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TWO SEQUENCES OF OPERATOR MONOTONE FUNCTIONS UNDER STRICTLY CHAOTIC ORDER

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Abstract. Based on the well-known operator monotone function $\log t$ the chaotic order was defined by $A \gg B$, e.g. $\log A \geqslant \log B$, and thereby the operator monotonicity of functions under strictly chaotic order has been introduced. However, until now the revealed operator functions with such property have been less than those functions under usual order. In this study, we investigate two sequences of operator monotone functions under strictly chaotic order, following the trace of Professor T. Furuta, S. Izumino and so on. However, the method is different from that of the relative results on

$$f_0(t) = \frac{t-1}{\log t}, f_p(t) = \frac{t - pf_{p-1}(t)}{\log t}, \ p = 1, 2, \cdots$$

due to S. Izumino and N. Nakamura.

1. Introduction

A bounded linear operator A on a Hilbert space H is positive, in symbol $A \geqslant 0$, if $(Ax,x) \geqslant 0$ for all $x \in H$. In particular, A is strictly positive, in symbol $A > 0$, if A is positive invertible. The well-known Löwner-Heinz inequality[2] says that if $A \geqslant B \geqslant 0$, then $A^\alpha \geqslant B^\alpha$ for $0 \leqslant \alpha \leqslant 1$. This means that the function $t \to t^\alpha$ $(0 \leqslant \alpha \leqslant 1)$ on $[0,\infty)$ is operator monotone. Another known example of operator monotone function is $t \to \log t$ on $(0,1) \cup (1,+\infty)$ [3]. Based on this fact, the chaotic order $A \gg B$, which is weaker than the usual order $A \geqslant B$, is defined by $\log A \geqslant \log B$ between strictly positive operators A and B. Similarly, the strictly chaotic order is defined by $\log A > \log B$.

Recently, Professor T. Furuta, by using Löwner-Heinz inequality, showed the following operator monotonicity of functions

$$\varphi(t) = \frac{t-1}{\log t}, \quad \psi(t) = \frac{t\log t - t + 1}{\log^2 t}$$

under strictly chaotic order:

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THEOREM F. ([4]) *Let A and B be two strictly positive operators on a Hilbert space H. If $\log A > \log B$ with $1 \notin \sigma(A), \sigma(B)$, then there exists $\beta \in (0,1]$ such that $\varphi(A^\alpha) > \varphi(B^\alpha)$ and $\psi(A^\alpha) > \psi(B^\alpha)$ for all $\alpha \in (0,\beta)$.*

The first author and L. Zhang extended the result about $\psi(t)$ in Theorem F to a sequence of functions

$$\psi_k(t) = \frac{t^k \log^k t - (t-1)^k}{\log^{k+1} t} \ (k = 1,2,\cdots):$$

THEOREM J-Z. ([5]) *Assume the conditions as in Theorem F, then for all integers $k \geqslant 1$, there exists $\beta \in (0,1]$ such that $\psi_k(A^\alpha) > \psi_k(B^\alpha)$ for all $\alpha \in (0,\beta)$.*

Since $\psi_1(t) = \psi(t)$, we know that Theorem J-Z is an essential extension of Theorem F.

Another extension of Theorem F was given by Professor S. Izumino and N. Nakamura:

THEOREM N-I. ([1]) *Assume the conditions as in Theorem F, then for all integers $p \geqslant 0$, there exists $\beta \in (0,1]$ such that $f_p(A^\alpha) > f_p(B^\alpha)$ for all $\alpha \in (0,\beta)$, where*

$$f_0(t) = \frac{t-1}{\log t}, \ f_p(t) = \frac{t - p f_{p-1}(t)}{\log t}, \ p = 1,2,\cdots.$$

By direct calculation we have $f_0(t) = \varphi(t)$ and $f_1(t) = \psi(t)$, so Theorem I-N is also an essential extension of Theorem F.

In this paper, we study the operator monotonicity of two sequences of functions $f_{pk}(t)$ and $h_{pk}(t)$ $(p,k = 1,2,\cdots)$ under strictly chaotic order. We prove that $f_{p1}(t)$ is just $f_p(t)$, and therefore more general results are obtained.

