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ON THE COMPLETE MONOTONICITY OF QUOTIENT OF GAMMA FUNCTIONS

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Abstract. In this paper, we concern with a conjecture involving the ratio of two gamma functions posed by Qi, Guo and Chen [Math. Inequal. Appl. 9(3)(2006), 427-436]. We also generalize a result of Qi [Theorem 2, J. Comput. Appl. Math. 214 (2008), 610-616].

1. Introduction

The Euler gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty e^{-t}e^{x-1}dt.$$

The psi or digamma function, the logarithmic derivative of the gamma function is expressed by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \int_0^\infty \frac{e^{-t}-e^{-xt}}{1-e^{-t}}dt = -\gamma + \frac{1}{x} + \sum_{k=1}^\infty \frac{x}{k(x+k)} \tag{1}$$

and the polygamma function $\psi^{(n)}(x)$ can be expressed as

$$\psi^{(n)}(x) = (-1)^{n+1}\int_0^\infty \frac{t^n}{1-e^{-t}}e^{-xt}dt = (-1)^{n+1}n!\sum_{k=0}^\infty \frac{1}{(x+k)^{n+1}} \tag{2}$$

for all $x > 0$ and $n \in \mathbb{N}$, where $\gamma = 0.5772156649\cdots$ is the Euler-Mascheroni constant. There exists a very extensive literature on the gamma function, about its characterizations, applications and history, for example, see [1, 2, 3, 4, 6, 7, 8, 12, 16] and the related references therein.

A function f is said to be *completely monotonic* on an interval I, if it has all derivatives of all orders on I and satisfies $(-1)^n f^{(n)}(x) \geqslant 0$, for all $n \geqslant 0$ and $x \in I$. The function f is said to be *completely monotonic function of order* k, $k \in \mathbb{N}$, on an interval I if $f^{(k)}$ is completely monotonic and $f^{(k-1)}$ is not completely monotonic on

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I. A positive function f is said to be *logarithmically completely monotonic* on an interval I, if it has derivatives of all orders on I and its logarithm $\ln f$ is completely monotonic on I. Several general properties for the set of (logarithmically) completely monotonic functions have been studied. For instance, in [13, 17], it was recovered that any logarithmically completely monotonic function f on interval I must be completely monotonic on I, but not conversely. Moreover, in [9], it was shown that every completely monotonic function on interval $(0,+\infty)$ is logarithmically convex.

Define the functions

$$\omega_{s,t}(x) = \frac{\psi(x+t)-\psi(x+s)}{t-s} \text{ and } \Psi_{s,t}(x) = \begin{cases} \left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{\frac{1}{t-s}}, & s \neq t, \\ \psi(x+s), & s=t. \end{cases} \tag{3}$$

Qi, Guo and Chen [16] studied the function $\Psi_{s,t}(x)$ and showed the following result.

PROPOSITION 1. (See [16]) *Let s,t be nonnegative numbers and $\rho = \min\{s,t\}$, the function $\frac{1}{\Psi_{s,t}(x)}$ is logarithmically completely monotonic on $x \in (-\rho,\infty)$.*

In addition, they posed the following conjecture.

CONJECTURE 1. (See [16, Open problem 2]) *Let s and t be nonnegative real numbers such that $s \neq t$ and let $\rho = \min(s,t)$. The functions*

$$\mathscr{P}_{s,t}(x) = \Psi_{s,t}(x)\ \omega_{s,t}(x) \text{ and } \mathscr{Q}_{s,t}(x) = \frac{1}{\omega_{s,t}(x)} - \Psi_{s,t}(x)$$

are completely monotonic for $x \in (-\rho,\infty)$.

The main result of this paper can be formulated as follows, which gives a negative answer for the above conjecture.

THEOREM 1. *Neither the function $\mathscr{P}_{s,t}(x)$ nor $\mathscr{Q}_{s,t}(x)$ is completely monotonic on $(-\rho,\infty)$.*

For positive numbers p and q such that $p \neq q$, we define the *logarithmic mean* of p,q to be

$$L(p,q) = \frac{p-q}{\ln p - \ln q}.$$

In 2008, Qi and his coathors [11, 18] showed the following result.

