)]B_1(t),$$

$$B_1(t) = \left[w(\sigma(t),t)\tilde{Q}_1(t) + \int_{t_0}^{t} w_1^{\Delta}(t,s)\tilde{Q}_2(s)\Delta s\right],$$

and

$$Q_i(t) = g(t)a(t) + \lambda k_i^{\lambda-1}(t)h^{\alpha}(t)a^{\sigma}(t) + (1-\lambda)k_i^{\lambda}(t)h^{\beta}(t), \quad i=1,2.$$

Proof. Define a function

$$z(t) = \int_{t_0}^{t} k(t,s)\Delta s, \quad t \in \mathbb{T}^{\kappa},$$

where

$$k(t,s) = w(t,s)[g(s)x(s) + h(s)(x^{\sigma}(s))^{\lambda}].$$

Then $z(t_0) = 0$, $z(t)$ is nondecreasing, and

$$x(t) \leqslant a(t) + b(t)z(t), \quad t \in \mathbb{T}^{\kappa}.$$

Similar to the arguments in Theorem 1, by Lemma 3 and Lemma 4, we have

$$\begin{aligned}
z^{\Delta}(t) &= k(\sigma(t),t) + \int_{t_0}^{t} k_1^{\Delta}(t,s)\Delta s \\
&= w(\sigma(t),t)[g(t)x(t) + h(t)(x^{\sigma}(t))^{\lambda}] \\
&\quad + \int_{t_0}^{t} w_1^{\Delta}(t,s)[g(s)x(s) + h(s)(x^{\sigma}(s))^{\lambda}]\Delta s \\
&\leqslant w(\sigma(t),t)\left[g(t)b(t)z(t) + \lambda k_1^{\lambda-1}(t)h^{\alpha}(t)b^{\sigma}(t)z^{\sigma}(t) + \tilde{Q}_1(t)\right] \\
&\quad + \int_{t_0}^{t} w_1^{\Delta}(t,s)\left[g(s)b(s)z(s) + \lambda k_2^{\lambda-1}(s)h^{\alpha}(s)b^{\sigma}(s)z^{\sigma}(s) + \tilde{Q}_2(s)\right]\Delta s \\
&\leqslant A_1(t)z(t) + W(t)z^{\sigma}(t) + B_1(t) \\
&= A_1(t)z(t) + \frac{A_2(t)}{1+\mu(t)A_2(t)}z^{\sigma}(t) + B_1(t), \quad t \in \mathbb{T}^{\kappa},
\end{aligned}$$

which implies (13) by Lemma 2. □

REMARK 3. In theorem 2, we need to guarantee that $\mu(t)W(t) < 1$ on $\mathbb{T}^{\kappa}$ for appropriate rd-continuous functions $k_1(t) > 0$ and $k_2(t) > 0$. It is not difficult to verify that such a condition holds for sufficiently small $k_1(t)$ and $k_2(t)$ for given λ, w, h and b.

THEOREM 3. *Assume a,b,x are nonnegative rd-continuous functions defined on $\mathbb{T}^\kappa$. Let $f:\mathbb{T}^\kappa\times\mathbb{R}_+\to\mathbb{R}_+$ be a continuous function satisfying*

$$0\leqslant f(t,x)-f(t,y)\leqslant \phi(t,y)(x-y) \tag{14}$$

for $t\in\mathbb{T}^\kappa$ and $x\geqslant y\geqslant 0$, where $\phi:\mathbb{T}^\kappa\times\mathbb{R}_+\to\mathbb{R}_+$ is a continuous function. Suppose also that there exists a rd-continuous function $k(t)>0$ such that $\mu(t)\Phi(t)<1$ on $\mathbb{T}^\kappa$, where

$$\Phi(t)=\lambda k^{\lambda-1}(t)h^{\alpha}(t)\phi[t,(1-\lambda)k^{\lambda}(t)h^{\beta}(t)]b^{\sigma}(t).$$

Then for any nonnegative constants α and β satisfying $\lambda\alpha+(1-\lambda)\beta=1$, inequality (III) implies

$$x(t)\leqslant a(t)+b(t)\int_{t_0}^{t}e_M(t,\sigma(s))N(s)\Delta s,\quad t\in\mathbb{T}^\kappa, \tag{15}$$

where

$$M(t)=\frac{\Phi(t)}{1-\mu(t)\Phi(t)},\quad N(t)=[1+\mu(t)M(t)]N_1(t),$$

and

$$N_1(t)=\lambda k^{\lambda-1}(t)h^{\alpha}(t)\phi[t,(1-\lambda)k^{\lambda}(t)h^{\beta}(t)]a^{\sigma}(t)+f[t,(1-\lambda)k^{\lambda}(t)h^{\beta}(t)].$$