2. Main Result

First of all, we define two sequences of real numbers $\{u_{jk}\}_{j,k=1}^{\infty}$ and $\{v_{jk}\}_{j,k=1}^{\infty}$ by

$$\begin{cases} e^{kt} = (1 + t + \frac{t^2}{2!} + \cdots)^k = 1 + u_{1k}t + u_{2k}t^2 + \cdots \\ \qquad (1 + \frac{t}{2!} + \frac{t^2}{3!} + \cdots)^k = 1 + v_{1k}t + v_{2k}t^2 + \cdots \end{cases}$$

It's easy to know that $0 < v_{jk} < u_{jk} = \frac{k^j}{j!}$ and $v_{1k} = \frac{k}{2}$, $v_{j1} = \frac{1}{(j+1)!}$, where $j,k = 1,2,\cdots$.

Now the main result in this paper is given below:

THEOREM 2.1. *Let A and B be two strictly positive operators on a Hilbert space H. If $\log A > \log B$ with $1 \notin \sigma(A), \sigma(B)$, then for all integers $p \geqslant 1, k \geqslant 1$, there exists $\beta \in (0,1]$ such that $f_{pk}(A^\alpha) > f_{pk}(B^\alpha)$ for all $\alpha \in (0,\beta)$, where $v_{0k} = 1$,*

$$f_{pk}(t) = \frac{(-1)^p p!}{\log^{p+k} t}[t^k \log^k t \sum_{j=1}^{p} (-1)^j v_{j-1,k} \log^{j-1} t + (t-1)^k], \quad p,k = 1,2,\cdots.$$

To achive this theorem, we should give some lemmas, involving the discussion of the relation between $\{f_p(t)\}_{p=1}^{\infty}$ in Theorem I-N and $\{f_{pk}(t)\}_{p,k=1}^{\infty}$ in our theorem.

LEMMA 2.2. *$f_{p1}(t) = f_p(t)$ for all $t \in (0,1)\cup(1,+\infty)$ and $p = 1,2,\cdots$.*

Proof. By the definition we know that

$$f_{11}(t) = \frac{(-1)}{\log^2 t}[t\log t(-1v_{01}) + (t-1)] = \psi(t) = f_1(t).$$

Supposing $f_{j1}(t) = f_j(t)$ for $j = 1,2,\cdots,p-1 (p \geqslant 2)$, we have

$$\begin{aligned} f_p(t) &= \frac{t - pf_{p-1}(t)}{\log t} = \frac{t - pf_{p-1,1}(t)}{\log t} \\ &= \frac{1}{\log t}\left\{t - p\frac{(-1)^{p-1}(p-1)!}{\log^p t}[t\log t\sum_{j=1}^{p-1}(-1)^j v_{j-1,1}\log^{j-1} t + (t-1)]\right\} \\ &= \frac{(-1)^p p!}{\log^{p+1} t}\left[t\log t\sum_{j=1}^{p-1}(-1)^j v_{j-1,1}\log^{j-1} t + (t-1) + \frac{(-1)^p}{p!}t\log^p t\right] \\ &= \frac{(-1)^p p!}{\log^{p+1} t}\left\{t\log t\,[\sum_{j=1}^{p-1}(-1)^j v_{j-1,1}\log^{j-1} t + (-1)^p v_{p-1,1}\log^{p-1} t] + (t-1)\right\} \\ &= \frac{(-1)^p p!}{\log^{p+1} t}\left[t\log t\sum_{j=1}^{p}(-1)^j v_{j-1,1}\log^{j-1} t + (t-1)\right] = f_{p1}(t). \end{aligned}$$

And hence, we get the conclusion by inductive method. □

According to Lemma 2.2, we know that our theorem is an essential extension of Theorem I-N!

LEMMA 2.3. *If we denote*

$$F_{pk}(t) = \frac{(-1)^p p!}{t^{p+k}}[e^{kt}t^k\sum_{j=1}^{p}(-1)^j v_{j-1,k}t^{j-1} + (e^t-1)^k]\ ,$$

then

$$F_{pk}(t) = p!e^{kt}\sum_{j=0}^{i}(-1)^j v_{j+p,k}t^j = p!\sum_{i=0}^{\infty} w_{pk}^{(i)}t^i\ (p,k = 1,2,\cdots)\ ,$$

in which $u_{0k} = 1$ and

$$w_{pk}^{(i)} = \sum_{j=0}^{i}(-1)^j u_{i-j,k}v_{j+p,k}\ (i = 0,1,2,\cdots; \quad p,k = 1,2,\cdots)$$

