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A PART–METRIC VARIANT OF NEWTON'S INEQUALITIES

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Abstract. This note gives a part-metric variant of Newton's inequalities. A particular case proved useful recently in the study of difference equations involving ratios of elementary symmetric polynomials.

1. Introduction

This short note provides a part-metric variant of Newton's inequalities. In particular, for fixed $k \geqslant 0$, consider the elementary symmetric polynomials, $\{e_{j,k}\}$ in the k variables $X_1, X_2, \ldots, X_k$, i.e., $e_{0,k}(X_1, X_2, \ldots, X_k) = 1$ and

$$e_{j,k}(X_1, X_2, \ldots, X_k) = \sum_{1 \leqslant i_1 < i_2 < \ldots < i_j \leqslant k} X_{i_1} X_{i_2} \ldots X_{i_j}, \qquad 1 \leqslant j \leqslant k. \tag{1.1}$$

When the inputs are clear from the context, we may denote $e_{j,k}(X_1, X_2, \ldots, X_k)$ by $e_j(X_1, X_2, \ldots, X_k)$ or simply e_j for $0 \leqslant j \leqslant k$. Note that every symmetric polynomial can be written as a polynomial in elementary symmetric polynomials (see for instance [17] or [20]). For some further discussion on the importance of symmetric polynomials see for instance [14] and the extensive references in [10].

The well-known Newton's inequalities for elementary symmetric polynomials are the following.

THEOREM 1. (Newton's inequalities) *For fixed $k \geqslant 1$ and $\{S_i\}$ defined via*

$$S_i = S_i(X_1, X_2, \ldots, X_k) = \frac{e_i(X_1, X_2, \ldots, X_k)}{\binom{k}{i}}, \tag{1.2}$$

we have for $X_1, X_2, \ldots, X_k > 0$ that

$$\frac{S_0}{S_1} \leqslant \frac{S_1}{S_2} \leqslant \cdots \leqslant \frac{S_{k-1}}{S_k}, \tag{1.3}$$

with equalities in (1.3) if and only if $X_1 = X_2 = \cdots = X_k$.

In addition, it follows that

$$\frac{e_0}{e_1} < \frac{e_1}{e_2} < \cdots < \frac{e_{k-1}}{e_k}. \tag{1.4}$$

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See for instance [5], [6], [15], [18], [19] and [22] and the references therein, for further discussion and generalizations of Theorem 1.

In [3], the following theorem regarding convergence of positive solutions to rational difference equations was proven. The case $i = k-1$, was proven in [9] (see also [7], [13], [8] and Section 4.5 in [11]).

THEOREM 2. *Suppose* $0 \leqslant i \leqslant k-1$ *and* $\{y_n\}$ *satisfies*

$$y_n = \left(\frac{e_{i,k}}{e_{i+1,k}}\right)(y_{n-t_1}, y_{n-t_2}, \dots, y_{n-t_k}), \tag{1.5}$$

where $t_l \geqslant 1$ *for* $1 \leqslant l \leqslant k$, $\gcd(t_1, t_2, \dots, t_k) = 1$ *and* $y_{-s}, y_{-s+1}, \dots, y_{-1} \in \mathbb{R}^+$, *with*

$$s = \max\{t_1, t_2, \dots, t_k\}. \tag{1.6}$$

If at least one t_l *is even, then* $\{y_n\}$ *converges to the unique equilibrium*

$$c = \sqrt{\frac{i+1}{k-i}} = \sqrt{\frac{\binom{k}{i}}{\binom{k}{i+1}}}. \tag{1.7}$$

Otherwise, $\{y_n\}$ *is asymptotically periodic with (not necessarily prime) period two.*

Instrumental in the proof of Theorem 2 in [3] was the following variant of Theorem 1.