THEOREM 2. (See [11, Theorem 2]) *For $s,t \in \mathbb{R}$ with $s \neq t$, the function*

$$\mathscr{O}_{s,t}(x) = \left[\frac{\Gamma(x+s)}{\Gamma(x+t)}\right]^{\frac{1}{(s-t)}} \cdot \frac{1}{e^{\psi(L(x+s,x+t))}}$$

is decreasing on $x > -\min\{s,t\}$.

The second main result of this paper is to improve Theorem 2.

THEOREM 3. *For $s,t \in \mathbb{R}$ with $s \neq t$, the function $\mathscr{O}_{s,t}(x)$ is logarithmically convex with respect to $x > -\min\{s,t\}$.*

2. Proofs of theorems

In order to prove our main results, we need the following lemmas.

LEMMA 1. (see [2, 9])
(i) The sum, the product, and the pointwise limit of (logarithmically) completely monotonic functions are also (logarithmically) completely monotonic.
(ii) A logarithmically completely monotonic function must be completely monotonic.
(iii) Let $f:(0,\infty)\to\mathbb{R}$ and $g:I\to(0,\infty)$. If f and g' are completely monotonic, then $x\mapsto f(g(x))$ is also completely monotonic.

LEMMA 2. (See [4, 5, 16]) *Let $s,t\in\mathbb{R}$ with $t-s\neq\pm 1$ and define*

$$z_{s,t}(x) = \begin{cases} \Psi_{s,t}(x)-x, & s\neq t, \\ e^{\psi(x+s)}, & s=t, \end{cases}$$

for $x\in(-\alpha,\infty)$ with $\alpha=\min\{s,t\}$. Then the function $z_{s,t}(x)$ is either convex and decreasing for $|t-s|<1$ or concave and increasing for $|t-s|>1$.

LEMMA 3. ((See [12, 15])) *Let $t>s\geqslant 0$ and $k\in\mathbb{N}$. Then the double inequality*

$$\frac{(k-1)!}{(x+\alpha)^k}\leqslant\frac{(-1)^{k-1}[\psi^{(k-1)}(x+t)-\psi^{(k-1)}(x+s)]}{t-s}\leqslant\frac{(k-1)!}{(x+\beta)^k},\ \text{for } x>-\min\{s,t\},$$

holds if $\alpha\geqslant\max\{s,\frac{s+t-1}{2}\}$ and $0\leqslant\beta\leqslant\min\{s,\frac{s+t-1}{2}\}$.

LEMMA 4. (see [10, Theorem 2]) *For $x\in(0,\infty)$, we have*

$$-\frac{(n-1)!}{x^n}-\frac{n!}{x^{n+1}}<(-1)^n\psi^n(x)<-\frac{(n-1)!}{x^n}-\frac{n!}{2x^{n+1}}. \tag{4}$$

2.1. The proof of the Theorem 1

Firstly, we give the proof of the prior part. Since

$$\omega_{s,t}(x)=(\ln\Psi_{s,t}(x))'=\frac{\Psi'_{s,t}(x)}{\Psi_{s,t}(x)},$$

we have

$$\mathscr{P}_{s,t}(x)=\Psi'_{s,t}(x). \tag{5}$$

According to Lemma 2, it is immediate that $\mathscr{P}'_{s,t}(x) > 0$ for $|t-s| < 1$ and $\mathscr{P}'_{s,t}(x) < 0$ for $|t-s| > 1$, which implies that $\mathscr{P}_{s,t}(x)$ is not completely monotonic for $|t-s| < 1$.