Proof. Since

$$\frac{1}{t^k}(1-e^{-t})^k = \frac{1}{t^k}\left[1 - \sum_{j=0}^{\infty}\frac{(-t)^j}{j!}\right]^k = \left[\sum_{j=0}^{\infty}\frac{(-t)^j}{(j+1)!}\right]^k = \sum_{j=0}^{\infty}v_{jk}(-t)^j = \sum_{j=0}^{\infty}(-1)^j v_{jk}t^j,$$

we have

$$F_{pk}(t) = \frac{(-1)^p p! e^{kt}}{t^p}\left[\sum_{j=1}^{p}(-1)^j v_{j-1,k}t^{j-1} + \frac{1}{t^k}(1-e^{-t})^k\right] = \frac{(-1)^p p! e^{kt}}{t^p}\sum_{j=p}^{\infty}(-1)^j v_{jk}t^j$$
$$= p! e^{kt}\sum_{j=p}^{\infty}(-1)^{j-p}v_{jk}t^{j-p} = p!\left(\sum_{i=0}^{\infty}u_{ik}t^i\right)\left[\sum_{j=0}^{\infty}(-1)^j v_{j+p,k}t^j\right]$$
$$= p!\sum_{i=0}^{\infty}\left[\sum_{j=0}^{i}(-1)^j u_{i-j,k}v_{j+p,k}\right]t^i = p!\sum_{i=0}^{\infty}w_{pk}^{(i)}t^i.$$

□

Now, we should know more details about $\{v_{jk}\}$ and $\{w_{pk}^{(i)}\}$. It is well-known that $(x-1)^k = \sum_{q=0}^{k} C_k^q x^q(-1)^{k-q}$ leads to $\sum_{q=0}^{k} C_k^q(-1)^{k-q} = 0$, by this means and inductive method it is easy to get the lemma below:

LEMMA 2.4. *$\sum_{q=1}^{k} C_k^q(-1)^{k-q}q^j = 0$ for all integers $j : 1 \leqslant j < k$; $\frac{1}{k!}[\sum_{q=1}^{k} C_k^q(-1)^{k-q}q^k] = 1$.*

LEMMA 2.5. *According to the definition of $\{v_{jk}\}$, we have*

$$v_{jk} = \frac{(-1)^k}{(j+k)!}\sum_{q=1}^{k}(-1)^q q^{j+k}C_k^q\,,$$

as well as the inductive relation

$$v_{jk} = \frac{k}{j+k}\left(v_{j-1,k} + v_{j,k-1}\right)\ (v_{01} = 1, v_{10} = 0)$$

for $j,k = 1,2,\cdots$.

Proof. By the definition and Lemma 2.4 we know that

$$t^k(1+\sum_{j=1}^{\infty}v_{jk}t^j) = t^k(1+\frac{t}{2!}+\frac{t^2}{3!}+\cdots)^k = (t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots)^k = (e^t-1)^k$$
$$= \sum_{q=0}^{k}C_k^q e^{qt}(-1)^{k-q} = \sum_{q=0}^{k}\left[1+\sum_{j=1}^{\infty}\frac{(qt)^j}{j!}\right]C_k^q(-1)^{k-q}$$
$$= \sum_{q=0}^{k}C_k^q(-1)^{k-q} + \sum_{j=1}^{\infty}\frac{1}{j!}\left[\sum_{q=1}^{k}C_k^q(-1)^{k-q}q^j\right]t^j$$
$$= \sum_{j=1}^{\infty}\frac{1}{j!}\left[\sum_{q=1}^{k}C_k^q(-1)^{k-q}q^j\right]t^j$$
$$= \frac{1}{k!}\left[\sum_{q=1}^{k}C_k^q(-1)^{k-q}q^k\right]t^k + \sum_{j\neq k}^{\infty}\frac{1}{j!}\left[\sum_{q=1}^{k}C_k^q(-1)^{k-q}q^j\right]t^j$$

$$
\begin{aligned}
&= t^k + \sum_{j\neq k}^{\infty} \frac{1}{j!}\left[\sum_{q=1}^{k} C_k^q (-1)^{k-q} q^j\right] t^j \\
&= t^k + \sum_{j=k+1}^{\infty} \frac{1}{j!}\left[\sum_{q=1}^{k} C_k^q (-1)^{k-q} q^j\right] t^j \\
&= t^k + \sum_{j=1}^{\infty} \frac{(-1)^k}{(j+k)!}\left[\sum_{q=1}^{k} (-1)^q q^{j+k} C_k^q\right] t^{j+k}.
\end{aligned}
$$

And hence it follows that

$$
v_{jk} = \frac{(-1)^k}{(j+k)!} \sum_{q=0}^{k} (-1)^q q^{j+k} C_k^q
$$

by comparing with the coefficient of t^j.