THEOREM 3. *For* $X > 0$, *define the transformed value* X^* *via*

$$X^* = \max\left\{\frac{X}{c}, \frac{c}{X}\right\}, \tag{1.8}$$

where c *is as defined in* (1.7)*, and suppose that* $X_1, X_2, \dots, X_k > 0$. *Then,*

$$\left(\frac{e_i(X_1, X_2, \dots, X_k)}{e_{i+1}(X_1, X_2, \dots, X_k)}\right)^* \leqslant \frac{e_1(X_1^*, X_2^*, \dots, X_k^*)}{k}. \tag{1.9}$$

The part-metric or Thompson's metric, p, on $(\mathbb{R}^+)^r$ is defined for any $X = (x_1, x_2, \dots, x_r) \in (\mathbb{R}^+)^r$ and $Y = (y_1, y_2, \dots, y_r) \in (\mathbb{R}^+)^r$ via

$$p(X,Y) = -\log_2 \min\left\{\frac{x_i}{y_i}, \frac{y_i}{x_i} : 1 \leqslant i \leqslant r\right\}. \tag{1.10}$$

For some recent work regarding global asymptotic stability which employ part-metric-type techniques see for instance [1], [2], [4], [12], [16], [23], [24], [25] and the references therein. From a part-metric perspective, it would have sufficed for the purposes in [3] to replace the inequality in (1.9) with the weaker

$$\left(\frac{e_i(X_1, X_2, \dots, X_k)}{e_{i+1}(X_1, X_2, \dots, X_k)}\right)^* \leqslant \max\{X_1^*, X_2^*, \dots, X_k^*\},$$

but the bound in (1.9) is stronger and somewhat more natural in the given context.

In the present paper, we will prove the following.

THEOREM 4. *For $X > 0$ and $0 \leqslant i < j \leqslant k$, define the transformed value X^* via*

$$X^* = \max\left\{\frac{X}{c}, \frac{c}{X}\right\}, \tag{1.11}$$

where

$$c = c_{i,j} = \left(\frac{\binom{k}{i}}{\binom{k}{j}}\right)^{\frac{1}{j-i+1}} = \left(\frac{j(j-1)\dots(i+1)}{(k-i)(k-(i+1))\dots(k-j+1)}\right)^{\frac{1}{j-i+1}}, \tag{1.12}$$

and suppose that $X_1, X_2, \dots, X_k > 0$. Then, for $0 \leqslant h \leqslant i$,

$$c\frac{S_h}{S_{j-i+h}}(X_1^*, X_2^*, \dots, X_k^*) \leqslant \frac{e_i}{e_j}(X_1, X_2, \dots, X_k), \tag{1.13}$$

with equality if and only if $X_1 = X_2 = \dots = X_k = c$ when $0 \leqslant h < i$, and if and only if $X_1, X_2, \cdots, X_k \geqslant c$ when $h = i$. Similarly, for $0 \leqslant h \leqslant k - j$,

$$\frac{e_i}{e_j}(X_1, X_2, \dots, X_k) \leqslant c\frac{S_{j-i+h}}{S_h}(X_1^*, X_2^*, \dots, X_k^*), \tag{1.14}$$

with equality if and only if $X_1 = X_2 = \dots = X_k = c$ when $0 \leqslant h < k - j$ and if and only if $X_1, X_2, \cdots, X_k \leqslant c$ when $h = k - j$.

Since, by Theorem 1, $S_a/S_{a+h} \leqslant S_{a+1}/S_{a+h+1}$ for (a,h) satisfying $a, h \geqslant 0$ and $a + h + 1 \leqslant k$, we have the following corollary.

COROLLARY 1. *Suppose the hypotheses of Theorem 4 are satisfied, and set*

$$D^*_{a,b} = \frac{S_a}{S_b}(X_1^*, X_2^*, \dots, X_k^*), \tag{1.15}$$

for $0 \leqslant a, b \leqslant k$. Then,

$$D^*_{0,j-i} \leqslant D^*_{1,j-i+1} \leqslant \dots \leqslant D^*_{i,j} \leqslant \frac{e_i}{ce_j} \leqslant D^*_{k-i,k-j} \leqslant D^*_{k-i+1,k-j+1} \leqslant \dots \leqslant D^*_{j-i,0}, \tag{1.16}$$