Proof. Define a function $u(t)$ by

$$u(t)=\int_{t_0}^{t}f[s,(x^{\sigma}(s))^{\lambda}]\Delta s.$$

Then $u(t_0)=0$ and $x(t)\leqslant a(t)+b(t)u(t)$. Based on the straightforward computation, from (14) and Lemma 3 we get

$$\begin{aligned}
u^{\Delta}(t)&=f[t,(x^{\sigma}(t))^{\lambda}]\\
&\leqslant f[t,\lambda k^{\lambda-1}(t)h^{\alpha}(t)x^{\sigma}(t)+(1-\lambda)k^{\lambda}(t)h^{\beta}(t)]\\
&\leqslant \lambda k^{\lambda-1}(t)h^{\alpha}(t)\phi[t,(1-\lambda)k^{\lambda}(t)h^{\beta}(t)]x^{\sigma}(t)+f[t,(1-\lambda)k^{\lambda}(t)h^{\beta}(t)]\\
&\leqslant \Phi(t)u^{\sigma}(t)+N_1(t)\\
&=\frac{M(t)}{1+\mu(t)M(t)}u^{\sigma}(t)+N_1(t),\quad t\in\mathbb{T}^\kappa.
\end{aligned}$$

This together with Lemma 2 imply (15). □

REMARK 4. For some particular cases of $\mathbb{T}$, α and β, Theorems 1-3 include some results in [14] as special cases.

5. Applications

To illustrate the usefulness of the results we state the corresponding theorems in the previous section for the special case $\mathbb{T}=\mathbb{Z}$. It is not difficult to provide similar results for other specific time scales of interest.

COROLLARY 1. *Let* $\mathbb{T}=\mathbb{Z}$ *and* $a,b,g,h,x:\mathbb{N}_0=\{t_0,t_0+1,\cdots\}\to\mathbb{R}_+$ *be continuous. Then for any nonnegative constants* α *and* β *satisfying* $\lambda\alpha+(1-\lambda)\beta=1$, *inequality (I) implies*

$$x(t)\leqslant a(t)+b(t)\sum_{s=t_0}^{t-1}\left(\prod_{\tau=s+1}^{t-1}(1+P(\tau))\right)Q(s),\quad t\in\mathbb{N}_0,$$

where $P(t)$ *and* $Q(t)$ *are defined as in Theorem 1 with* $\mu(t)=1$.

COROLLARY 2. *Assume* $\mathbb{T}=\mathbb{Z}$, $a,b,g,h,x:\mathbb{N}_0\to\mathbb{R}_+$, *and* $w(t,s)$ *is a nonnegative function on* $\mathbb{N}_0$ *such that* $w_1^{\Delta}(t,s)\geqslant 0$ *for* $t\geqslant s$ *and* (8) *holds. Suppose also that there exist rd-continuous functions* $k_1(t)>0$ *and* $k_2(t)>0$ *such that* $W(t)<1$ *for* $t\in\mathbb{N}_0$. *Then for any nonnegative constants* α *and* β *satisfying* $\lambda\alpha+(1-\lambda)\beta=1$, *inequality (II) implies*

$$x(t)\leqslant a(t)+b(t)\sum_{s=t_0}^{t-1}\left(\prod_{\tau=s+1}^{t-1}(1+A(\tau))\right)B(s),\quad t\in\mathbb{N}_0,$$

where $W(t)$, $A(t)$ *and* $B(t)$ *are the same as in Theorem 2 with* $\mu(t)=1$.

COROLLARY 3. *Assume* $\mathbb{T}=\mathbb{Z}$ *and* a,b,x *are nonnegative functions on* $\mathbb{N}_0$. *Let* $f:\mathbb{N}_0\times\mathbb{R}_+\to\mathbb{R}_+$ *be a function satisfying* (14). *Suppose also that there exists a rd-continuous function* $k(t)>0$ *such that* $\Phi(t)<1$ *on* $\mathbb{N}_0$. *Then for any nonnegative constants* α *and* β *satisfying* $\lambda\alpha+(1-\lambda)\beta=1$, *inequality (III) implies*

$$x(t)\leqslant a(t)+b(t)\sum_{s=t_0}^{t-1}\left(\prod_{\tau=s+1}^{t-1}(1+M(\tau))\right)N(s),\quad t\in\mathbb{N}_0,$$

where $\Phi(t)$, $M(t)$ *and* $N(t)$ *are defined as in Theorem 3 with* $\mu(t)=1$.

In this paper, we have presented a method to study certain class of nonlinear integral inequalities involving $x^{\sigma}(t)$. Based on this method, some other results can be similarly obtained when $a(t)$ and $b(t)$ satisfy additional assumptions such as monotonicity. We here do not enumerate them in detail. At the end of this paper, we apply this method to a numerical example. Consider the following initial value problem on time scales

$$x^{\Delta}(t)=H(t,x(t),(x^{\sigma}(t))^{\lambda}),\quad x(t_0)=x_0,\ t\in\mathbb{T}^{\kappa},\tag{16}$$

where $H:\mathbb{T}^{\kappa}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying

$$|H(t,x(t),(x^{\sigma}(t))^{\lambda})|\leqslant g(t)|x(t)|+h(t)|(x^{\sigma}(t))^{\lambda}|,\quad t\in\mathbb{T}^{\kappa},$$