Moreover, when $|t-s| > 1$, since $\mathscr{P}_{s,t}(x) = \mathscr{P}_{t,s}(x)$, without loss of generality, we assume that $t-s > 1$ and $(-1)^k(\mathscr{P}_{s,t}(x))^{(k)} \geqslant 0$ for $k < n$, according to (5), which is equivalent to $(-1)^j\Psi_{s,t}^{(j)}(x) \leqslant 0$ for all $j = 1,2,\ldots n$. By the mean value theorem and (2), it is immediate that

$$(-1)^k\omega_{s,t}^{(k)}(x) = (-1)^k\psi^{(k+1)}(x+\varepsilon) > 0, \tag{6}$$

where $\varepsilon \in (s,t)$. Thus $\omega_{s,t}(x)$ is completely monotonic on $(-\rho,+\infty)$. Note that this result can be derived from [16, Proposition 3]. On the other hand, by Leibnitz's rule and induction on n, we obtain

$$\begin{aligned}(-1)^n\mathscr{P}_{s,t}^{(n)}(x) &= (-1)^n\sum_{k=0}^{n}\binom{n}{k}\Psi_{s,t}^{(k)}(x)\omega_{s,t}^{(n-k)}(x)\\ &= \sum_{k=0}^{n}\binom{n}{k}(-1)^k\Psi_{s,t}^{(k)}(x)\cdot(-1)^{n-k}\omega_{s,t}^{(n-k)}(x)\\ &\triangleq \Psi_{s,t}(x)\cdot H_{s,t}(x) + \sum_{k=2}^{n}\binom{n}{k}(-1)^k\Psi_{s,t}^{(k)}(x)\cdot(-1)^{n-k}\omega_{s,t}^{(n-k)}(x),\end{aligned}$$

where

$$H_{s,t}(x) = (-1)^n\omega_{s,t}^{(n)}(x) - n\cdot\omega_{s,t}(x)\cdot(-1)^{n-1}\cdot\omega_{s,t}^{(n-1)}(x).$$

By the mean value theorem and using Lemma 4, we have

$$\omega_{s,t}(x) = \psi'(x+\xi) > \frac{1}{x+\xi} + \frac{1}{2(x+\xi)^2} \tag{7}$$

$$\omega_{s,t}^{(n-1)}(x) = \psi^{(n)}(x+\eta), \tag{8}$$

where $\xi,\eta \in (s,t)$. Consequently, by Lemma 3 and Lemma 4, we obtain that

$$\begin{aligned}H_{s,t}(x) &< \frac{n!}{(x+\beta)^{n+1}} + n\cdot\omega_{s,t}(x)\cdot(-1)^n\psi^n(x+\eta)\\ &< \frac{n!}{(x+\beta)^{n+1}} - n\cdot\omega_{s,t}(x)\left\{(n-1)!\cdot\frac{1}{(x+\eta)^n} + \frac{n!}{2(x+\eta)^{n+1}}\right\}\\ &= n!\cdot\left\{\frac{1}{(x+\beta)^{n+1}} - \omega_{s,t}(x)\cdot\frac{1}{(x+\eta)^n}\left(1+\frac{n}{2(x+\eta)}\right)\right\}\\ &\triangleq D_{s,t}(x).\end{aligned}$$

where $\beta = s$. Let $D_{s,t}(x) < 0$, it leads to

$$\omega_{s,t}(x) > \frac{(x+\eta)^n}{(x+\beta)^{n+1}\cdot(1+\frac{n}{2(x+\eta)})}.$$

According to the inequality (7), let

$$\frac{1}{x+\xi}+\frac{1}{2(x+\xi)^2}\geqslant\frac{(x+\eta)^n}{(x+\beta)^{n+1}\cdot(1+\frac{n}{2(x+\eta)})}.$$

we have

$$x+\xi+\frac{1}{2}\geqslant\frac{(x+\xi)^2(x+\eta)^n}{(x+\beta)^{n+1}\cdot(1+\frac{n}{2(x+\eta)})}. \tag{9}$$

For the fixed value x, take $n\geqslant(x+t)^3$, then there must exist some interval I such that for $x\in I$, the inequality (9) is found, which indicates that $H_{s,t}(x)<0$ and $(-1)^n\mathscr{P}_{s,t}^{(n)}(x)$ <0. Thus it leads to the contradiction, so it is concluded that the function $\mathscr{P}_{s,t}(x)$ is not completely monotonic on $(-\rho,\infty)$ and its completely monotonic property has nothing to do with the parameters s and t.