In particular, since (see Lemma 1, below),

$$D^*_{j-i,0} \leqslant (\max\{X_1^*, X_2^*, \dots, X_k^*\})^{j-i}, \tag{1.17}$$

$$c\left(\min_{1\leqslant t\leqslant k}\left\{\frac{X_t}{c}, \frac{c}{X_t}\right\}\right)^{j-i} \leqslant \frac{e_i}{e_j} \leqslant c\left(\max_{1\leqslant t\leqslant k}\left\{\frac{X_t}{c}, \frac{c}{X_t}\right\}\right)^{j-i}. \tag{1.18}$$

For inequalities similar in type to that in (1.18), see for instance [24] and the references therein.

Note that the constant $c = c_{i,j}$ as defined in (1.12) is the unique equilibrium for the equation

$$y_n = \left(\frac{e_{i,k}}{e_{j,k}}\right)(y_{n-t_1}, y_{n-t_2}, \ldots, y_{n-t_k}), \qquad n \geqslant 0. \tag{1.19}$$

Theorem 4 leads to the following extension of Theorem 3.

THEOREM 5. *For $X > 0$ and $0 \leqslant i < j \leqslant k$, define the transformed value X^* via (1.11). Suppose that $X_1, X_2, \ldots, X_k > 0$. Then, for $0 \leqslant h \leqslant \min\{i, k-j\}$*

$$\left(\frac{e_i(X_1, X_2, \ldots, X_k)}{e_j(X_1, X_2, \ldots, X_k)}\right)^* \leqslant \frac{S_{j-i+h}(X_1^*, X_2^*, \ldots, X_k^*)}{S_h(X_1^*, X_2^*, \ldots, X_k^*)}, \tag{1.20}$$

with equality if and only if $X_1 = X_2 = \cdots = X_k = c$.

In the next section, we will prove Theorem 4.

2. Proof of Theorem 4

In this section we will prove Theorem 4. Essential to the proof will be the following elementary lemma which follows directly from Theorem 1.

LEMMA 1. *The ratio $R = R_{i,j}$ defined via*

$$R(X_1, X_2, \ldots, X_k) = \frac{e_{i,k}(X_1, \ldots, X_k)}{e_{j,k}(X_1, \ldots, X_k)}, \qquad 0 \leqslant i < j \leqslant k \tag{2.1}$$

is decreasing in each of its arguments.

Proof. The lemma clearly holds if $i = 0$. Otherwise, we have

$$\begin{aligned} R(X_1, X_2, \ldots, X_k) &= \frac{X_k e_{i-1,k-1}(X_1, \ldots, X_{k-1}) + e_{i,k-1}(X_1, \ldots, X_{k-1})}{X_k e_{j-1,k-1}(X_1, \ldots, X_{k-1}) + e_{j,k-1}(X_1, \ldots, X_{k-1})} \\ &= \frac{X_k e_{i-1,k-1} + e_{i,k-1}}{X_k e_{j-1,k-1} + e_{j,k-1}}. \end{aligned} \tag{2.2}$$

Hence

$$\begin{aligned} \frac{dR}{dX_k}(X_1 \ldots, X_k) &= \frac{e_{i-1,k-1}(X_k e_{j-1,k-1} + e_{j,k-1}) - (X_k e_{i-1,k-1} + e_{i,k-1}) e_{j-1,k-1}}{(X_k e_{j-1,k-1} + e_{j,k-1})^2} \\ &= \frac{e_{i-1,k-1} e_{j,k-1} - e_{j-1,k-1} e_{i,k-1}}{(X_k e_{j-1,k-1} + e_{j,k-1})^2} \\ &= \frac{\frac{e_j}{e_i}\left(\frac{e_{i-1}}{e_i} - \frac{e_{j-1}}{e_j}\right)}{(X_k e_{j-1,k-1} + e_{j,k-1})^2} < 0, \end{aligned} \tag{2.3}$$

by (1.4), and since R is symmetric in its arguments, the result follows. $\square$

We are now in a position to prove Theorem 4.

