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FACTORIZATION OF FACTORIALS AND A RESULT OF HARDY AND RAMANUJAN

MEHDI HASSANI

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Abstract. We obtain an explicit approximation for the sum of prime powers in the factorization of $n!$ into prime numbers. This reproves, as more as gives an explicit version to, a well-known result of Hardy and Ramanujan concerning the summation $\sum_{k\leqslant n}\Omega(k)$.

1. Introduction

Let $v_p(n!)$ be the power of the prime p in the factorization of $n!$ into prime numbers. In this paper, we find some bounds for the arithmetic function

$$\mathscr{S}(n)=\sum_{p\leqslant n} v_p(n!),$$

where in this sum (and what follows below) p runs over primes. Indeed, we show the following result.

THEOREM 1.1. *For $n\geqslant 2$ we have*

$$n\log\log n+M'n-\frac{814921n}{\log n}<\mathscr{S}(n)<n\log\log n+M'n+\frac{n}{\log^2 n}. \tag{1.1}$$

where M' is a constant defined by

$$M'=\gamma+\sum_p\left(\log\left(1-p^{-1}\right)+(p-1)^{-1}\right)=1.0346538818\cdots.$$

We note that γ refers to Euler's constant. Moreover, the constant M' has been known before (see [2]), and to evaluate it with high numerical precision we can use the representation

$$M'=\gamma+\sum_{k=2}^{\infty}\frac{\varphi(k)\log\zeta(k)}{k}.$$

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Now, we let $\Omega(k)$ be the total number of prime factors of k. A result of Hardy and Ramanujan [3] asserts that

$$\sum_{k\leqslant n}\Omega(k)=n\log\log n+M'n+O\Big(\frac{n}{\log n}\Big). \tag{1.2}$$

By considering the relation

$$\sum_{k\leqslant n}\Omega(k)=\Omega(n!)=\mathscr{S}(n),$$

we observe that the truth of Theorem 1.1 gives an explicit version of (1.2). This improves the previously known [1] bound

$$\Big|\sum_{k\leqslant n}\Omega(k)-n\log\log n\Big|<23n \qquad (n\geqslant 3).$$

2. Required explicit bounds

In this section, we state some required explicit bounds, which allow us to obtain explicit approximation of $\mathscr{S}(n)$. Starting point of the proof of Theorem 1.1 is the following lemma, which is Theorem 2.1 from [4].

LEMMA 2.1. *For every $n\in\mathbb{N}$ and prime p with $p\leqslant n$ we have*

$$\frac{n-p}{p-1}-\frac{\log n}{\log p}<v_p(n!)\leqslant\frac{n-1}{p-1}. \tag{2.1}$$

Then, let us set

$$\mathscr{A}(n):=\sum_{p\leqslant n}\frac{1}{p},\quad \mathscr{B}(n):=\sum_{p\leqslant n}\frac{1}{p-1},\quad \mathscr{C}(n):=\sum_{p\leqslant n}\frac{1}{\log p},\quad \mathscr{D}(n):=\sum_{p\leqslant n}\frac{1}{p(p-1)}.$$

Theorem 5 from [5] gives the following bound for $\mathscr{A}(n)$.

LEMMA 2.2. *For every $n\geqslant 2$ we have*

$$\log\log n+M-\frac{1}{2\log^2 n}<\mathscr{A}(n)<\log\log n+M+\frac{1}{\log^2 n}, \tag{2.2}$$

where M is a constant defined by

$$M=\gamma+\sum_{p}\Big(\log\big(1-p^{-1}\big)+p^{-1}\Big)=0.2614972128\cdots.$$

The constant M is known as the Meissel-Mertens constant. To compute M we may apply rapidly converging series

$$M=\gamma+\sum_{k=2}^{\infty}\frac{\mu(k)\log\zeta(k)}{k}.$$

LEMMA 2.3. *For $n \geqslant 2$ we have*

$$\mathscr{D}(n) = (M' - M) - \mathscr{E}(n),$$

where $\mathscr{E}(n)$ is an error term satisfying the bound $1/(3n^2) < \mathscr{E}(n) < 1/n$.