Furthermore, by the direct calculation, we have

$$
\begin{aligned}
&\frac{k}{j+k}(v_{j-1,k} + v_{j,k-1}) \\
&= \frac{k}{j+k}\left[\frac{(-1)^k}{(j+k-1)!}\sum_{q=1}^{k}(-1)^q q^{j+k-1} C_k^q + \frac{(-1)^{k-1}}{(j+k-1)!}\sum_{q=1}^{k-1}(-1)^q q^{j+k-1} C_{k-1}^q\right] \\
&= \frac{(-1)^k k}{(j+k)!}\left[\sum_{q=1}^{k-1}(-1)^q q^{j+k-1}\left(C_k^q - C_{k-1}^q\right) + (-1)^k k^{j+k-1}\right] \\
&= \frac{(-1)^k}{(j+k)!}\left[\sum_{q=1}^{k-1}(-1)^q q^{j+k-1} k C_{k-1}^{q-1} + (-1)^k k^{j+k}\right] \\
&= \frac{(-1)^k}{(j+k)!}\left[\sum_{q=1}^{k-1}(-1)^q q^{j+k} C_k^q + (-1)^k k^{j+k}\right] = v_{jk}.
\end{aligned}
$$

□

LEMMA 2.6. *$w_{1k}^{(i)} = u_{i+1,k} - v_{i+1,k}$ for $i = 0,1,2,\cdots; k = 1,2,\cdots$ and $w_{p1}^{(i)} = \frac{1}{(i+p+1)i!p!}$ for $i = 0,1,2,\cdots;\ p = 1,2,\cdots$.*

Proof. By the definition of $\{u_{jk}\}, \{v_{jk}\}, \{w_{pk}^{(i)}\}$ and Lemma 2.3, we have

$$
\begin{aligned}
F_{1k}(t) &= \frac{1}{t^{k+1}}\left[t^k e^{kt} - (e^t - 1)^k\right] \\
&= \frac{1}{t}\left[(1 + u_{1k}t + u_{2k}t^2 + u_{3k}t^3 + \cdots) - (1 + v_{1k}t + v_{2k}t^2 + v_{3k}t^3 + \cdots)\right] \\
&= (u_{1k} - v_{1k}) + (u_{2k} - v_{2k})t + (u_{3k} - v_{3k})t^2 + \cdots
\end{aligned}
$$

and hence, we get $w_{1k}^{(i)} = u_{i+1,k} - v_{i+1,k}$, for $i = 0,1,2,\cdots; k = 1,2,\cdots$.

While

$$
\begin{aligned}
w_{p1}^{(i)} &= \sum_{j=0}^{i}(-1)^j u_{i-j,1}v_{j+p,1} = \sum_{j=0}^{i}\frac{(-1)^j}{(i-j)!(j+p+1)!} \\
&= \frac{1}{(i+p+1)!}\sum_{j=0}^{i}(-1)^j C_{i+p+1}^{j+p+1} \\
&= \frac{1}{(i+p+1)!}\left[\sum_{j=0}^{i-1}(-1)^j\left(C_{i+p}^{j+p}+C_{i+p}^{j+p+1}\right)+(-1)^i\right] \\
&= \frac{1}{(i+p+1)!}\left[\sum_{j=0}^{i}(-1)^j C_{i+p}^{j+p}-\sum_{j=1}^{i}(-1)^j C_{i+p}^{j+p}\right] \\
&= \frac{1}{(i+p+1)!}C_{i+p}^{p} = \frac{1}{(i+p+1)i!p!}.
\end{aligned}
$$

This completes the proof. □

LEMMA 2.7. *According to the definition of* $\{w_{pk}^{(i)}\}$ *in Lemma 2.3, we have the inductive relation*

$$
w_{pk}^{(i)} = \frac{k}{p+k+i}\left[w_{p-1,k}^{(i)}+w_{pk}^{(i-1)}+\sum_{q=0}^{i}\frac{1}{(i-q)!}w_{p,k-1}^{(q)}\right] \quad \text{for} \quad i=1,2,\cdots;\ p,k=2,3,\cdots.
$$

where $w_{pk}^{(0)} = v_{pk}$, $w_{1k}^{(i)} = u_{i+1,k}-v_{i+1,k}$, $w_{p1}^{(i)} = \frac{1}{(i+p+1)i!p!}$ *for* $i=0,1,2,\cdots;\ p,k = 1,2,\cdots$.