Proof of Theorem 4. We need to show that for fixed $0 \leqslant r \leqslant k$ and all positive $X_1, X_2, \ldots, X_k$ satisfying

$$X_1, X_2, \ldots, X_r \geqslant c \text{ and } X_{r+1}, X_{r+2}, \ldots, X_k \leqslant c, \tag{2.4}$$

$$Q_1 \stackrel{def}{=} \frac{S_{j-i+h}}{S_h}\left(\frac{X_1}{c}, \ldots, \frac{X_r}{c}, \frac{c}{X_{r+1}}, \ldots, \frac{c}{X_k}\right) - \frac{1}{c}\frac{e_i}{e_j}(X_1, X_2, \ldots, X_k) \geqslant 0 \tag{2.5}$$

and

$$Q_2 \stackrel{def}{=} \frac{S_{j-i+h}}{S_h}\left(\frac{X_1}{c}, \ldots, \frac{X_r}{c}, \frac{c}{X_{r+1}}, \ldots, \frac{c}{X_k}\right) - c\frac{e_j}{e_i}(X_1, X_2, \ldots, X_k) \geqslant 0. \tag{2.6}$$

To prove (2.5), note that by Lemma 1, for $0 \leqslant h \leqslant k-j$

$$\begin{aligned} Q_1 &\geqslant \frac{S_{j-i+h}}{S_h}\left(1, \ldots, 1, \frac{c}{X_{r+1}}, \ldots, \frac{c}{X_k}\right) - \frac{1}{c}\frac{e_i}{e_j}(c, \ldots, c, X_{r+1}, \ldots, X_k) \\ &= c^{j-i}\frac{S_{j-i+h}}{S_h}\left(\frac{1}{c}, \ldots, \frac{1}{c}, \frac{1}{X_{r+1}}, \ldots, \frac{1}{X_k}\right) - \frac{1}{c}\frac{e_i}{e_{i+1}}(c, \ldots, c, X_{r+1}, \ldots, X_k) \\ &= c^{j-i}\left(\frac{S_{k-(j-i+h)}}{S_{k-h}}(c, \ldots, c, X_{r+1}, \ldots, X_k) - \frac{1}{c^{j-i+1}}\frac{e_i}{e_j}(c, \ldots, c, X_{r+1}, \ldots, X_k)\right) \\ &= c^{j-i}\left(\frac{S_{k-(j-i+h)}}{S_{k-h}}(c, \ldots, c, X_{r+1}, \ldots, X_k) - \frac{S_i}{S_j}(c, \ldots, c, X_{r+1}, \ldots, X_k)\right) \geqslant 0. \end{aligned} \tag{2.7}$$

Similarly, for $0 \leqslant h \leqslant i$

$$\begin{aligned} Q_2 &\geqslant \frac{S_{j-i+h}}{S_h}\left(\frac{X_1}{c}, \ldots, \frac{X_r}{c}, 1, \ldots, 1\right) - c\frac{e_j}{e_i}(X_1, \ldots, X_r, c, \ldots, c) \\ &= \frac{1}{c^{j-i}}\left(\frac{S_{j-i+h}}{S_h}(X_1, \ldots, X_r, c, \ldots, c) - c^{j-i+1}\frac{e_j}{e_i}(X_1, \ldots, X_r, c, \ldots, c)\right) \\ &= \frac{1}{c^{j-i}}\left(\frac{S_{j-i+h}}{S_h}(X_1, \ldots, X_r, c, \ldots, c) - \frac{S_j}{S_i}(X_1, \ldots, X_r, c, \ldots, c)\right) \geqslant 0. \end{aligned} \tag{2.8}$$

The statements regarding equality follow upon noting that, by Lemma 1, the first inequality in (2.7) is strict unless $X_1, X_2, \ldots X_r = c$, and by Theorem 1, the final inequality is strict unless $X_{r+1}, \ldots, X_k = c$ or $h = k - j$. A similar argument applies to the inequalities in (2.8), and the theorem follows. □

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