Proof. We have

$$\mathscr{D}(n) = \sum_p \frac{1}{p(p-1)} - \sum_{p>n} \frac{1}{p(p-1)} = M' - M - \mathscr{E}(n),$$

where we let $\mathscr{E}(n) = \sum_{p>n} 1/(p(p-1))$. We note that

$$\mathscr{E}(n) < \sum_{j=n+1}^{\infty} \frac{1}{j(j-1)} = \frac{1}{n}.$$

Also, we write

$$\mathscr{E}(n) = \sum_{j=0}^{\infty} \left(\sum_{2^j n < p \leqslant 2^{j+1} n} \frac{1}{p(p-1)} \right).$$

Bertrand's postulate asserts that for any integers $j \geqslant 0$ and $n \geqslant 2$ there exists at least one prime p with $2^j n < p < 2^{j+1} n$. Thus, we obtain

$$\mathscr{E}(n) > \sum_{j=0}^{\infty} \frac{1}{(2^{j+1} n)^2} = \frac{1}{3n^2}.$$

This implies the stated result.

It seems that the true order of $\mathscr{E}(n)$ is $1/(n \log n)$. Also, since we have $\mathscr{B}(n) = \mathscr{A}(n) + M' - M - \mathscr{E}(n)$, we can easily get some explicit bounds for $\mathscr{B}(n)$ by using Lemma 2.2 and Lemma 2.3. About, $\mathscr{C}(n)$, we have the following result, which is Proposition 2.2 from [1].

LEMMA 2.4. *For every $n \geqslant 2$, we have*

$$\left| \mathscr{C}(n) - \left(\frac{n}{\log^2 n} + \frac{2n}{\log^3 n} + \frac{6n}{\log^4 n} \right) \right| < 271382 \frac{n}{\log^5 n}. \tag{2.3}$$

Finally, let us recall the statement of Theorem 1 from [5], which gives an upper bound for the function $\pi(x) = \sum_{p \leqslant x} 1$.

LEMMA 2.5. *For $n \geqslant 2$ we have*

$$\pi(n) < \frac{n}{\log n} \left(1 + \frac{3}{2 \log n} \right). \tag{2.4}$$

3. Proof of Theorem 1.1

3.1. Lower bound

By considering the left hand side of (2.1), we have

$$\mathscr{S}(n) > \sum_{p \leqslant n} \left(\frac{n-p}{p-1} - \frac{\log n}{\log p} \right) = (n-1)\mathscr{B}(n) - \pi(n) - \mathscr{C}(n)\log n.$$

We apply bounds for the functions $\mathscr{A}(n)$, $\mathscr{C}(n)$, $\mathscr{E}(n)$ and $\pi(n)$, which all hold for $n \geqslant 2$. We obtain

$$\mathscr{S}(n) > n\log\log n + M'n + R_\ell(n),$$

where

$$R_\ell(n) = -\frac{2n}{\log n} - \frac{4n}{\log^2 n} - \frac{6n}{\log^3 n} - \frac{271382n}{\log^4 n} - \log\log n - M' - 1 + \frac{1}{n} + \frac{1}{2\log^2 n}.$$

Now, we let $g(n) = R_\ell(n) + 814921n/\log n$. Since $g(2) > 0$ and for $n \geqslant 2$ we have $g'(n) > 0$, we obtain $R_\ell(n) > -814921n/\log n$. This proves the left hand side of (1.1).

3.2. Upper bound

By considering the right hand side of (2.1), we have

$$\mathscr{S}(n) \leqslant \sum_{p \leqslant n} \left(\frac{n-1}{p-1} \right) = (n-1)\mathscr{B}(n) = (n-1)\left(\mathscr{A}(n) + M' - M - \mathscr{E}(n)\right).$$

Now, we apply bounds for the functions $\mathscr{A}(n)$ and $\mathscr{E}(n)$, which hold for $n \geqslant 2$. We obtain

$$\mathscr{S}(n) < n\log\log n + M'n + R_u(n),$$

where $R_u(n) = (n-1)/\log^2 n - \log\log n - M' - (n-1)/(3n^2)$ and trivially we have $R_u(n) < n/\log^2 n$ for $n \geqslant 2$. This proves the right hand side of (1.1), and thus, completes the proof of Theorem 1.1.

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Mehdi Hassani
Department of Mathematics
Zanjan University
P.O.Box : 313
45371-38111 Zanjan
Iran
e-mail: mehdi.hassani@znu.ac.ir