Secondly, by the same consideration, we have

$$\mathscr{Q}_{s,t}(x)=\frac{1-\Psi'_{s,t}(x)}{\omega_{s,t}(x)}$$
$$\mathscr{Q}'_{s,t}(x)=-\frac{\omega_{s,t}^3(x)\Psi_{s,t}(x)+\omega'_{s,t}(x)}{\omega_{s,t}^2(x)}\triangleq\frac{-G_{s,t}(x)}{\omega_{s,t}^2(x)}.$$

In the case $|t-s|>1$, by Lemma 2 we have $\Psi'_{s,t}(x)>1$, which implies that $\mathscr{Q}_{s,t}(x)<0$ and $\mathscr{Q}_{s,t}(x)$ is not completely monotonic on $(-\rho,+\infty)$ for all s,t with $|t-s|>1$.

In the case $|t-s|<1$, assume that $\mathscr{Q}_{s,t}(x)$ is completely monotonic. According to (6) and Lemma 1, it is immediately that $G_{s,t}(x)$ is completely monotonic on $(-\rho,\infty)$, and by simple computation, we have

$$G'_{s,t}(x)=\omega_{s,t}(x)\cdot\Psi'_{s,t}(x)\cdot(\omega_{s,t}^2(x)+3\omega'_{s,t}(x))+\omega''_{s,t}(x).$$

By Lemma 2, it leads to $0<\Psi'_{s,t}(x)<1$, thus for $x\in(-\rho,\infty)$,

$$\omega_{s,t}^2(x)+3\omega'_{s,t}(x)<0.$$

On the other hand, since

$$\frac{G_{s,t}(x)}{\Psi'_{s,t}(x)}=\omega_{s,t}^2(x)+\frac{\omega'_{s,t}(x)}{\Psi'_{s,t}(x)}>0,$$

here exists an interval I, such that for $x\in I$ and $\delta\geqslant 3$, $\omega_{s,t}^2(x)+\delta\omega'_{s,t}(x)>0$, which leads to the contradiction. Thus $\mathscr{Q}_{s,t}(x)$ is not completely monotonic on $(-\rho,\infty)$ and has nothing to do with the parameters s and t too, which completes the proof of Theorem 1. □

2.2. Proof of Theorem 3

Taking logarithm of the function $\mathscr{O}_{s,t}(x)$ and using the mean value theorem gives

$$\ln \mathscr{O}_{s,t}(x) = \frac{1}{s-t}\int_t^s \psi(x+u)du - \psi(L(x+s,x+t)). \tag{10}$$

Substituting (1) into (10) gives

$$\ln \mathscr{O}_{s,t}(x) = \sum_{i=1}^{\infty}\left[\frac{1}{i+L(x+s,x+t)} - \frac{1}{s-t}\int_t^s \frac{1}{i+x+u}du\right]$$

and

$$[\ln \mathscr{O}_{s,\,t}(x)]' = \sum_{i=1}^{\infty}\left[\frac{1}{s-t}\int_t^s \frac{1}{(i+x+u)^2}du - \frac{L^2(x+s,x+t)}{(x+s)(x+t)}\frac{1}{(i+L(x+s,x+t))^2}\right].$$

Let $A = A(s,t;x) = x+\frac{s+t}{2}$, and let us write $L(x+s,x+t)$ as L for short. By simple computation, we have (see [11, Page 615])

$$L' = \frac{L^2}{(x+s)(x+t)} \text{ and } L'' = \frac{2L^2}{(x+s)^2(x+t)^2}(L-A).$$