$g(t)$ and $h(t)$ are nonnegative rd-continuous functions on $\mathbb{T}^\kappa$. The solution of (16) satisfies the following integral inequality

$$x(t) = x_0 + \int_{t_0}^{t} H(s, x(s), (x^\sigma(s))^\lambda)\Delta s, \quad t \in \mathbb{T}^\kappa.$$

It yields

$$|x(t)| \leqslant |x_0| + \int_{t_0}^{t} [g(s)|x(s)| + h(s)|x^\sigma(s)|^\lambda]\Delta s, \quad t \in \mathbb{T}^\kappa. \tag{17}$$

Denote the right-hand side of (17) by $y(t)$. Then $|x(t)| \leqslant y(t)$, $y(t_0) = x_0$, and for any rd-continuous function $k(t) > 0$

$$\begin{aligned} y^\Delta(t) &= g(t)|x(t)| + h(t)|x^\sigma(t)|^\lambda \\ &\leqslant g(t)y(t) + \lambda k^{\lambda-1}(t)h^\alpha(t)y^\sigma(t) + (1-\lambda)k^\lambda(t)h^\beta(t), \quad t \in \mathbb{T}^\kappa. \end{aligned}$$

Let $k(t) = [1 + \lambda\mu(t)h^\alpha(t)]^{1/(1-\lambda)}$. Then, for $t \in \mathbb{T}^\kappa$ we have

$$y^\Delta(t) \leqslant g(t)y(t) + \frac{\lambda h^\alpha(t)}{1 + \lambda\mu(t)h^\alpha(t)} y^\sigma(t) + (1-\lambda)[1 + \lambda\mu(t)h^\alpha(t)]^{\lambda/(1-\lambda)}h^\beta(t).$$

By Lemma 2, it yields

$$x(t) \leqslant |x_0|e_p(t, t_0) + \int_{t_0}^{t} e_p(t, \sigma(s))q(s)ds, \quad t \in \mathbb{T}^\kappa,$$

where

$$p(t) = g(t) + \lambda h^\alpha(t) + \lambda\mu(t)g(t)h^\alpha(t),$$

and

$$q(t) = (1-\lambda)[1 + \lambda\mu(t)h^\alpha(t)]^{1/(1-\lambda)}h^\beta(t).$$

REFERENCES

[1] S. HILGER, *Analysis on measure chains-A unified approach to continuous and discrete calculus*, Results Math. **18** (1990), 18–56.

[2] R. AGARWAL, M. BOHNER, A. PETERSON, *Inequalities on time scales: a survey*, Math. Inequal. Appl. **4** (2001), 535–557.

[3] E. AKIN-BOHNER, M. BOHNER, F. AKIN, *Pachpatte inequalities on time scales*, Journal of Inequalities in Pure Applied Mathematics **6** (2005), article 6, 23 pages.

[4] W. N. LI, *Some new dynamic inequalities on time scales*, J. Math. Anal. Appl. **319** (2006), 802–814.

[5] W. N. LI, *Some Pachpatte type inequalities on time scales*, Computers and Mathematics with Applications **57** (2009), 275–282.

[6] W. N. LI, W. SHENG, *Some nonlinear integral inequalities on time scales*, Journal of Inequalities and Applications (2007), 15 pages.

[7] W. N. LI, W. SHENG, *Some nonlinear dynamic inequalities on time scales*, Proc. Indian Acad. Sci. Math. Sci. **117** (2007), 545–554.

[8] W. N. LI, *Some delay integral inequalities on time scales*, Computers and Mathematics with Applications **59** (2010), 1929–1936.

[9] D. R. ANDERSON, *Nonlinear dynamic integral inequalities in two independent variables on time scale pairs*, Advances in Dynamical Systems and Applications **3** (2008), 1–13.

[10] D. R. ANDERSON, *Dynamic double integral inequalities in two independent variables on time scales*, J. Math. Inequal., **2** (2008), 163–184.
[11] D. B. PACHPATTE, *Explicit estimates on integral inequalities with time scale*, Journal of Inequalities in Pure Applied Mathematics **7** (2006), article 143, 8 pages.
[12] F. H. WONG, C. C. YEH, C.H. HONG, *Gronwall inequalities on time scales*, Math. Inequal. Appl., **9** (2006), 75–86.
[13] F. WONG, C. C. YEH, S. L YU, C. H. HONG, *Young's inequality and related results on time scales*, Appl. Math. Lett. **18** (2005), 983–988.
[14] B. G. PACHPATTE, *On some new inequalities related to a certain inequality arising in the theory of differential equations*, J. Math. Anal. Appl. **251** (2000), 736–751.
[15] M. BOHNER, A. PETERSON, *Dynamic Equations on Time Scales, An Introduction with Applications*, Birkhäuser, Boston, 2001.
[16] M. BOHNER, A. PETERSON (EDS.), *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.
[17] F. W. MENG, D. JI, *On some new nonlinear discrete inequalities and their applications*, J. Comput. Appl. Math. **208** (2007), 425–433.
[18] F. W. MENG, W. N. LI, *On some new nonlinear discrete inequalities and their applications*, J. Comput. Appl. Math. **158** (2003), 407–417.

(Received August 30, 2010)

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