Proof. By the definition of $\{u_{jk}\}, \{w_{pk}^{(i)}\}$ and Lemma 2.5, we have

$$
\begin{aligned}
\sum_{j=0}^{i}(-1)^j u_{i-j,k}\left(v_{j+p-1,k}+v_{j+p,k-1}\right) &= \sum_{j=0}^{i}(-1)^j u_{i-j,k}\frac{j+p+k}{k}v_{j+p,k} \\
&= \sum_{j=0}^{i}(-1)^j\frac{p+k+i-(i-j)}{k}u_{i-j,k}v_{j+p,k} \\
&= \frac{p+k+i}{k}w_{pk}^{(i)}-w_{pk}^{(i-1)}
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{j=0}^{i}(-1)^j u_{i-j,k}v_{j+p,k-1} &= \sum_{j=0}^{i}(-1)^j\frac{(k-1+1)^{i-j}}{(i-j)!}v_{j+p,k-1} \\
&= \sum_{j=0}^{i}\frac{(-1)^j}{(i-j)!}\left[\sum_{h=0}^{i-j}C_{i-j}^{h}(k-1)^h\right]v_{j+p,k-1} \\
&= \sum_{j=0}^{i}(-1)^j\left[\sum_{h=0}^{i-j}\frac{(k-1)^h}{h!(i-j-h)!}\right]v_{j+p,k-1}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=0}^{i}(-1)^j\left[\sum_{h=0}^{i-j}\frac{1}{(i-j-h)!}u_{h,k-1}\right]v_{j+p,k-1}\\
&\overset{q=i-j-h}{=} \sum_{j=0}^{i}(-1)^j\left[\sum_{q=0}^{i-j}\frac{1}{q!}u_{i-j-q,k-1}\right]v_{j+p,k-1}\\
&= \sum_{q=0}^{i}\frac{1}{(i-q)!}\left[\sum_{j=0}^{q}(-1)^j u_{q-j,k-1}v_{j+p,k-1}\right]\\
&= \sum_{q=0}^{i}\frac{1}{(i-q)!}w_{p,k-1}^{(q)}.
\end{aligned}
$$

Therefore

$$
w_{p-1,k}^{(i)}=\sum_{j=0}^{i}(-1)^j u_{i-j,k}v_{j+p-1,k}=\frac{p+k+i}{k}w_{pk}^{(i)}-w_{pk}^{(i-1)}-\sum_{q=0}^{i}\frac{1}{(i-q)!}w_{p,k-1}^{(q)}.
$$

While, we have $w_{pk}^{(0)}=v_{pk}$ by direct calculation, and $w_{1k}^{(i)}=u_{i+1,k}-v_{i+1,k}$, $w_{p1}^{(i)}=\frac{1}{(i+p+1)i!p!}$ for $i=0,1,2,\cdots;\ p,k=1,2,\cdots$ by Lemma 2.6, then the proof is completed. □

LEMMA 2.8. *$0<w_{pk}^{(i)}\leqslant u_{ik}v_{pk}$ for $i=0,1,2,\cdots;p,k=1,2,\cdots$.*

Proof. Firstly, we have known that $w_{pk}^{(0)}=v_{pk}>0$ $(p,k=1,2,\cdots)$ by the definition and $w_{1k}^{(i)}=u_{i+1,k}-v_{i+1,k}>0$ $(i=0,1,2,\cdots;\ k=1,2,\cdots)$, $w_{p1}^{(i)}=\frac{1}{(i+p+1)i!p!}>0$ $(i=0,1,2\cdots;\ p=1,2,\cdots)$ by Lemma 2.6. And hence it follows $w_{pk}^{(i)}>0$ $(i=0,1,2,\cdots;\ p,k=1,2,\cdots)$ by Lemma 2.7 and induction.