Also, we have

$$\begin{aligned}
(\ln \mathscr{O}_{s,t}(x))'' &= \sum_{i=1}^{\infty}\frac{-2}{s-t}\int_t^s \frac{1}{(i+x+u)^3}du - \left(\sum_{i=1}^{\infty}\frac{1}{(i+L)^2}L'\right)' \\
&= \sum_{i=1}^{\infty}\frac{-2}{s-t}\int_t^s \frac{1}{(i+x+u)^3}du - \frac{2L^2}{(x+s)^2(x+t)^2}(L-A)\sum_{i=1}^{\infty}\frac{1}{(i+L)^2} \\
&\quad + \sum_{i=1}^{\infty}\frac{2L^4}{(i+L)^3(x+s)^2(x+t)^2} \\
&= \frac{1}{s-t}\sum_{i=1}^{\infty}\frac{2(i+x)(t-s)+t-s}{(i+x+s)^2(i+x+t)^2} + \frac{2L^4}{(x+s)^2(x+t)^2}\sum_{i=1}^{\infty}\frac{1}{(i+L)^3} \\
&\quad + \frac{2L^2}{(x+s)^2(x+t)^2}(A-L)\sum_{i=1}^{\infty}\frac{1}{(i+L)^2}.
\end{aligned}$$

By the standard argument (see [11]), we obtain $\sqrt{(x+s)(x+t)} < L < A$, which implies

$$\begin{aligned}
&\left(\frac{\ln \mathscr{O}_{s,t}(x)}{2}\right)'' \\
&\quad > \frac{L^4}{(x+s)^2(x+t)^2}\sum_{i=1}^{\infty}\frac{1}{(i+L)^3} + \frac{A-L}{(x+s)(x+t)}\sum_{i=1}^{\infty}\frac{1}{(i+x+s)(i+x+t)} \\
&\qquad\quad - \sum_{i=1}^{\infty}\frac{i+A}{(i+x+s)^2(i+x+t)^2} \\
&\quad > \frac{L^4}{(x+s)^2(x+t)^2}\sum_{i=1}^{\infty}\frac{1}{(i+L)^3} + \frac{A-L}{(x+s)(x+t)}\sum_{i=1}^{\infty}\frac{(x+s)(x+t)}{(i+x+s)^2(i+x+t)^2} \\
&\qquad\quad - \sum_{i=1}^{\infty}\frac{i+A}{(i+x+s)^2(i+x+t)^2} \\
&\quad = \sum_{i=1}^{\infty}\left[\frac{L^4}{(x+s)^2(x+t)^2(i+L)^3} - \frac{i+L}{(i+x+s)^2(i+x+t)^2}\right].
\end{aligned}$$

In order to prove the convexity property of (10), it is sufficient to verify that

$$\frac{i\sqrt{(x+s)(x+t)}}{\sqrt{(i+x+s)(i+x+t)}-\sqrt{(x+s)(x+t)}} < L, \tag{11}$$

for $s,t \in \mathbb{R}$ and $x > -\min\{s,t\}$ with $s \neq t$. However, (11) can follow clearly from the inequality $\sqrt{(i+x+s)(i+x+t)}-\sqrt{(x+s)(x+t)} > i$, which can be obtained easily by simple computation. Obviously, if a positive function f is logarithmically convex on the interval I, then it is also convex on I, which completes the proof of Theorem 3. □

We end this section by noting that we obtain the following property for the function $\frac{1}{e^{\psi(L(s,t;\, x))}}$.

THEOREM 4. *For $s,t \in \mathbb{R}$ with $s \neq t$, the function*

$$\mathscr{K}_{s,t}(x) = \frac{1}{e^{\psi(L(s,t;\, x))}}$$

is logarithmically completely monotonic for $x > -\min\{s,t\}$.

Proof. By simple computation, we have

$$(-\ln \mathscr{K}_{s,\,t}(x))' = \psi'(L(x+s,x+t)) \cdot L'(x+s,x+t).$$

By general properties of logarithmically completely monotonic functions, it is sufficient to the condition that $L'(x+s,x+t)$ is completely monotonic for $x > -\min\{s,t\}$, which is equivalent to show that $L(x+s,x+t)$ is a completely monotonic function (of the first order) for $x > -\min\{s,t\}$ and $s,t \in \mathbb{R}$ with $s \neq t$. Whereas, this has been given in [14], which completes the proof.

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