Secondly, by direct calculation, we have $w_{pk}^{(0)}=u_{0k}v_{pk}$ $(u_{0k}=1)$ and

$$
\begin{aligned}
w_{pk}^{(i)} &= \sum_{j=0}^{i}(-1)^j u_{i-j,k}v_{j+p,k}=u_{ik}v_{pk}-\sum_{j=1}^{i}(-1)^{j-1}u_{i-j,k}v_{j+p,k}\\
&= u_{ik}v_{pk}-\sum_{j=0}^{i-1}(-1)^j u_{i-1-j,k}v_{j+p+1,k}=u_{ik}v_{pk}-w_{p+1,k}^{(i-1)}\\
&< u_{ik}v_{pk} \quad \text{for} \quad i,p,k=1,2,\cdots.
\end{aligned}
$$

□

LEMMA 2.9. *Let A be a self-adjoint operator on a Hilbert space H. If $a,b\in R$ such that $0<a\leqslant b$, then $aA\geqslant -b\|A\|I$.*

Proof. For $\mu\in\sigma(aA+b\|A\|I)$, there exists $\lambda\in\sigma(A)$ such that $\mu=a\lambda+b\|A\|$. It is well-known that $|\lambda|\leqslant\|A\|$, so $\mu=a\lambda+b\|A\|\geqslant b\|A\|-a|\lambda|\geqslant 0$. Noticing that $aA+b\|A\|I$ is also self-adjoint, it gives $aA+b\|A\|I\geqslant 0$. □

LEMMA 2.10. *Let A and B be two self-adjoint operators on a Hilbert space H. If $A > B$, then for all integers $p \geqslant 1$, $k \geqslant 1$, there exists $\beta \in (0,1]$, which is independent of p, such that $F_{pk}(\alpha A) > F_{pk}(\alpha B)$ for all $\alpha \in (0,\beta)$, where $F_{pk}(t)$ $(p,k = 1,2,\cdots)$ are defined in Lemma 2.3.*

Proof. As $A > B$ holds, there exists $\varepsilon > 0$ such that $A - B \geqslant \varepsilon I > 0$. Denote $\mu_k := \inf_{p \geqslant 1} \frac{w_{pk}^{(1)}}{v_{pk}}$ and choose

$$\beta = \min\left\{ \frac{\mu_k \varepsilon}{(e^{k\|A\|} + e^{k\|B\|})},\ 1 \right\}.$$

Now, it is easy to see, from Lemma 2.5, (i) $v_{p+1,k} = \frac{k}{p+1+k}(v_{p,k} + v_{p+1,k-1})$, and by definition, (ii) $v_{p+1,k-1} < v_{p+1,k}$. thus, by (i) and (ii), we can obtain $v_{p+1,k} < \frac{k}{p+1} v_{p,k}$. Note that by Lemmas 2.3 and 2.8, $\omega_{pk}^{(1)} = u_{1k}v_{pk} - u_{0k}v_{p+1,k} = kv_{pk} - v_{p+1,k} > 0$, thus $\frac{\omega_{pk}^{(1)}}{v_{pk}} < k$; while, $\frac{\omega_{pk}^{(1)}}{v_{pk}} = k - \frac{v_{p+1,k}}{v_{pk}} > k - \frac{k}{p+1} = \frac{pk}{p+1} \geqslant \frac{k}{2}$, then $\frac{k}{2} \leqslant \mu_k < k$. And hence, β can be independently from p and $\beta \in (0,1]$.

Next, by Lemma 2.3, Lemma 2.8 and Lemma 2.9, for all $\alpha \in (0,\beta)$, we have

$$\begin{aligned}
F_{pk}(\alpha A) - F_{pk}(\alpha B) &= p! \sum_{i=0}^{\infty} w_{pk}^{(i)} \left[(\alpha A)^i - (\alpha B)^i \right] \\
&= \alpha p! \left[w_{pk}^{(1)}(A-B) + \alpha \sum_{i=2}^{\infty} w_{pk}^{(i)} \alpha^{i-2}(A^i - B^i) \right] \\
&\geqslant \alpha p! \left[w_{pk}^{(1)} \varepsilon - \alpha \sum_{i=2}^{\infty} u_{ik} v_{pk} \|A^i - B^i\| \right] I \\
&\geqslant \alpha p! \left[w_{pk}^{(1)} \varepsilon - \alpha v_{pk} \sum_{i=0}^{\infty} u_{ik} \left(\|A\|^i + \|B\|^i \right) \right] I \\
&= \alpha p! \left[w_{pk}^{(1)} \varepsilon - \alpha v_{pk} (e^{k\|A\|} + e^{k\|B\|}) \right] I > 0.
\end{aligned}$$

It gives the conclusion. □

To prove Theorem 2.1 in this paper, we have only to replace A and B by $\log A$ and $\log B$ respectively in Lemma 2.10.

LEMMA 2.11. *If we denote*

$$H_{pk}(t) = \frac{e^{kt}}{t^p} \left[\left(\frac{e^t - 1}{t} \right)^k - \sum_{j=1}^{p} v_{j-1,k} t^{j-1} \right],$$

then

$$H_{pk}(t) = \sum_{i=0}^{\infty} \varpi_{pk}^{(i)} t^i \quad \textit{for} \quad p,k = 1,2,\cdots$$

where

$$\varpi_{pk}^{(i)} = \sum_{j=0}^{i} u_{i-j,k} v_{j+p,k} \quad \textit{in which} \quad u_{0k} = 1$$

as well as the relation $0 < \varpi_{pk}^{(i)} \leqslant \frac{(2k)^{i+p}}{(i+p)!}$ *for* $i = 0, 1, 2, \cdots; p, k = 1, 2, \cdots$.

LEMMA 2.12. *Let A and B be two self-adjoint operators on a Hilbert space H. If $A > B$, then for all integers $p \geqslant 1$, $k \geqslant 1$, there exists $\beta \in (0,1]$ such that $H_{pk}(\alpha A) > H_{pk}(\alpha B)$ for all $\alpha \in (0,\beta)$.*

Proof. As $A > B$ holds, there exists $\varepsilon > 0$ such that $A - B \geqslant \varepsilon I > 0$. Choosing

$$\beta = \min\left\{ \frac{\varpi_{pk}^{(1)} \varepsilon}{\frac{e^{2k\|A\|}}{\|A\|^p} + \frac{e^{2k\|B\|}}{\|B\|^p}},\ 1 \right\},$$

we obtain, by Lemma 2.9 and Lemma 2.11, for all $\alpha \in (0,\beta)$

$$\begin{aligned}
H_{pk}(\alpha A) - H_{pk}(\alpha B) &= \sum_{i=0}^{\infty} \varpi_{pk}^{(i)} \left[(\alpha A)^i - (\alpha B)^i \right] \\
&= \alpha \left[\varpi_{pk}^{(1)} (A - B) + \alpha \sum_{i=2}^{\infty} \varpi_{pk}^{(i)} \alpha^{i-2} (A^i - B^i) \right] \\
&\geqslant \alpha \left[\varpi_{pk}^{(1)} \varepsilon - \alpha \sum_{i=2}^{\infty} \frac{(2k)^{i+p}}{(i+p)!} \|A^i - B^i\| \right] I \\
&\geqslant \alpha \left[\varpi_{pk}^{(1)} \varepsilon - \alpha \sum_{i=0}^{\infty} \frac{(2k)^{i+p}}{(i+p)!} \left(\frac{\|A\|^{i+p}}{\|A\|^p} + \frac{\|B\|^{i+p}}{\|B\|^p} \right) \right] I \\
&\geqslant \alpha \left[\varpi_{pk}^{(1)} \varepsilon - \alpha \left(\frac{e^{2k\|A\|}}{\|A\|^p} + \frac{e^{2k\|B\|}}{\|B\|^p} \right) \right] I > 0.
\end{aligned}$$

It gives the conclusion. □

THEOREM 2.13. *Let A and B be two strictly positive operators on a Hilbert space H. If $\log A > \log B$ with $1 \notin \sigma(A), \sigma(B)$, then for all integers $p \geqslant 1, k \geqslant 1$, there exists $\beta \in (0,1]$ such that $h_{pk}(A^\alpha) > h_{pk}(B^\alpha)$ for all $\alpha \in (0,\beta)$, where $v_{0k} = 1$,*

$$h_{pk}(t) = \frac{t^k}{\log^p t} \left[\left(\frac{t-1}{\log t} \right)^k - \sum_{j=1}^{p} v_{j-1,k} \log^{j-1} t \right], \quad p, k = 1, 2, \cdots .$$

Proof. We have only to replace A and B by $\log A$ and $\log B$ respectively in Lemma 2.12. □

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